Liouville Quantum Field Theory

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1based on joint works with: C. Guillarmou, F. David, A. Kupiainen, H. Lacoin, V. Vargas
Outline

Motivation in 2d-quantum gravity

Liouville quantum field theory (LQFT)

Semiclassical limit
Plan of the talk

Motivation in 2d-quantum gravity

Liouville quantum field theory (LQFT)

Semiclassical limit
General principles in physics

Euclidean field theories are described by a functional (called *action*):

\[ S : f \in \Sigma \mapsto S(f) \in \mathbb{R} \]

where \( \Sigma \) is a functional space.

- **Classical physics**: (principle of least action) study the minima of this functional

- **Quantum physics**: (probabilistic approach) construct the measure

\[ e^{-S(f)} Df \]

where \( Df \) is a "measure" on the functional space \( \Sigma \).
Einstein-Hilbert action for gravity and matter fields

Fix a manifold $M$ and denote $\text{Met}(M)$ the set of Riemannian metrics $g$ on $M$.

- General relativity gives the action for the gravitational field

$$S_{\text{EH}}(g) = \frac{1}{2\kappa} \int_M R_g \, dV_g + \mu \int_M dV_g$$

where the functional space is $\text{Met}(M)$.

Notations: $\triangle_g = \text{Laplacian}$, $\nabla_g = \text{gradient}$, $R_g = \text{Ricci curvature}$, $V_g = \text{volume form}$
Einstein-Hilbert action for gravity and matter fields

Fix a manifold $M$ and denote $\text{Met}(M)$ the set of Riemannian metrics $g$ on $M$.

- General relativity gives the action for the gravitational field
  
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  where the functional space is $\text{Met}(M)$.

- **Coupling with matter:** consider an action for matter field
  
  $$\psi \mapsto S_m(g, \psi)$$

  defined on some functional space $\Theta$ made up of maps $\psi : M \to \mathbb{R}^p$.

  The total action for Gravity+matter is
  
  $$S(g, \psi) = S_{\text{EH}}(g) + S_m(g, \psi)$$

  acting on the functional space $\Sigma = \text{Met}(M) \times \Theta$.

Notations: $\triangle_g = \text{Laplacian}, \nabla_g = \text{gradient}, \quad R_g = \text{Ricci curvature}, \quad V_g = \text{volume form}$
Quantum gravity

Action for Gravity+matter is

\[ S(g, \psi) = S_{EH}(g) + S_m(g, \psi). \]

- classical gravity: study the minima of this functional (yields Einstein’s field equation)
- quantum gravity: construct the measure

\[ e^{-S(g, \psi)} DgD\psi \]

on the functional space \( \text{Met}(M) \times \Theta \) where \( Dg \) is the volume form of the DeWitt metric.

Features:
- non renormalizable in 4d.
- Einstein-Hilbert functional is trivial in 2d, hence more tractable.
Polyakov’s breakthrough (1981)  
(works if the matter field possesses conformal symmetries in 2d).

Roughly and on the Riemann sphere:
- Decompose each metric $g$ as
  \[ g = d^*(e^h \hat{g}) \]
  where $d$ is a diffeomorphism, $h$ is a scalar function and $\hat{g}$ is the round metric.
- apply the corresponding change of variables to the DeWitt metric and get
  \[
  \int F(g) e^{-S(g, \psi)} DgD\psi = C(\hat{g}) \int F(e^{\gamma X} \hat{g}) e^{-S_L(\hat{g}, X)} DX
  \]
  where $S_L$ is the Liouville action functional
  \[
  \frac{1}{4\pi} \int_M \left( |\partial_g X|^2 + QR_g X + 4\pi \mu e^{\gamma X} \right) dV_g
  \]
  with $\gamma \in (0, 2)$, $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$ and $\mu > 0$.
  \[\Rightarrow\] law of the metric described by Liouville QFT

Notations: $\triangle_g = \text{Laplacian}$, $\partial_g = \text{gradient}$, $R_g = \text{Ricci curvature}$, $V_g = \text{volume form}$
Sketchy state of the art

1 Polyakov’s decomposition
   - Regularized determinants (Ray, Singer, D’Hoker, Phong, Kurzepa, Sarnak...)
   - Random planar maps
   - Ferrari, Klevtsov, Zelditch’s (2011) approach based on Kähler geometry

2 Construction of Liouville QFT
   - Albeverrio, Hoegh Krohn, Paycha, Scarlatti (‘92) Hoegh-Krohn model
   - Takhtajan (‘93) geometrical approach
     Perturbative series expansion around the minimum of the action
   - David, Kupiainen, Rhodes, Vargas (2014): Construction on the sphere
   - Guillarmou, Rhodes, Vargas (2016): Construction on hyperbolic surfaces
towards a rigorous construction of string theory

3 DOZZ formula (quantitative study of Liouville QFT)
   - formula proposed by Dorn/Otto/Zamolodchikov brothers
   - convincing arguments given by Teschner under a set of "axioms" supposedly satisfied by the Liouville QFT.
Plan of the talk

Motivation in $2d$-quantum gravity

Liouville quantum field theory (LQFT)

Semiclassical limit
Classical Liouville theory (uniformization)

(a) E. Picard

(b) H. Poincaré

Goal: Given a 2d Riemann manifold \((M, g_0)\), find a metric \(g\) conformally equivalent to \(g_0\), i.e. of the form

\[ g = e^{\gamma X} g_0 \]

for some \(\gamma \in \mathbb{R}\), with constant Ricci scalar curvature \(-2\pi \gamma^2 \mu\)

\[ R_g = -2\pi \gamma^2 \mu \]  \((*)\)

Notations: \(\Delta_g = \text{Laplacian}, \ \partial_g = \text{gradient}, \ \ R_g=\text{Ricci curvature}, \ \ V_g=\text{volume form}\)
Classical Liouville theory (uniformization)

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g = e^{\gamma X} g_0
\]

for some \(\gamma \in \mathbb{R}\), with constant Ricci scalar curvature \(-2\pi \gamma^2 \mu\)

\[
R_g = -2\pi \gamma^2 \mu
\]  \(\star\)

**Liouville equation:** If \(X : M \to \mathbb{R}\) solves the equation

\[
\gamma \Delta_{g_0} X - R_{g_0} = 2\pi \mu \gamma^2 e^{\gamma X}
\]

then the metric \(g = e^X g_0\) satisfies \((\star)\).

**Liouville action:** Find such a \(X\) by minimizing the functional

\[
X \mapsto S_L(g_0, X) = \frac{1}{4\pi} \int_M \left( |\partial_{g_0} X|^2 + Q_{\text{cl}} R_{g_0} X + 4\pi \mu e^{\gamma X} \right) dV_{g_0}
\]

with \(Q_{\text{cl}} = \frac{2}{\gamma}\)

Notations: \(\Delta_g = \text{Laplacian},\ \partial_g = \text{gradient},\ \ R_g=\text{Ricci curvature},\ \ V_g=\text{volume form}\)
LQFT on the Riemann sphere \( S^2 = \mathbb{C} \cup \{ \infty \} \)

We see the Riemann sphere as the complex plane \( \mathbb{C} \) equipped with the round metric

\[
\hat{g}(z) = \frac{4}{(1 + |z|^2)^2},
\]

hence \( R_{\hat{g}} = 2 \).

Fix \( \gamma \in ]0, 2] \), \( Q = \frac{2}{\gamma} + \frac{\gamma}{2} \) and \( \mu > 0 \).

**Theorem (DKRV, 2014)**

Pick a \( g \) in the conformal class of \( \hat{g} \). One can make sense of

\[
e^{-S_L(X, g)} \, DX
\]

where \( DX \) is the "uniform measure" on maps \( X : S^2 \to \mathbb{R} \) and \( S_L \) Liouville action:

\[
S_L(g, X) := \frac{1}{4\pi} \int_{S^2} \left( |\partial_g X|^2 + QR_{\hat{g}}X + 4\pi \mu e^{\gamma X} \right) \, dV_{\hat{g}}
\]

as a measure on \( H^{-s}(S^2, g) \) for \( s > 0 \).

Notations: \( \triangle_g = \text{Laplacian}, \ \partial_g = \text{gradient}, \ \ R_g=\text{Ricci curvature}, \ \ V_g=\text{volume form} \).
Idea of the construction

\[ \nu_L(DX) = e^{-\frac{Q}{4\pi} \int_{S^2} R_g X \, dV_g - \mu \int_{S^2} e^{\gamma X} \, dV_g} \times e^{-\frac{1}{4\pi} \int_{S^2} |\partial_g X|^2 \, dV_g} DX \]

Gaussian measure (GFF) on \( H^{-s}(S^2, g) \)

The exponential is ill defined on \( H^{-s}(S^2, g) \) with \( s > 0 \) ⇒ make sense of

\[ \int_{S^2} e^{\gamma X} \, dV_g \]

by means of renormalization theory:

**Gaussian multiplicative chaos**, Kahane 1985

Figure: J.-P. Kahane
Definition of the correlation functions

Important feature of Quantum Field Theories: the correlation functions

- they stand for observables of the system
- they are in some cases exactly computable

In Liouville QFT, they are defined for distinct $z_i \in S^2$ and $\alpha_i \in \mathbb{R}$ by

$$\langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle_g := \int \left( \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \right) e^{-S_L(X,g)} DX$$

where $\phi$ is the Liouville field

$$\phi(z) = X(z) + \frac{Q}{2} \ln g(z).$$
Link with classical Liouville theory

Formally

\[
\langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle_g := C \int e^{-S_{L}^{(z_i, \alpha_i)}(X, g)} \, DX
\]

where

\[
S_{L}^{(z_i, \alpha_i)}(X, g) = \frac{1}{4\pi} \int_{\Sigma} \left( |\partial_g X|^2 + Q R_g X + 4\pi \mu \, e^{\gamma X} \right) \, dV_g - \sum_{i} \alpha_i X(z_i)
\]

Classically, the minimum \( X \) of this functional solves the Liouville equation with sources

\[
\gamma \Delta_g X - R_g = 2\pi \mu \gamma^2 e^{\gamma X} - 2\pi \gamma \sum_{i} \alpha_i \delta_{z_i}.
\]

The metric \( e^{\gamma X} g \) has uniformized curvature on \( \mathbb{S}^2 \setminus \{z_1, \ldots, z_n\} \) and conical singularities at the points \((z_i)_i\).
Existence of the correlation functions

**Theorem (DKRV, 2014)**

The correlation \( \langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle \) exists and is non trivial if and only if:

\[ \forall i, \; \alpha_i < Q \quad \text{and} \quad \sum_{i=1}^{n} \alpha_i > 2Q \quad \text{(Seiberg bounds)} \]

In particular, existence implies \( n \geq 3! \)
Axiomatic of Conformal Field Theories

Let \((\alpha_i)_i\) satisfy the Seiberg bounds then

Theorem (DKRV, 2014)

1) **Conformal covariance:** If \(\psi\) is a Mobius transform, we have

\[
\langle \prod_{i=1}^{n} e^{\alpha_i \phi(\psi(z_i))} \rangle_g = \prod_{i=1}^{n} |\psi'(z_i)|^{-2\Delta_{\alpha_i}} \langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle_g
\]

where \(\Delta_{\alpha_i} = \frac{\alpha_i}{2} (Q - \frac{\alpha_i}{2})\) is called conformal weight of \(e^{\alpha_i X(z)}\).

2) **Conformal anomaly:** If \(g' = e^{\varphi} g\) then

\[
\ln \frac{\langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle_{g'}}{\langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle_g} = -\frac{1 + 6Q^2}{96\pi} \int_{\mathbb{C}} \left( |\partial_g \varphi|^2 + 2R_g \varphi \right) dV_g
\]
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Semiclassical limit
Context: Quantify how the quantum theory is dominated by the classical one

Setup: consider the following probability law \( \tilde{P} \) on \( H^{-s}(S^2, g) \) with expectation

\[
\tilde{E}[F(\phi)] := C \int F(\phi) \left( \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \right) e^{-S_L(X, g)} DX
\]

where

- \( \phi = X + \frac{Q}{2} \ln g \) is the Liouville field
- \( F \) is a continuous functional on \( H^{-s}(S^2, g) \)
- \( C \) is a renormalization constant such that \( \tilde{E}[1] = 1 \)

Goal: Find the limit in law of the field \( \gamma \phi \) under the probability law \( \tilde{P} \) when

\[
\gamma \to 0 \text{ and } \mu = \frac{\Lambda}{\gamma^2}, \quad \text{and} \quad \forall i, \quad \alpha_i = \frac{\chi_i}{\gamma}
\]

for fixed \( \chi_i < 2 \).
Semiclassical limit (LRV, 2014)

As $\gamma \to 0$, the field $\gamma \phi$ converges in probability under $\tilde{\mathbb{P}}$ towards the solution $\phi_*$ to the equation

$$\Delta \phi_* = 2\pi \Lambda e^{\phi_*} - 2\pi \sum_i \chi_i \delta z_i$$

Fluctuations (LRV, 2014)

As $\gamma \to 0$, the field $\phi - \frac{1}{\gamma} \phi_*$ converges in law under $\tilde{\mathbb{P}}$ towards a Massive Free Field in the metric $e^{\phi_*} (z) dz^2$

One can also prove a large deviation principle.

Remark: This result combined with the BPZ equations provides a proof of Polyakov’s conjecture for the accessory parameters (see V. Vargas’ talk)
Make the change of variables $\gamma X - \rightarrow X'$ in the correlation functions

$$\int \left( \prod_{i=1}^{n} e^{\alpha_i X(z_i)} \right) e^{-S_L(X,g)} DX$$

to get for small $\gamma$

$$\int \left( \prod_{i=1}^{n} e^{\alpha_i X(z_i)} \right) e^{-S_L(X,g)} DX \sim \int e^{-\frac{1}{\gamma^2} S(X')} DX'$$

with

$$S(h) := \frac{1}{4\pi} \int \left( |\partial_g h|^2 + 2R_g h + 4\pi \Lambda e^h \right) dV_g - \sum_{i} \chi_i h(z_i).$$

Hence (Laplace’s method)

$$\int \left( \prod_{i=1}^{n} e^{\alpha_i X(z_i)} \right) e^{-S_L(X,g)} DX \sim e^{-\frac{1}{\gamma^2} S(h_*)}$$

where

$$S(h_*) = \min_{h} S(h)$$
Take a linear functional $L$ on $H^{-s}$. Similarly $(\gamma X - > X')$

$$\int e^{L(X/\gamma)} \left( \prod_{i=1}^{n} e^{\alpha_i X(z_i)} \right) e^{-S_L(X,g)} DX \sim \int e^{-\frac{1}{\gamma^2} (S(X') - L(X'))} DX'$$
Take a linear functional $L$ on $H^{-s}$. Similarly ($\gamma X > X'$)

$$\int e^{L(X/\gamma)} \left( \prod_{i=1}^{n} e^{\alpha_i X(z_i)} \right) e^{-S_L(X,g)} DX \sim \int e^{-\frac{1}{2\gamma^2} \left( S(X') - L(X') \right)} DX'$$

Hence (Laplace’s method)

$$\int e^{L(X/\gamma)} \left( \prod_{i=1}^{n} e^{\alpha_i X(z_i)} \right) e^{-S_L(X,g)} DX \sim e^{-\frac{1}{2\gamma^2} \min_{h} \left( S(h) - L(h) \right)}.$$ 

Otherwise stated

$$\gamma^2 \ln \tilde{E}[e^{-\gamma L(\gamma X)}] \sim -\min_{h} \left( S(h) - S(h_*) - L(h) \right).$$
Take a linear functional $L$ on $H^{-s}$. Similarly $(\gamma X - X')$

$$\int e^{L(X/\gamma)} \left( \prod_{i=1}^{n} e^{\alpha_i X(z_i)} \right) e^{-S_L(X,g)} DX \sim \int e^{-\frac{1}{\gamma^2} \left( S(X') - L(X') \right)} DX'$$

Hence (Laplace’s method)

$$\int e^{L(X/\gamma)} \left( \prod_{i=1}^{n} e^{\alpha_i X(z_i)} \right) e^{-S_L(X,g)} DX \sim e^{-\frac{1}{\gamma^2} \min_h \left( S(h) - L(h) \right)}.$$

Otherwise stated

$$\gamma^2 \ln \tilde{E}[e^{\gamma^{-2}L(\gamma X)}] \sim -\min_h \left( S(h) - S(h_*) - L(h) \right).$$

By standard large deviation results

$$\tilde{P}[\gamma X - h_* \in A] \sim e^{-\frac{1}{\gamma^2} \min_{h \in A} (S(h+h_*) - S(h_*))}$$

for $A \subset H^{-s}$. 
Thanks!