Nodal Sets of High-Energy Arithmetic Random Waves

Maurizia Rossi
UR Mathématiques, Université du Luxembourg

Probabilistic Methods in Spectral Geometry and PDE

Montréal – August 22-26, 2016

This talk is mainly based on

Non-Universality of nodal length distribution for arithmetic random waves,
joint work with
Domenico Marinucci, Giovanni Peccati, Igor Wigman

and

Phase singularities in complex arithmetic random waves,
joint work with
Federico Dalmao, Ivan Nourdin, Giovanni Peccati.

Outline

1. Berry's Random Wave Model
2. Arithmetic Random Waves
3. Nodal Length
   - Mean & Variance
   - Asymptotic Distribution: Non-Universal NCLT
4. Nodal Intersections Number
   - Phase Singularities
   - Mean, Variance and Distribution

$(\mathcal{M}, g)$ compact Riemannian manifold of dimension 2

Δ_g Laplace-Beltrami operator

$$\Delta_g f + Ef = 0$$

eigenvalues $E_0 \le E_1 \le E_2 \le E_3 \le \dots$

eigenfunctions $f_0, f_1, f_2, f_3, \dots$ o.b. of $L^2(M)$

Behavior of f_j for large j : the zeroes $f_j^{-1}(0)$

f_j real $\Rightarrow$ zeroes are disjoint union of smooth curves (length)

For "generic" chaotic surfaces, compare

$$f_j \quad \longleftrightarrow \quad W_j$$

$W_j = (W_j(x))_{x\in\mathbb{R}^2}$ centered Gaussian field on the plane

$$\mathrm{Cov}\,(W_j(x), W_j(y)) = J_0(\sqrt{E_j}\|x-y\|), \quad x, y \in \mathbb{R}^2.$$

$\Rightarrow$ stationary and isotropic!

Q : Compare???

Example: nodal length on $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$ (the 2-torus).

Take a representative planar domain $U \subset \mathbb{R}^2$ and study nodal length of W_j restricted to U (r.v.)

- Expected nodal length per unit area of W_j : $\sqrt{E_j}/2\sqrt{2}$
- Predicted variance : $\approx \log E_j$ [Berry, 2002]

$\mathbb{T} := \mathbb{R}^2/\mathbb{Z}^2$ 2-torus Δ Laplacian

$$\Delta f + E f = 0$$

Eigenvalues $E_n = 4\pi^2 n,$ $n =$ sum of two integer squares

$$S = \{n = {\lambda_1}^2 + {\lambda_2}^2, \quad \exists \lambda_1, \lambda_2 \in \mathbb{Z}\}$$

Set of frequencies $\Lambda_n = \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2 : {\lambda_1}^2 + {\lambda_2}^2 = n\}$

$|\Lambda_n| =: \mathcal{N}_n$ (grows *on average* as $\sqrt{\log n}$...)

<u>Eigenfunctions</u> ($\lambda \in \Lambda_n$)

$$\mathrm{e}_\lambda(x) := \mathrm{e}^{i2\pi\langle\lambda, x\rangle}, \quad x \in \mathbb{T}$$

L^2-o.b.

Probability measure on $\mathbb{S}^1$ induced by Λ_n

$$\mu_n = \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \delta_{\frac{\lambda}{\sqrt{n}}}$$

$\exists$ density-1 sequence $\{n_j\}_j \subset \{n\}$ s.t.

$$\mu_{n_j} \Rightarrow d\theta/2\pi.$$

$\exists$ other weak-* partial limit of $\{\mu_n\}_n$ [Kurlberg-Wigman, 2016].

$$\mu(k) := \int_{\mathbb{S}^1} z^{-k} \, d\mu(z), \quad k \in \mathbb{Z}, \quad \text{Fourier coefficients}$$

$\forall \eta \in [-1, 1], \exists \{n_j\}_j$ s.t.

$$\widehat{\mu}_{n_j}(4) \to \eta$$

If $\mu_{n_j} \Rightarrow d\theta/2\pi$, then $\widehat{\mu}_{n_j}(4) \to 0$.

$n \in S$

$$T_n(x) = \frac{1}{\sqrt{\mathcal{N}_n}} \sum_{\lambda \in \Lambda_n} a_\lambda \mathrm{e}_\lambda(x), \quad x \in \mathbb{T}$$

$\{a_\lambda\}_{\lambda \in \Lambda_n}$ iid complex-Gaussian except for $\overline{a_\lambda} = a_{-\lambda} (\Rightarrow T_n$ is real !!!)

$$\mathbb{E}[a_\lambda] = 0 \qquad \mathbb{E}[|a_\lambda|^2] = 1$$

T_n centered Gaussian random field

$$r_n(x-y) = \mathrm{Cov}\left(T_n(x), T_n(y)\right) = \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \cos(2\pi\langle \lambda, x-y \rangle), \quad x, y \in \mathbb{T}$$

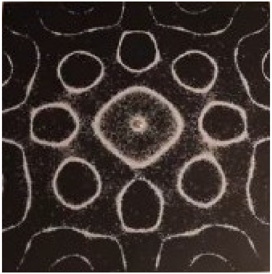
$\Rightarrow T_n$ stationary

- ARW approximate Berry's RWM

NODAL LENGTH

Nodal lines $T_n^{-1}(0) = \{x \in \mathbb{T} : T_n(x) = 0\}$
disjoint union of smooth curves

$$\mathcal{L}_n := \text{length}(T_n^{-1}(0))$$



AIM: to study the sequence of random variables $\{\mathcal{L}_n\}_{n \in S}$

Theorem [Rudnick-Wigman, 2008] For fixed $n \in S$,

$$\mathbb{E}[\mathcal{L}_n] = \frac{1}{2\sqrt{2}}\sqrt{E_n} = \frac{1}{2\sqrt{2}}\sqrt{4\pi^2 n}.$$

Consistent with the expected nodal length for the RWM!

Theorem [Krishnapur-Kurlberg-Wigman, 2013] As $\mathcal{N}_n \to +\infty$

$$\mathrm{Var}(\mathcal{L}_n) = c_n \frac{E_n}{\mathcal{N}_n^2}(1 + o(1))$$

$$c_n = \frac{1 + \widehat{\mu}_n(4)^2}{512}$$

$\widehat{\mu}_n(4) =$ 4-th Fourier coefficient of μ_n

$|\widehat{\mu}_n(4)| \le 1 \quad \Rightarrow$ variance order of magnitude $E_n/\mathcal{N}_n^2$

- Natural guess: $E_n/\mathcal{N}_n$ Berry's cancellation phenomenon
- Non-Universality

$\forall \eta \in [-1,1], \exists \{n_j\}_j$ s.t.

$$\widehat{\mu}_{n_j}(4) \to \eta$$

$$\Rightarrow \mathrm{Var}(\mathcal{L}_{n_j}) \sim \frac{1+\eta^2}{512} \frac{E_{n_j}}{\mathcal{N}_{n_j}^2}$$

Integral form

$$\mathcal{L}_n = \text{length}(T_n^{-1}(0)) = \int_{\mathbb{T}} \delta_0(T_n(x)) \|\nabla T_n(x)\| \, dx$$

gradient $\nabla T_n = (\partial_1 T_n, \partial_2 T_n), \qquad \partial_i := \partial/\partial x_i \ \ (i = 1, 2).$

$\mathcal{L}_n \in L^2(\mathbb{P})$ and functional of a Gaussian field

---

$\Rightarrow$ **Wiener-Itô chaos expansion in** $L^2(\mathbb{P})$

$$L^2(\mathbb{P}) = \bigoplus_{q=0}^{+\infty} C_q \ \Rightarrow \ \mathcal{L}_n = \sum_{q=0}^{+\infty} \mathcal{L}_n[q] = \sum_{q=0}^{+\infty} \text{Proj}\left(\mathcal{L}_n \mid C_q\right)$$

if $q \neq m$, then $C_q \perp C_m \ \Rightarrow \ \text{Cov}\left(\mathcal{L}_n[q], \mathcal{L}_n[m]\right) = 0$

$0 -$ order projection is the mean: $\mathcal{L}_n[0] = \mathbb{E}[\mathcal{L}_n]$.

$$\mathcal{L}_n = \mathbb{E}[\mathcal{L}_n] + \sum_{q=1}^{+\infty} \mathcal{L}_n[q], \quad \mathbb{E}[\mathcal{L}_n] = \frac{1}{2\sqrt{2}}\sqrt{E_n}.$$

Proposition

odd chaoses $\quad \mathcal{L}_n[2q+1] = 0, \quad q \geq 0$

$$q \geq 2, \quad \mathcal{L}_n[2q] = \sqrt{\frac{E_n}{2}} \sum_{u=0}^{q} \sum_{k=0}^{u} \frac{\alpha_{2k,2u-2k}\beta_{2q-2u}}{(2k)!(2u-2k)!(2q-2u)!} \times$$

$$\times \int_{\mathbb{T}} H_{2q-2u}(T_n(x)) H_{2k}(\widetilde{\partial}_1 T_n(x)) H_{2u-2k}(\widetilde{\partial}_2 T_n(x))\, dx,$$

$$\mathrm{Var}(\partial_i T_n(x)) = \frac{E_n}{2} \;\rightarrow\; \widetilde{\partial}_i := \sqrt{\frac{2}{E_n}}\partial_i \quad (i = 1, 2).$$

$$\mathcal{L}_n[2] = \sqrt{\frac{E_n}{2}}\Big(\frac{\alpha_{0,0}\beta_2}{2!}\int_{\mathbb{T}} H_2(T_n(x))\,dx +$$

$$+\frac{\alpha_{2,0}\beta_0}{2!}\int_{\mathbb{T}} H_2(\widetilde{\partial}_1 T_n(x))\,dx + \frac{\alpha_{0,2}\beta_0}{2!}\int_{\mathbb{T}} H_2(\widetilde{\partial}_2 T_n(x))\,dx\Big)$$

Lemma [R. (2015), Marinucci, Peccati, R., Wigman (2016)]

$$\mathcal{L}_n[2] = 0$$

Proof

$$H_2(t) = t^2 - 1$$

$$\int_{\mathbb{T}} (\partial_i T_n(x))^2\,dx = -\int_{\mathbb{T}} T_n(x)\partial_{ii} T_n(x)\,dx \quad (\partial_{ii} := \partial^2/\partial_{x_i^2})$$

$$\partial_{11} T_n + \partial_{22} T_n = \Delta T_n = -E_n T_n$$

$$\alpha_{2,0} = \alpha_{0,2}$$

$$\mathcal{L}_n - \mathbb{E}[\mathcal{L}_n] = \mathcal{L}_n[4] + \sum_{q\ge 3} \mathcal{L}_n[2q]$$

$$\Rightarrow \mathrm{Var}(\mathcal{L}_n) = \mathrm{Var}(\mathcal{L}_n[4]) + \sum_{q\ge 3} \mathrm{Var}(\mathcal{L}_n[2q]).$$

Lemma

$$\lim_{n\to+\infty} \frac{\mathrm{Var}(\mathcal{L}_n[4])}{\mathrm{Var}(\mathcal{L}_n)} = 1.$$

$\Rightarrow$ as $\mathcal{N}_n \to +\infty$,

$$\frac{\mathcal{L}_n - \mathbb{E}[\mathcal{L}_n]}{\sqrt{\mathrm{Var}(\mathcal{L}_n)}} = \frac{\mathcal{L}_n[4]}{\sqrt{\mathrm{Var}(\mathcal{L}_n[4])}} + o_{\mathbb{P}}(1)$$

4th chaos dominates the whole series.

4th chaos component (distribution)

$$\mathcal{L}_n[4] = \sqrt{\frac{E_n}{2}} \left(\frac{\alpha_{0,0}\beta_4}{4!} \int_{\mathbb{T}} H_4(T_n(x))\, dx + \dots \right)$$

$$W(n) = \begin{pmatrix} W_1(n) \\ W_2(n) \\ W_3(n) \\ W_4(n) \end{pmatrix} = \frac{1}{n\sqrt{\mathcal{N}_n/2}} \sum_{\substack{\lambda=(\lambda_1,\lambda_2)\in\Lambda_n \\ \lambda_2>0}} \left(|a_\lambda|^2 - 1\right) \begin{pmatrix} n \\ \lambda_1^2 \\ \lambda_2^2 \\ \lambda_1\lambda_2 \end{pmatrix}$$

Proposition

As $\mathcal{N}_n \to +\infty$,

$$\mathcal{L}_n[4] = \sqrt{\frac{E_n}{512\,\mathcal{N}_n^2}} \Big(1 + W_1(n)^2 - 2W_2(n)^2 - 2W_3(n)^2 - 4W_4(n)^2 + o_{\mathbb{P}}(1) \Big).$$

Lemma

For $\{n_j\} \subseteq S$ s.t. $\mathcal{N}_{n_j} \to +\infty$ and $\widehat{\mu}_{n_j}(4) \to \eta$,

$$W(n_j) \xrightarrow{\mathrm{d}} Z(\eta) = \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{pmatrix},$$

where $Z(\eta)$ is a centered Gaussian vector with covariance

$$\Sigma = \Sigma(\eta) = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{3+\eta}{8} & \frac{1-\eta}{8} & 0 \\ \frac{1}{2} & \frac{1-\eta}{8} & \frac{3+\eta}{8} & 0 \\ 0 & 0 & 0 & \frac{1-\eta}{8} \end{pmatrix}.$$

The eigenvalues of Σ are $0, \frac{3}{2}, \frac{1-\eta}{8}, \frac{1+\eta}{4} \Rightarrow \Sigma$ is singular.

$\eta \in [0,1]$

$$\mathcal{M}_\eta := \frac{1}{2\sqrt{1+\eta^2}}(2-(1+\eta)X_1^2-(1-\eta)X_2^2),$$

$X=(X_1,X_2)$ standard bivariate Gaussian.
$\eta_1 \neq \eta_2 \Rightarrow$ the distributions of $\mathcal{M}_{\eta_1}$ and $\mathcal{M}_{\eta_2}$ are $\neq$.

Theorem [Marinucci, Peccati, R., Wigman (2016)]

For $\{n_j\} \subseteq S$ s.t. $\mathcal{N}_{n_j} \to +\infty$ and $|\widehat{\mu}_{n_j}(4)| \to \eta$

$$\frac{\mathcal{L}_{n_j} - \mathbb{E}[\mathcal{L}_{n_j}]}{\sqrt{\mathrm{Var}(\mathcal{L}_{n_j})}} \stackrel{law}{\to} \mathcal{M}_\eta$$

Non-Universal NonCLT.

$n \in S$. T_n and $\widehat{T}_n$ two independent Gaussian Laplace eigenfunctions.

$$\mathcal{I}_n := |T_n^{-1}(0) \cap \widehat{T}_n^{-1}(0)|$$

- $\mathcal{I}_n$ is the number of zeroes of complex arithmetic random waves!

$$\Theta_n(x) := \frac{1}{\sqrt{\mathcal{N}_n}} \sum_{\lambda \in \Lambda_n} v_\lambda e_\lambda(x), \quad x \in \mathbb{T},$$

$\{v_\lambda\}_{\lambda \in \Lambda_n}$ i.i.d. complex-Gaussian: $\Re(v_\lambda)$ and $\Im(v_\lambda)$ i.i.d. $\mathcal{N}(0,1)$

$$\rightarrow \quad T_n(x) = \Re(\Theta_n(x)) \qquad \widehat{T}_n(x) = \Im(\Theta_n(x))$$

The set of zeroes of Θ_n coincides with $T_n^{-1}(0) \cap \widehat{T}_n^{-1}(0)$

Phase singularities

Complex ARW approximate complex Berry's RWM

complex centered Gaussian field, whose real and imaginary parts are independent (real) RWM

$$\mathrm{Cov}\,(W_j(x), W_j(y)) = J_0(\sqrt{E_j}\|x-y\|), \quad x, y \in \mathbb{R}^2.$$

[Berry, 2002]

Expected number of phase singularities $E_j/(4\pi)$

Predicted variance $\approx E_j \cdot \log E_j$

Theorem [Dalmao, Nourdin, Peccati, R. (2016)]

$$(i)\ n \in S, \quad \mathbb{E}[I_n] = \frac{E_n}{4\pi}.$$

$$(ii) \text{ As } \mathcal{N}_n \to +\infty, \quad \mathrm{Var}(I_n) = d_n \frac{E_n^2}{\mathcal{N}_n^2}(1 + o(1)),$$

$$d_n = \frac{3\widehat{\mu}_n(4)^2 + 5}{128\pi^2}.$$

$$(iii) \text{ As } \mathcal{N}_{n_j} \to +\infty \text{ and } |\widehat{\mu}_{n_j}(4)| \to \eta,$$

$$\widetilde{I}_{n_j} \Rightarrow \frac{1}{2\sqrt{10 + 6\eta^2}} \left(\frac{1+\eta}{2} A + \frac{1-\eta}{2} B - 2(C-2) \right),$$

$$A, B, C \text{ ind}, \quad A \stackrel{law}{=} 2X^2 + 2Y^2 - 4Z^2 \stackrel{law}{=} B, \qquad C \stackrel{law}{=} X^2 + Y^2$$

$$(X, Y, Z) \sim \mathcal{N}(0, I_{3\times 3}).$$

STEPS OF THE PROOF

$\mathbf{T}_n := (T_n, \widehat{T}_n)$ 2-dim. Gaussian field on $\mathbb{T}$

1. $I_n = \int_{\mathbb{T}} \delta_{\mathbf{0}}(\mathbf{T}_n(x)) \, |J_{\mathbf{T}_n}(x)| \, dx$ (integral form)
2. $I_n = \mathbb{E}[I_n] + \sum_{q\geq 1} I_n[2q]$ (chaotic expansion)
3. $I_n[2] = 0$ (Berry's cancellation)
4. As $\mathcal{N}_n \to +\infty$, $\mathrm{Var}(I_n[4]) = d_n \frac{E_n^2}{\mathcal{N}_n^2}(1 + o(1))$
5. As $\mathcal{N}_n \to +\infty$, $\mathrm{Var}\left(\sum_{q\geq 3} I_n[2q]\right) = o(\mathrm{Var}(I_n[4]))$
6. As $\mathcal{N}_n \to +\infty$, $\widetilde{I}_n = \widetilde{I}_n[4] + o_{\mathbb{P}}(1)$
7. NonCentral and NonUniversal asymptotic distribution of $\widetilde{I}_n[4]$.

Proposition

$$\mathrm{Var}\Big(\underbrace{\sum_{q\ge 3} I_n[2q]}_{=:\mathrm{proj}(I_n|C_{\ge 6})}\Big) = o(\mathrm{Var}(I_n[4])), \quad \text{as } \mathcal{N}_n \to +\infty$$

Sketch of the proof

$\mathbb{T}$ = disjoint union of little squares Q (translation of $Q_0 = [0, 1/M) \times [0, 1/M)$, where $M \approx \sqrt{E_n}$, along k/M, $k \in \mathbb{Z}^2$)

$$\mathrm{proj}(I_n|C_{\ge 6}) = \sum_Q \mathrm{proj}\left(I_{n_{|Q}}|C_{\ge 6}\right)$$

$$\mathrm{Var}(\mathrm{proj}(I_n|C_{\ge 6})) = \sum_{Q,Q'} \mathrm{Cov}\left(\mathrm{proj}\left(I_{n_{|Q}}|C_{\ge 6}\right), \mathrm{proj}\left(I_{n_{|Q'}}|C_{\ge 6}\right)\right)$$

Split the sum into "good" and "bad" pairs of squares

Fix $\varepsilon_1 > 0$ small number

Definition [$\sim$ Rudnick, Wigman (2014)] (Q, Q') is singular if $\exists (x,y) \in Q \times Q'$ s.t. $|r_n(x-y)| > \varepsilon_1$ or $|\partial_1 r_n(x-y)| > \varepsilon_1 \sqrt{n}$ or $|\partial_2 r_n(x-y)| > \varepsilon_1 \sqrt{n}$ or $|\partial_{11} r_n(x-y)| > \varepsilon_1 n$ or $|\partial_{12} r_n(x-y)| > \varepsilon_1 n$ or $|\partial_{22} r_n(x-y)| > \varepsilon_1 n$.

$$\sum_{Q,Q' \text{ sing.}} \left| \mathrm{Cov}\left(\text{proj}\left(I_{n_{|Q}} | C_{\geq 6} \right), \text{proj}\left(I_{n_{|Q'}} | C_{\geq 6} \right) \right) \right| \leq$$

$$\sum_{Q,Q' \text{ sing.}} \sqrt{\mathrm{Var}\left(\text{proj}\left(I_{n_{|Q}} | C_{\geq 6} \right) \right) \mathrm{Var}\left(\text{proj}\left(I_{n_{|Q'}} | C_{\geq 6} \right) \right)} \leq$$

$$E_n \frac{\text{meas}(B_Q)}{1/E_n} \mathrm{Var}\left(\text{proj}\left(I_{n_{|Q_0}} | C_{\geq 6} \right) \right).$$

$B_Q =$ union of all squares Q' s.t. (Q, Q') is a singular pair.

Lemma

$$\text{meas}(B_Q) = O\left(\int_{\mathbb{T}} r_n(x)^6 \, dx\right).$$

Lemma

$$\mathrm{Var}\left(\text{proj}\left(I_{n_{|Q_0}} | C_{\geq 6}\right)\right) = O(1).$$

Proof

$$\mathrm{Var}\left(\text{proj}\left(I_{n_{|Q_0}} | C_{\geq 6}\right)\right) \leq \mathbb{E}[I^2_{n_{|Q_0}}] =$$
$$= \underbrace{\mathbb{E}[I^2_{n_{|Q_0}}] - \mathbb{E}[I_{n_{|Q_0}}]}_{=:A} + \underbrace{\mathbb{E}[I_{n_{|Q_0}}]}_{=:B \ (fine!)}.$$

$$A = \text{2nd fact moment} = \int_{Q_0}\int_{Q_0} K_2(x,y)\, dxdy = \frac{1}{M^2}\int_{\tilde{Q}_0} K_2(x)\, dx,$$

$$K_2(x) := p_{(\mathbf{T}_n(x),\mathbf{T}_n(0))}(\mathbf{0},\mathbf{0})\mathbb{E}\left[|J_{\mathbf{T}_n}(x)|\,|J_{\mathbf{T}_n}(0)| \,\Big|\, \mathbf{T}_n(x) = \mathbf{T}_n(0) = \mathbf{0}\right],$$

where $p_{(\mathbf{T}_n(x),\mathbf{T}_n(0))}$ is the density of $(\mathbf{T}_n(x),\mathbf{T}_n(0))$.

KAC-RICE AROUND THE ORIGIN

$$K_2(x) \ll \frac{\det \Omega_n(x)}{1 - r_n(x)^2}, \qquad x \in \mathbb{T},$$

where

$$\det(\Omega_n(x)) = \frac{E}{2}\left(\frac{E}{2} - \frac{(\partial_1 r_n(x))^2 + (\partial_2 r_n(x))^2}{1 - r_n(x)^2}\right) =: \Psi_n(x)$$

Taylor: as $\|x\| \to 0$

$$\Psi_n(x) = E_n^3 \|x\|^2 + E_n^4\, O(\|x\|^4)$$

$$\Rightarrow K_2(x) \ll E_n^2$$

Recall $M \approx \sqrt{E_n}$

$$\frac{1}{E_n} \int_{\widetilde{Q}_0} K_2(x)\, dx \ll \frac{1}{E_n^2} E_n^2 = O(1).$$

So far

$$\mathrm{Var}\Big(\underbrace{\sum_{q\ge 3} I_n[2q]}_{=:\mathrm{proj}(I_n|C_{\ge 6})}\Big) = O\left(E_n^2 \int_{\mathbb{T}} r_n(x)^6\,dx\right)$$

Now

$$\int_{\mathbb{T}} r_n(x)^6\,dx = \frac{|S_6(n)|}{\mathcal{N}_n^6}$$

[Bombieri-Bourgain, 2015]

$$|S_6(n)| = O(\mathcal{N}_n^{7/2}),$$

so that

$$\mathrm{Var}\Big(\sum_{q\ge 3} I_n[2q]\Big) = o(E_n^2/\mathcal{N}_n^2) = o(\mathrm{Var}(\mathcal{L}_n[4])).$$

Berry (2002) *Statistics of nodal lines and points in chaotic quantum billiards: perimeter corrections, fluctuations, curvature.*. J. Phys. A.

Bombieri, Bourgain (2015) *A problem on sums of two squares*. Int. Math. Res. Not.

Dalmao, Nourdin, Peccati, Rossi (2016) *Phase singularities in complex arithmetic random waves*. ArXiv: 1608.05631

Krishnapur, Kurlberg, Wigman (2013) *Nodal length fluctuations for arithmetic random waves*. Ann. of Math. (2)

Kurlberg, Wigman (2016) *On probability measures arising from lattice points on circles*. Math. Ann.

Marinucci, Peccati, Rossi, Wigman (2016) *Non-Universality of nodal length distribution for arithmetic random waves*. Geom. Funct. Anal.

Rossi (2015) *The geometry of spherical random fields*. PhD Thesis.

Rudnick, Wigman (2008) *On the volume of nodal sets for eigenfunctions of the Laplacian on the torus*. Ann. Henri Poincaré.

Rudnick, Wigman (2014) *Nodal intersections for random eigenfunctions on the torus*. Am. J. Math. (to appear)

FOR YOUR ATTENTION...