Nodal intersections of random toral eigenfunctions with a test curve

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1. Motivation & Background

Chladni plates video

General Setup

- (M, g) – Riemannian surface

  $\Delta = div \circ grad$ Laplace-Beltrami on M
- <u>Eigenfunctions:</u>

  $$\lambda_j \geq 0 \qquad \Delta\varphi_j + \lambda_j\varphi_j = 0$$
- Orthonormal basis of L^2(M,dVol), $\lambda_j \to \infty$

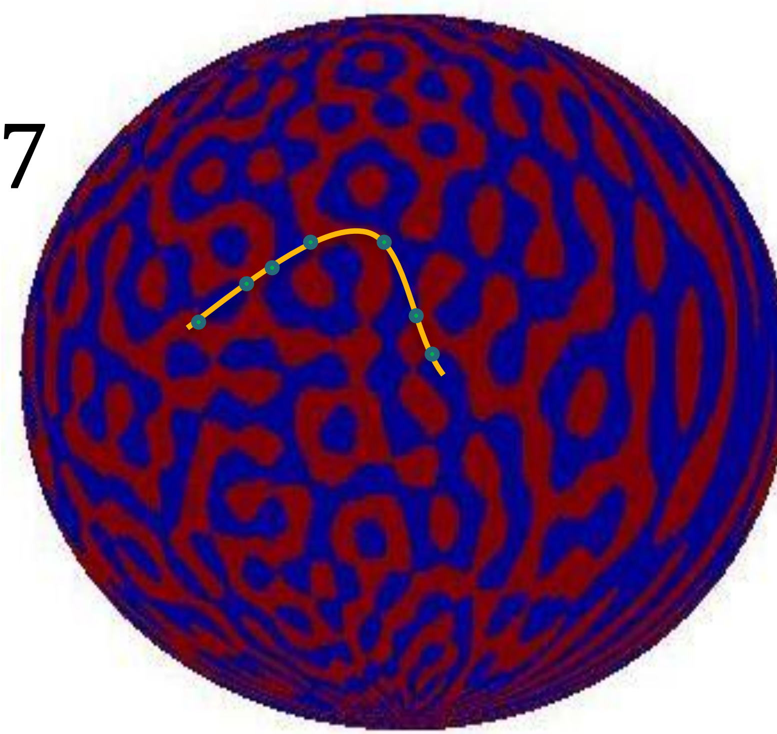
  $$\lambda_j \sim c_{area(M)} \cdot j \qquad \text{(Weyl law)}$$

Nodal intersections

- <u>Nodal set:</u> $\mathcal{N}(\varphi_j) = \varphi_j^{-1}(0)$, $j \to \infty$?
- Nodal intersections: $\mathcal{C} \subseteq M$ smooth curve

$$\mathcal{Z}_j = \mathcal{Z}_{\varphi_j}(\mathcal{C}) = |\mathcal{N}(\varphi_j) \cap \mathcal{C}|$$

$$\mathcal{Z}_j = 7$$



- $\mathcal{Z}_j$ =Zeros of $\varphi_j|_{\mathcal{C}}$

Expect $\mathcal{Z}_j \approx \sqrt{\lambda}$??

Toral Laplace eigenfunctions

- $\mathbb{T} = M = {}^{\mathbb{R}^2}/_{\mathbb{Z}^2}$ 2-torus, $x = (x_1, x_2) \in \mathbb{T}$
- $\mu \in \mathbb{Z}^2 \rightsquigarrow e(\langle x, \mu \rangle) = e^{2\pi i \langle x, \mu \rangle}$ eigenfunction (complex), $\lambda = 4\pi^2 \|\mu\|^2$

$$n = \|\mu\|^2 = \blacksquare + \blacksquare$$

- General $f_n(x) = \sum_{\|\mu\|^2 = n} a_\mu \cdot e(\langle x, \mu \rangle)$

  lattice points on circle
- Real-valued $\Leftrightarrow a_{-\mu} = \overline{a_\mu}$ (sines, cosines)

Toral nodal intersections

- $\mathbb{T} = M = {}^{\mathbb{R}^2}/_{\mathbb{Z}^2}$ standard 2-torus
- Toth-Zelditch (generic) + Bourgain-Rudnick (verified): $\mathcal{C}$ analytic $\mathcal{Z}_{\mathcal{C}} = O(\sqrt{\lambda})$,
- Lower $\mathcal{Z}_{\mathcal{C}} \gg \lambda^{1/2 - o(1)}$ analytic, curvature $\neq 0, \mathcal{Z}_{\mathcal{C}} \gg \lambda^{1/2}$ generic/conditional.
- Q1. Better in random scenario?
- Q2. Arithmetic ingredient inherent?

Random eigenfunctions

- $f_n(x) = \sum_{\|\mu\|^2=n} a_\mu \cdot e(\langle x, \mu \rangle)$
  a_μ standard Gaussian i.i.d. (save to $a_{-\mu} = \overline{a_\mu}$) "arithmetic random waves"
- $\mathcal{C} \subseteq \mathbb{T}$ smooth, $\mathcal{Z}_n = \mathcal{Z}_{f_n}(\mathcal{C})$ random variable, distribution?
- $E[\mathcal{Z}_n] = c_0 \cdot len(\mathcal{C}) \cdot \sqrt{n}$,
  $c_0 > 0$ universal (standard Kac-Rice)
- Variance?

2. Variance of nodal intersection number (joint with Zeev Rudnick)

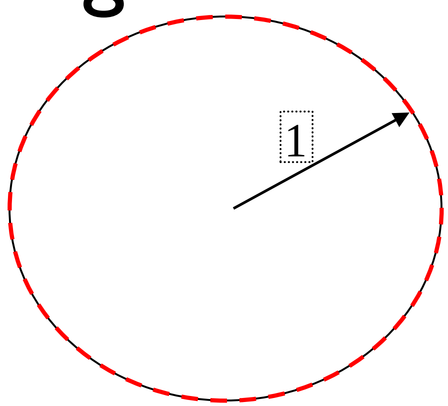
On the 2 squares problem

- $\mathcal{N}_n = r_2(n) = \#\{(a,b)\epsilon\mathbb{Z}^2 : a^2 + b^2 = n\}$
- On average $\mathcal{N}_n \sim \mathrm{c} \cdot \sqrt{\log(n)}$ (E. Landau)

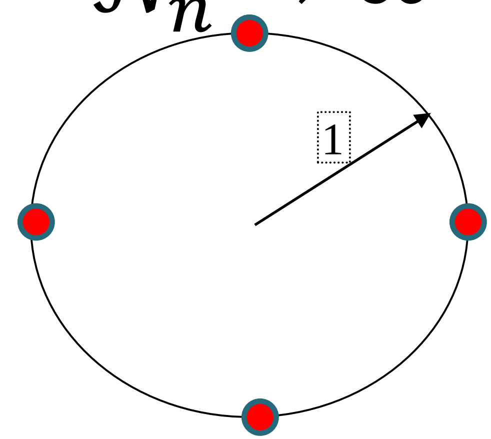
Assume $\mathcal{N}_n \to \infty$

- Equidistributed | Exceptional

generic



$\mathcal{N}_n \to \infty$



- Partial classification (P. Kurlberg-IW `15)

On the 2 squares problem

- $\tau_n = \frac{1}{\mathcal{N}_n} \sum_{\|\mu\|^2=n} \delta_{\mu/\sqrt{n}}$ probability on $\mathcal{S}^1$

  Equidistributed $\tau_{n_j} \Rightarrow \frac{d\theta}{2\pi}$ $(\tau_{n_j} \Rightarrow \tau)$
- Angular distribution $\leftrightsquigarrow$ Nodal structure (Nazarov-Sodin `12, Krishnapur-Kurlberg-W `13, Kurlberg-W `15, Buckley-W `16)
- Does $\mathcal{Z}_n = \mathcal{Z}_{f_n}(\mathcal{C})$ "feel" angular distribution, geometry of $\mathcal{C}$?

Nodal intersections variance

- Theorem (Rudnick–W): $\mathcal{C} \subseteq \mathbb{T}$ smooth, curvature $\neq 0, \mathcal{L} = len(\mathcal{C}), \gamma: [0, \mathcal{L}] \to \mathbb{T}$.

$$Var(Z_n) = c_n \cdot \frac{n}{\mathcal{N}_n} + O\left(\frac{n}{\mathcal{N}_n^{3/2}}\right)$$

- $c_n = 4B_n(\mathcal{C}) - \mathcal{L}^2 \in [0, \mathcal{L}^2]$

$$B_n(\mathcal{C}) = \int_{\mathcal{C}\times\mathcal{C}} \frac{dt_1 dt_2}{\mathcal{N}_n} \sum_{\|\mu\|^2=n} \left\langle \frac{\mu}{\sqrt{n}}, \dot{\gamma}(t_1) \right\rangle^2 \cdot \left\langle \frac{\mu}{\sqrt{n}}, \dot{\gamma}(t_2) \right\rangle^2$$

$$= \int_{\mathcal{C}\times\mathcal{C}} dt_1 dt_2 \int_{\mathcal{S}^1} \langle \theta, \dot{\gamma}(t_1) \rangle^2 \cdot \langle \theta, \dot{\gamma}(t_2) \rangle^2 \, d\tau_n(\theta)$$

On the limiting constant

- $B_n(\mathcal{C}) = \int_{\mathcal{C}\times\mathcal{C}} \frac{dt_1 dt_2}{\mathcal{N}_n} \int_{\mathcal{S}^1} \langle\theta, \dot{\gamma}(t_1)\rangle^2 \cdot \langle\theta, \dot{\gamma}(t_2)\rangle^2 d\tau_n(\theta)$

$= \int_{\mathcal{S}^1} d\tau_n(\theta) \left[\int_{\mathcal{C}} \langle\theta, \dot{\gamma}(t_1)\rangle^2 dt\right]^2$

- Depends on (limiting) angular distribution, geometry of $\mathcal{C}$.
- $Var(Z_n) = c_n \cdot \frac{n}{\mathcal{N}_n} + O\left(\frac{n}{\mathcal{N}_n^{3/2}}\right)$
- $c_n = 4B_n(\mathcal{C}) - \mathcal{L}^2$ may vanish (rare)
- Makes sense $B(\mathcal{C}, \tau)$; $B_n(\mathcal{C}) = B(\mathcal{C}, \tau_n)$

What if it vanishes?

- $B(\mathcal{C},\tau)=\int_{\mathcal{S}^1} d\tau(\theta)\left[\int_{\mathcal{C}} \langle\theta,\dot{\gamma}(t_1)\rangle^2 dt\right]^2$

$c_n = 4B(\mathcal{C},\tau_n)-\mathcal{L}^2$ may vanish, $\mathcal{N}_n \to \infty$

- a. $\mathcal{C} \subseteq \mathbb{T}$ is given (smooth, curvature$\neq 0$),

  $4B(\mathcal{C},\tau) \equiv \mathcal{L}^2, \forall\tau$ probability on $\mathcal{S}^1$

  $\Leftrightarrow 4B\left(\mathcal{C},\frac{d\theta}{2\pi}\right)=\mathcal{L}^2$, e.g. semi-circle

- b. $\tau_{n_j} \Rightarrow \tau$ and $4B(\mathcal{C},\tau)=\mathcal{L}^2$.

  Very restrictive on τ (Cilleruelo/tilted)

3. Limit distribution of nodal intersections (joint with Maurizia Rossi, in progress)

Limit distribution

- Theorem 1 (M. Rossi-IW, in progress)

$\mathcal{C} \subseteq \mathbb{T}$ smooth, curvature $\neq 0, \mathcal{L} = len(\mathcal{C})$

- $\{n\}$ such that $c_n = 4B_n(\mathcal{C}) - \mathcal{L}^2 > 0$ bounded away from 0,
- $\mathcal{Z}_n = \mathcal{Z}_{f_n}(\mathcal{C})$.

Then $$\frac{\mathcal{Z}_n - \mathbb{E}[\mathcal{Z}_n]}{\sqrt{Var(\mathcal{Z}_n)}} \to \mathcal{N}(0,1)$$ (CLT)

- Non-Gaussianity in exceptional cases?

Non-Gaussianity:

- Recall $c_n = 4B_n(\mathcal{C}) - \mathcal{L}^2$ leading "constant"
- Theorem 2* (M. Rossi-IW, in progress)

Under (1)+(2):

- (1) Assume $4B(\mathcal{C},\tau) \equiv \mathcal{L}^2, \forall\tau$ ("scenario a")
- (2) Assume nearest neighbour lattice points: $\min_{\mu\neq\mu'}|\mu - \mu'| \gg n^{\delta}$, some $\delta > 0$. Generic, can take $\delta = \frac{1}{2} - \varepsilon$ (Bourgain-Rudnick).

1. $\mathcal{Var}(\mathcal{Z}_n) \sim d_n \cdot \frac{n}{\mathcal{N}_n{}^2}, 0 < d < d_n < D < \infty$
2. $\frac{\mathcal{Z}_n - \mathbb{E}[\mathcal{Z}_n]}{\sqrt{\mathcal{Var}(\mathcal{Z}_n)}}$ →Non-Gaussian, depends $\mathcal{C}, \tau$

Non-Gaussianity:

- Theorem 2* (M. Rossi-IW, in progress)
- Under (1) and (2)

$\frac{\mathcal{Z}_n - \mathbb{E}[\mathcal{Z}_n]}{\sqrt{\mathcal{Var}(\mathcal{Z}_n)}}$ →Non-Gaussian, depends on $\mathcal{C}, \tau$

- Example: $\mathcal{C}$=circle, $\tau = \frac{d\theta}{2\pi}$,

(Z_1, Z_2) - standard Gaussian i.i.d.

$$\frac{\mathcal{Z}_n - \mathbb{E}[\mathcal{Z}_n]}{\sqrt{\mathcal{Var}(\mathcal{Z}_n)}} \to 1 - \frac{Z_1^2 + Z_2^2}{2}$$

4. Outline of proofs

Variance computation: Kac-Rice

- $\mathcal{Z}_{f_n}(\mathcal{C}) \leftrightsquigarrow$Zeros of $\mathrm{g}(t) := f_n(\gamma(t))$

$\looparrowright$ Kac-Rice

- g – Gaussian process, covariance function

$$\mathrm{r}(t_1, t_2) = \mathbb{E}[g(t_1) \cdot g(t_2)]$$

- 2-point correlation function $K_2(t_1, t_2) = \mathbb{E}[|g'(t_1) \cdot g'(t_2)||g(t_1) = g(t_2) = 0]$

In principle may be expressed in r, r', r''

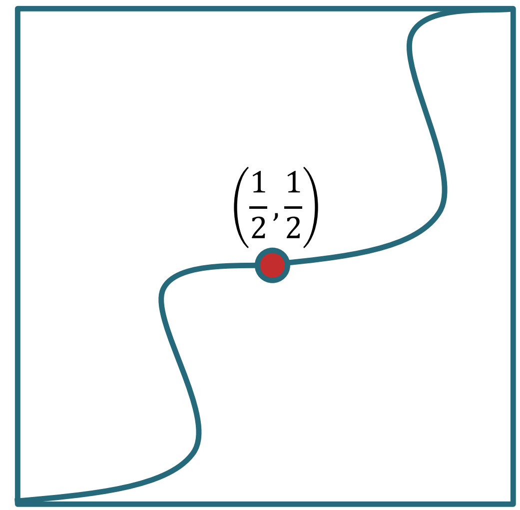
Proof outline: Kac-Rice

- 2-point correlation function $K_2(t_1, t_2) = \mathbb{E}[|g'(t_1) \cdot g'(t_2)||g(t_1) = g(t_2) = 0]$
  In principle may be expressed in r, r', r''
- Factorial moment (meta-theorem)
  $\mathbb{E}[\mathcal{Z}_n(\mathcal{Z}_n - 1)] = \int_{\mathcal{C}\times\mathcal{C}} K_2(t_1, t_2) dt_1 dt_2$
  Under non-degeneracy of $\left(g(t_1), g(t_2)\right)$
- May not hold, Kac-Rice may fail

Proof outline (cont.)

- Example: $\mathcal{C} = 2\mathcal{C}'$
- $\mathcal{Z}_n = \mathcal{Z}_{f_n}(\mathcal{C}) = 2\mathcal{Z}_{f_n}(\mathcal{C}')$
- Application of Kac-Rice on $\mathcal{C}, \mathcal{C}' \Longrightarrow$

$$\mathbb{E}[\mathcal{Z}_n(\mathcal{Z}_n - 1)] = \mathbb{E}[\mathcal{Z}_n(\mathcal{Z}_n - 2)]\ !!!$$



- Solution: divide curve into pieces of length $\frac{1}{\sqrt{n}}$, apply Kac-Rice separately
- Bound the contribution of singular covariance by Cauchy-Schwartz

Proof outline (cont.)

- Expand

$K_2(t_1,t_2) \approx \mathbb{E}^2 + r^2 - * \,(r_1^2 + r_2^2) + r_{12}^2$

- Approximate Kac-Rice:

$Var(\mathcal{Z}_n) \leftrightsquigarrow 2^{nd}$ moment: r, r_1,r_2,r_{12}

- Involves evaluating oscillatory integrals
- Main contribution comes from diagonal

Use curvature not vanishing to bound off-diagonal

- $\int_{\mathcal{C}\times\mathcal{C}} K_2(t_1,t_2)dt_1dt_2 = \mathbb{E}[\mathcal{Z}_n(\mathcal{Z}_n - I(\mathcal{C}))]$?

Limit distribution

- Wiener chaos expansion

$\mathcal{Z}_{f_n}(\mathcal{C}) - \mathbb{E}\left[\mathcal{Z}_{f_n}(\mathcal{C})\right] = \sum_{q \geq 1} \mathcal{Z}_{f_n}(\mathcal{C})[2q]$
$= \mathcal{Z}_{f_n}(\mathcal{C})[2] + \mathcal{Z}_{f_n}(\mathcal{C})[4] + \cdots$

- Fact: If $c_n = 4B_n(\mathcal{C}) - \mathcal{L}^2$ bounded away from 0, $\mathcal{Z}_{f_n}(\mathcal{C})[2]$ dominates.
- Less clear if c_n small.
- Under assumption of Theorem 2 (non-Gaussianity), $\mathcal{Z}_{f_n}(\mathcal{C})[4]$ dominates both $\mathcal{Z}_{f_n}(\mathcal{C})[2]$, higher order terms.