

Heavenly constructions of quaternionic manifolds

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1. Background and motivation
2. The quaternionic Feix–Kaledin construction
3. Examples and applications

Joint work with Alexandra Borowka

1. H-space with a cosmological constant

A real-analytic 3-manifold with Riemannian conformal metric is naturally the conformal infinity of a germ-unique real-analytic 4-manifold with real-analytic Riemannian metric satisfying the self-dual Einstein equations with cosmological constant -1 .

—LeBrun 1982.

Heavenly construction (via Penrose twistor theory)

- ▶ Complexify, work in holomorphic category, add real structures.
- ▶ The *ambitwistor space* of null geodesics in a (suitably convex) holomorphic conformal 3-manifold B is holomorphic contact 3-manifold Z .
- ▶ The null geodesics through a point in B define a (legendrian) *twistor line* \mathcal{C} in Z , i.e., a rational curve with normal bundle $\mathcal{O}(1) \otimes \mathbb{C}^2$.
- ▶ Thus Z is a *twistor space*: by Kodaira theory, the moduli space of generic (non-legendrian) deformations of \mathcal{C} is a holomorphic self-dual Einstein 4-manifold.

1. Hyperkähler metrics on cotangent bundles

The cotangent bundle T^*S of a complex manifold S is a holomorphic symplectic manifold, and is often hyperkähler, e.g. S is $\mathbb{C}P^n$ or a coadjoint orbit (Calabi, Kronheimer).

General picture (Feix 2001, Kaledin 1999): for S a real-analytic Kähler manifold, \exists a germ-unique $U(1)$ -invariant hyperkähler metric on a tubular nbhd of the zero section in T^*S .

Heavenly construction (via HKLR twistor theory)

- ▶ Complexify, work in holomorphic category, add real structures.
- ▶ Given a holomorphic Kähler $2n$ -manifold S^c , construct a holomorphic $2n + 1$ manifold Z from $\hat{Z} := S^c \times \mathbb{C}P^1$ by blowing down the zero section along the $(1, 0)$ -foliation and the infinity section along the $(0, 1)$ -foliation.
- ▶ A $\mathbb{C}P^1$ fibre of $\hat{Z} \rightarrow S^c$ projects to a twistor line \mathcal{C} in Z , i.e., a rational curve with normal bundle $\mathcal{O}(1) \otimes \mathbb{C}^{2n}$.
- ▶ The moduli space of deformations of \mathcal{C} is a holomorphic $4n$ -manifold with a holomorphic hyperkähler metric, since Z has a fibration over $\mathbb{C}P^1$ with symplectic leaves.

1. Plan: generalizations and an analogy

Generalizations

Feix and Kaledin: when S has a complex affine structure with type $(1,1)$ curvature, can construct a hypercomplex structure on nbhd of zero section in TS .

Question: generalize to construction of quaternionic structures?

Complexification: an analogy

A real-analytic n -manifold M has a germ-unique *complexification*: a holomorphic n -manifold M^c with an antiholomorphic involution whose fixed point set is M .

Underlying complex manifold $(M^c_{\mathbb{R}}, J)$ has M as a *totally real submanifold*, i.e., $TM \cap J(TM) = 0$, so $J(TM) \cong TM$ is the normal bundle to M in $M^c_{\mathbb{R}}$.

Szoke and Bielawski: embed $M^c_{\mathbb{R}}$ as nbhd of zero section in TM .

Theorem. *A real-analytic projective manifold M has a complexification $M^c_{\mathbb{R}} \subseteq TM$ s.t. for any geodesic $\gamma \subseteq M$, $M^c_{\mathbb{R}} \cap T\gamma$ is a complex submanifold.*

1. Projective structures

Idea: seek common framework in projective geometry.

Let M be a (real) n -manifold. A (real) *affine connection* is a connection D on TM (e.g., $D = \nabla^g$ for a riemannian metric g).

Affine connections D and \tilde{D} on TM are *projectively equivalent* iff $\exists \gamma \in \Omega^1(M)$, a 1-form, with

$$\tilde{D}_X - D_X = \llbracket X, \gamma \rrbracket^r \in C^\infty(M, \mathfrak{gl}(TM)),$$

where $\llbracket X, \gamma \rrbracket^r(Y) := \frac{1}{2}(\gamma(X)Y + \gamma(Y)X)$.

We write $\tilde{D} = D + \gamma$ for short (instead of $\tilde{D} = D + \llbracket \cdot, \gamma \rrbracket^r$).

A *projective structure* on M^n ($n > 1$) is a projective class $\Pi^r = [D]$ of affine connections. Connections in Π^r have same unparametrized geodesics. (Also have same torsion, usually assumed zero.)

If M is a complex n -manifold, there is a holomorphic version of this notion, often called a complex projective structure. However, the Levi-Civita connection of a Kähler metric is not holomorphic.

1. Complex projective structures

Let (M, J) be an almost complex manifold of real dimension $n = 2m$. A *complex affine connection* is a connection D on TM with $DJ = 0$ (e.g., $D = \nabla^g$ for a hermitian metric g).

Complex affine connections D and \tilde{D} are *c-projectively equivalent* iff $\exists \gamma \in \Omega^1(M)$, a 1-form, with

$$\begin{aligned}\tilde{D}_X - D_X &= \llbracket X, \gamma \rrbracket^c \in C^\infty(M, \mathfrak{gl}(TM, J)), \\ \llbracket X, \gamma \rrbracket^c(Y) &:= \frac{1}{2}(\gamma(X)Y + \gamma(Y)X - \gamma(JX)JY - \gamma(JY)JX).\end{aligned}$$

We write $\tilde{D} = D + \gamma$ for short.

A *c-projective structure* on M^{2m} ($m > 1$) is an c-projective class $\Pi^c = [D]$ of complex affine connections. Connections in Π^c have the same torsion, often assumed type $(0, 2)$ —and then given by the Nijenhuis tensor of J .

1. Quaternionic structures

Let (M, Q) be a quaternionic manifold of real dimension $n = 4\ell$ (thus $Q \subset \mathfrak{gl}(TM)$, with fibres isomorphic to $\mathfrak{sp}(1)$, spanned by imaginary quaternions J_1, J_2, J_3). A *quaternionic affine connection* is a connection on TM preserving Q (e.g., $D = \nabla^g$ for a quaternion Kähler metric g on M).

Fact. For any two quaternionic connections D and \tilde{D} with the same torsion, $\exists \gamma \in \Omega^1(M)$ with

$$\begin{aligned}\tilde{D}_X - D_X &= \llbracket X, \gamma \rrbracket^q \in C^\infty(M, \mathfrak{gl}(TM, Q)), \\ \llbracket X, \gamma \rrbracket^q(Y) &:= \frac{1}{2} \left(\gamma(X)Y + \gamma(Y)X \right. \\ &\quad \left. - \sum_i (\gamma(J_i X)J_i Y + \gamma(J_i Y)J_i X) \right).\end{aligned}$$

An equivalence class of quaternionic connections may be denoted $\Pi^q = [D]$. Thus any quaternionic manifold has a distinguished class of torsion-free quaternionic connections.

1. Submanifolds

Observation 1. Let (M, J, Π^c) be a c-projective $2m$ -manifold, and let N be a maximal totally real submanifold, i.e., $J(TN) \cap TN = 0$ and $\dim N = m$ so that $TM|_N \cong TN \oplus J(TN)$.

By projecting c-projective connections onto TN , N inherits a projective structure: for $X, Y \in TN$, the projection of $\llbracket X, \gamma \rrbracket^c(Y)$ is $\llbracket X, i^* \gamma \rrbracket^r(Y)$, where $i: M \rightarrow N$ is the inclusion.

Observation 2. Let $(M^{4\ell}, Q)$ be a quaternionic manifold. A submanifold N is *totally complex* if TN is invariant under some $J \in Q$ (along N), but $I(TN) \cap TN = 0$ for any $I \in Q$ anticommuting with J . If N is a maximal ($\dim N = 2\ell$), then $TM|_N \cong TN \oplus TN^\perp$ where $TN^\perp = I(TN)$ for any such I .

This induces a c-projective structure on N : for $X, Y \in TN$, the projection of $\llbracket X, \gamma \rrbracket^q(Y)$ is $\llbracket X, i^* \gamma \rrbracket^c(Y)$.

Question: conversely, can complexify a real projective manifold, so can we quaternionify a c-projective manifold?

2. Model example

$S = \mathbb{C}P^m = P(\mathbb{C}^{m+1})$ has complexification $S^c = \mathbb{C}P^m \times \mathbb{C}P^m$.

Let $\hat{Z} = P(\mathcal{O}(-1, 0) \oplus \mathcal{O}(0, -1))$, a $\mathbb{C}P^1$ -bundle over S^c .

The map $\mathcal{O}(-1, 0) \oplus \mathcal{O}(0, -1) \rightarrow \mathbb{C}^{m+1} \times \mathbb{C}^{m+1}$

$$(\ell_1, v_1, \ell_2, v_2) \mapsto (v_1, v_2),$$

where $v_i \in \ell_i \leq \mathbb{C}^{m+1}$, induces a map $\hat{Z} \rightarrow Z = \mathbb{C}P^{2m+1}$.

(This is a partial blow-down of the zero and infinity sections of \hat{Z} .)

The $\mathbb{C}P^1$ fibres of $\hat{Z} \rightarrow S^c$ map to projective lines in $\mathbb{C}P^{2m+1}$,

which are twistor lines. The moduli space of such lines is

$\text{Gr}_2(\mathbb{C}^{2m+2})$, which is a complexification of $\mathbb{H}P^m$.

Theorem (HKLR, LeBrun, Pedersen–Poon). Let Z be a holomorphic $(2m+1)$ -manifold equipped with an antiholomorphic involution $\tau: Z \rightarrow Z$ containing a τ -invariant twistor line on which τ has no fixed points. Then the space of such twistor lines is a $4m$ -dimensional quaternionic manifold (M, Q) .

2. The main difficulty

To generalize the model, need to partially blow down 0 and ∞ sections of a projective line bundle over a complexification S^c of S .

Problem: S^c is not compact, nor even a complete variety.

Essence of problem: the blow-up of \mathbb{C}^{m+1} at the origin is the total space of $\pi: \mathcal{O}(-1) \rightarrow \mathbb{C}P^m$. Now let U be open in $\mathbb{C}P^m$.

- ▶ How to construct the blow-down knowing only $\pi^{-1}(U) \rightarrow U$?
- ▶ The image of $\pi^{-1}(U)$ under the blow-down is singular at 0 .

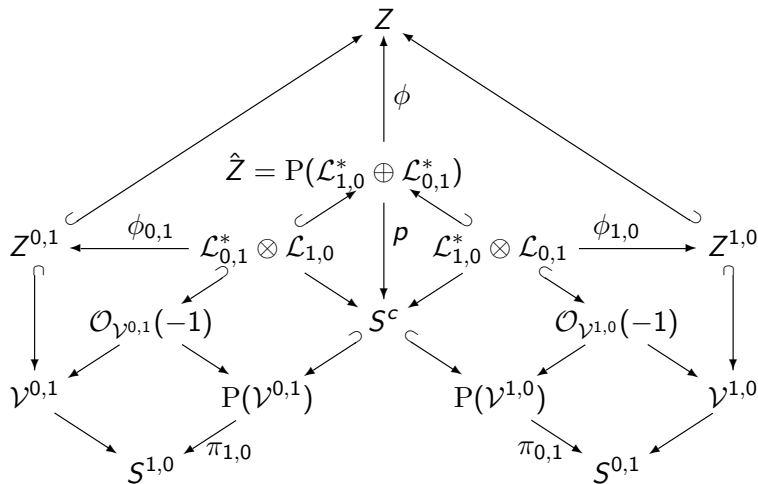
Resolution: U has a flat complex projective structure, hence a second order linear operator on sections of $\mathcal{O}(1)|_U$ whose solutions are “affine”. The vector space $\mathcal{A}(\mathcal{O}(1)|_U)$ of affine sections has evaluation maps $\mathcal{A}(\mathcal{O}(1)|_U) \rightarrow \mathcal{O}(1)_u$ for all $u \in U$.

- ▶ If $V = \mathcal{A}(\mathcal{O}(1)|_U)^*$ then for each $u \in U$, the image of the transpose $\mathcal{O}(-1)_u \rightarrow V$ is a 1-dimensional subspace, and this defines a developing map (local biholomorphism) $U \rightarrow P(V)$.
- ▶ Take the union of the image of $\pi^{-1}(U) = \mathcal{O}(-1)|_U$ in V with an open nbhd of the origin.

2. The main result

- ▶ (M^{4m}, Q) quaternionic, with a quaternionic $U(1)$ action.
- ▶ Fixed point set has a component S of dimension $2m$ with no triholomorphic points (where stabilizer commutes with Q).
- ▶ Then S is totally complex, with induced c-projective structure of type $(1, 1)$, i.e., c-projective curvature has type $(1, 1)$, i.e., complexification S^c has flat complex projective structures on $(1, 0)$ and $(0, 1)$ foliations.
- ▶ To any c-projective S of type $(1, 1)$ and any line bundle \mathcal{L} with connection of type $(1, 1)$, have a complexification S^c and a projective line bundle $\hat{Z} = P(\mathcal{L}_{0,1}^* \oplus \mathcal{L}_{1,0}^*) \rightarrow S^c$.
- ▶ \hat{Z} has a blow down which is a twistor space of a $U(1)$ -invariant quaternionic structure Q on a nbhd M of zero section in $TS \otimes (\mathcal{L}_{0,1}^* \otimes \mathcal{L}_{1,0})|_S$.
- ▶ Constructions are mutually inverse up to local isomorphism.

2. A picture of the construction of Z from S



Hooked arrows are open embeddings.

Other arrows are fibrations or (open embeddings of) blow-downs.

2. Twistor theory of complexified quaternionic manifolds

Let Z be a holomorphic $(2n + 1)$ -manifold containing twistor lines. Kodaira moduli space is a holomorphic $4n$ -manifold M^c , with incidence relation (twistor correspondence)

$$\begin{array}{ccc} & F_M := \{(z, u) \in Z \times M^c : z \in u\} & \\ & \swarrow \pi_Z \quad \searrow \pi_{M^c} & \\ Z & & M^c, \end{array}$$

where $u \in M^c$ is the twistor line $\pi_Z(\pi_{M^c}^{-1}(u)) \subseteq Z$.

Normal bundle to twistor lines defines a bundle $\mathcal{N} \rightarrow F_M$.

Locally over M^c , $\mathcal{N} \cong \pi_{M^c}^* \mathcal{E} \otimes \pi_Z^* \mathcal{O}_Z(1)$ where

- ▶ \mathcal{E} is a rank $2n$ bundle on M^c
- ▶ $\mathcal{O}_Z(1)$ is a line bundle on Z of degree 1 on each twistor line.

By Kodaira, $T_u M^c \cong H^0(u, \mathcal{N}|_u) \cong \mathcal{E}_u \otimes \mathcal{H}_u$, $\mathcal{H}_u = H^0(u, \mathcal{O}_Z(1))$.

Thus $TM^c \cong \mathcal{E} \otimes \mathcal{H}$; say $X \in TM^c$ is *null* if decomposable.

2. α -submanifolds

The fibre of F_M over $z \in Z$ projects to a submanifold α_z of M^c called an α -submanifold.

Thus $u \in \alpha_z$ iff $z \in u$, and then $T_u \alpha_z = \mathcal{E}_u \otimes \mathcal{O}_Z(-1)_z$, so that tangent spaces to α_z are null.

Twistor lines through $z \in u$ are determined by their tangent space at z , so α_z is isomorphic to an open submanifold of $P(T_z Z)$, and has a canonical flat projective structure.

Also have (complexified) quaternionic connections: torsion-free tensor product connections $D^{\mathcal{E}} \otimes D^{\mathcal{H}}$.

Prop. For any α -submanifold α_z in M^c , any quaternionic connection induces an affine connection on α_z compatible with its canonical flat projective structure.

2. Why are constructions mutually inverse?

Let Q be a $U(1)$ -invariant quaternionic structure on a nbhd M^{4m} of a fixed submanifold S^{2m} with no triholomorphic points.

- ▶ Weight space decomposition shows (S, J) is (maximal) totally complex submanifold of M , with J a section of $Q|_S$.
- ▶ $U(1)$ action lifts to holomorphic action on twistor space Z , generated by a vector field vanishing on sections $\pm J$ of $Z|_S$, denoted $S^{1,0}$ and $S^{0,1}$.
- ▶ Let $\phi: \hat{Z} \rightarrow Z$ be the blow-up of Z along $S^{1,0} \cup S^{0,1}$, with exceptional divisor $\underline{0} \cup \underline{\infty}$.
- ▶ $\phi^{-1}(Z|_S)$ has a neighbourhood foliated by a $2n$ -dimensional moduli space S^c of rational curves with trivial normal bundle.
- ▶ May assume \hat{Z} is a P^1 -bundle and S^c is an open nbhd of “diagonal” in $S^{1,0} \times S^{0,1}$.
- ▶ Lift of $U(1)$ action to \hat{Z} shows $\hat{Z} \setminus (\underline{0} \cup \underline{\infty})$ is a holomorphic principal \mathbb{C}^\times -bundle over S^c , hence $\hat{Z} \cong P(\mathcal{L}_{0,1}^* \otimes \mathcal{L}_{1,0}^*)$.
- ▶ By Proposition, induced c-projective structure has type $(1, 1)$.

3. Examples: complex grassmannians

Totally complex submanifolds S of quaternionic symmetric spaces (M, Q) fixed by a $U(1)$ action have been classified by Wolf: have many examples where (M, Q) is not even locally hypercomplex.

Example. Complex grassmannian $M = \text{Gr}_2(\mathbb{C}^{n+2})$ is a quaternionic symmetric space with twistor space $Z = F_{1,n+1}(\mathbb{C}^{n+2})$, the the flag manifold of pairs $B \leq W \leq \mathbb{C}^{n+2}$ with $\dim B = 1$ and $\dim W = n + 1$. The real structure on Z sends the flag $B \leq W$ to $W^\perp \leq B^\perp$.

Then $M^c \cong \{(U, V) \in \text{Gr}_2(\mathbb{C}^{n+2}) \times \text{Gr}_n(\mathbb{C}^{n+2}) : \mathbb{C}^{n+2} = U \oplus V\}$, and a fixed decomposition $\mathbb{C}^{n+2} = A \oplus \tilde{A}$, with $\dim A = 1$ and $\dim \tilde{A} = n + 1$, determines a submanifold $S^c = \{(U, V) \in M^c : A \leq U, V \leq \tilde{A}\}$ of M^c .

We find that S^c is an open submanifold of $P(\tilde{A}) \times P(\tilde{A}^*)$ and $\hat{Z} \cong P(\mathcal{O}_{\tilde{A}}(-1) \oplus \mathcal{O})|_{S^c} \cong P(\mathcal{O} \oplus \mathcal{O}_{\tilde{A}^*}(-1))|_{S^c}$.

3. Swann bundles and twisted Armstrong cones

Any quaternionic $4m$ -manifold (M, Q) has an associated hypercomplex cone \tilde{M} of dimension $4(m+1)$ fibering over it, called the *Swann bundle*: it is the \mathbb{C}^\times bundle over the twistor space Z of (M, Q) associated to the $m+1$ root of the anticanonical bundle K_Z^{-1} .

Question. If (M, Q) is constructed from a c-projective manifold S of type $(1, 1)$ and a line bundle \mathcal{L} of type $(1, 1)$, how can we construct \tilde{M} from S ?

Any c-projective $2m$ manifold S has a complex affine $2(m+1)$ -manifold fibering over it as the \mathbb{C}^\times bundle associated to the $m+1$ root of the anticanonical bundle K_S^{-1} (Armstrong).

To answer the question, we twist Armstrong's construction by \mathcal{L} .

Prop. If (M, Q) is constructed from S and \mathcal{L} , then its Swann bundle is constructed from the twisted Armstrong cone of (S, \mathcal{L}) .

3. Four dimensions

A c-projective structure on a complex surface S is the same thing as a Möbius structure (C, 1998). It automatically has type $(1, 1)$. Any such S , together with a line bundle \mathcal{L} with connection, gives rise to a self-dual conformal 4-manifold M with a $U(1)$ action having S as a component of the fixed point set.

By Jones–Tod (1985) and LeBrun (1990), the quotient $M/U(1)$ is (locally, near S) an *asymptotically hyperbolic Einstein–Weyl 3-manifold* with conformal infinity S (i.e., Möbius infinity S, \mathcal{L}).

This is a minitwistor version of the H-space construction. The flat model is hyperbolic 3-space \mathcal{H}^3 , which is a $U(1)$ quotient of the embedding of $\mathbb{C}P^1 \cong S^2$ in $\mathbb{H}P^1 \cong S^4$.

However, the following conjecture remains open.

Conjecture (LeBrun, 1990). If B is an asymptotically hyperbolic Einstein–Weyl 3-manifold on the interior of a compact manifold \bar{B} with conformal infinity $\partial\bar{B}$, then B is \mathcal{H}^3 , with the Einstein–Weyl structure of the hyperbolic metric.