Liouville Quantum Field Theory

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[1] based on joint works with: C. Guillarmou, F. David, A. Kupiainen, H. Lacoin, V. Vargas

Outline

Motivation in $2d$-quantum gravity

Liouville quantum field theory (LQFT)

Semiclassical limit

Plan of the talk

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General principles in physics

Euclidean field theories are described by a functional (called *action*):

$$S : f \in \Sigma \mapsto S(f) \in \mathbb{R}$$

where Σ is a functional space.

- **Classical physics**: (principle of least action) study the minima of this functional
- **Quantum physics**: (probabilistic approach) construct the measure

  $$e^{-S(f)} Df$$

  where Df is a "measure" on the functional space Σ.

Einstein-Hilbert action for gravity and matter fields

Fix a manifold M and denote $\mathrm{Met}(M)$ the set of Riemmannian metrics g on M.

- General relativity gives the action for the **gravitational field**

$$S_{\mathrm{EH}}(g) = \frac{1}{2\kappa} \int_M R_g \, dV_g + \mu \int_M dV_g$$

where the functional space is $\mathrm{Met}(M)$.

Notations: $\triangle_g = \mathit{Laplacian}$, $\nabla_g = \mathit{gradient}$, R_g=Ricci curvature, V_g=volume form

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- **Coupling with matter:** consider an action for matter field

$$\psi \mapsto S_{\mathrm{m}}(g, \psi)$$

defined on some functional space Θ made up of maps $\psi : M \to \mathbb{R}^p$.

The total action for Gravity+matter is

$$S(g, \psi) = S_{\mathrm{EH}}(g) + S_{\mathrm{m}}(g, \psi)$$

acting on the functional space $\Sigma = \mathrm{Met}(M) \times \Theta$.

Notations: $\triangle_g =$ *Laplacian*, $\nabla_g =$ *gradient*, R_g=Ricci curvature, V_g=volume form

Quantum gravity

Action for Gravity+matter is

$$S(g,\psi) = S_{\mathrm{EH}}(g) + S_{\mathrm{m}}(g,\psi).$$

- classical gravity: study the minima of this functional (yields Einstein's field equation)
- quantum gravity: construct the measure

$$e^{-S(g,\psi)} Dg D\psi$$

  on the functional space $\mathrm{Met}(M) \times \Theta$ where Dg is the volume form of the DeWitt metric.

Features:
- non renormalizable in $4d$.
- Einstein-Hilbert functional is trivial in $2d$, hence more tractable.

Polyakov's breakthrough (1981)

(works if the matter field possesses conformal symmetries in 2d).

Roughly and on the Riemann sphere:

- Decompose each metric g as

$$g = d^*(e^h \hat{g})$$

  where d is a diffeomorphism, h is a scalar function and $\hat{g}$ is the round metric.

- apply the corresponding change of variables to the DeWitt metric and get

$$\int F(g) e^{-S(g,\psi)} Dg D\psi = C(\hat{g}) \int F(e^{\gamma X} \hat{g}) e^{-S_{\mathrm{L}}(\hat{g},X)} DX$$

  where S_{L} is the Liouville action functional

$$\frac{1}{4\pi} \int_M \left(|\partial_g X|^2 + Q R_g X + 4\pi\mu e^{\gamma X} \right) dV_g$$

  with $\gamma \in (0,2)$, $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$ and $\mu > 0$.

$\Rightarrow$ law of the metric described by **Liouville QFT**

Notations: $\triangle_g = Laplacian$, $\partial_g = gradient$, R_g=Ricci curvature, V_g=volume form

Sketchy state of the art

1. **Polyakov's decomposition**
   - Regularized determinants (Ray, Singer, D'Hoker, Phong, Kurzepa, Sarnak...)
   - Random planar maps
   - Ferrari, Klevtsov, Zelditch's (2011) approach based on Kähler geometry

2. **Construction of Liouville QFT**
   - Albeverrio, Hoegh Krohn, Paycha, Scarlatti ('92) Hoegh-Krohn model
   - Takhtajian ('93) geometrical approach
     Perturbative series expansion around the minimum of the action
   - David, Kupiainen, Rhodes, Vargas (2014): Construction on the sphere
   - Guillarmou, Rhodes, Vargas (2016): Construction on hyperbolic surfaces
     towards a rigorous construction of string theory

3. **DOZZ formula** (quantitative study of Liouville QFT)
   - formula proposed by Dorn/Otto/Zamolodchikov brothers
   - convincing arguments given by Teschner under a set of "axioms" supposedly satisfied by the Liouville QFT.

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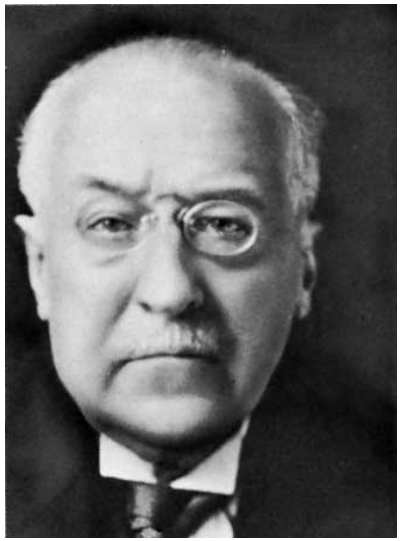
Classical Liouville theory (uniformization)



(a) E. Picard

(b) H.Poincaré

Goal: Given a $2d$ Riemann manifold (M, g_0), find a metric g conformally equivalent to g_0, i.e. of the form

$$g = e^{\gamma X} g_0$$

for some $\gamma \in \mathbb{R}$, with constant Ricci scalar curvature $-2\pi\gamma^2\mu$

$$R_g = -2\pi\gamma^2\mu \qquad (\star)$$

Notations: $\triangle_g = \textit{Laplacian}$, $\partial_g = \textit{gradient}$, R_g=Ricci curvature, V_g=volume form

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Liouville equation: If $X : M \to \mathbb{R}$ solves the equation

$$\gamma \triangle_{g_0} X - R_{g_0} = 2\pi\mu\gamma^2 e^{\gamma X}$$

then the metric $g = e^X g_0$ satisfies $(\star)$.

Liouville action: Find such a X by minimizing the functional

$$X \mapsto S_L(g_0, X) = \frac{1}{4\pi} \int_M \left(|\partial_{g_0} X|^2 + Q_{\mathrm{cl}} R_{g_0} X + 4\pi\mu e^{\gamma X} \right) dV_{g_0}$$

with $Q_{\mathrm{cl}} = \frac{2}{\gamma}$

Notations: $\triangle_g = \textit{Laplacian}$, $\partial_g = \textit{gradient}$, R_g=Ricci curvature, V_g=volume form

LQFT on the Riemann sphere $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$

We see the Riemann sphere as the complex plane $\mathbb{C}$ equipped with the round metric

$$\hat{g}(z) = \frac{4}{(1+|z|^2)^2},$$

hence $R_{\hat{g}} = 2$.

Fix $\gamma \in]0,2]$, $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$ and $\mu > 0$.

Theorem (DKRV, 2014)

Pick a g in the conformal class of $\hat{g}$. One can make sense of

$$e^{-S_L(X,g)} DX$$

where DX is the "uniform measure" on maps $X : \mathbb{S}^2 \to \mathbb{R}$ and S_L Liouville action:

$$S_L(g,X) := \frac{1}{4\pi} \int_{\mathbb{S}^2} \left(|\partial_g X|^2 + Q R_g X + 4\pi\mu e^{\gamma X} \right) dV_g$$

as a measure on $H^{-s}(\mathbb{S}^2, g)$ for $s > 0$.

Notations: $\triangle_g =$ *Laplacian*, $\partial_g =$ *gradient*, R_g=Ricci curvature, V_g=volume form

Idea of the construction

$$\nu_L(DX) = e^{-\frac{Q}{4\pi}\int_{\mathbb{S}^2} R_g X \, dV_g - \mu \int_{\mathbb{S}^2} e^{\gamma X} \, dV_g} \times \underbrace{e^{-\frac{1}{4\pi}\int_{\mathbb{S}^2} |\partial_g X|^2 \, dV_g} DX}_{\text{Gaussian measure (GFF) on } H^{-s}(\mathbb{S}^2, g)}$$

The exponential is ill defined
on $H^{-s}(\mathbb{S}^2, g)$ with $s > 0 \Rightarrow$ make sense of

$$\int_{\mathbb{S}^2} e^{\gamma X} \, dV_g$$

by means of renormalization theory:

Gaussian multiplicative chaos, Kahane 1985

Figure: J.-P. Kahane

Definition of the correlation functions

Important feature of Quantum Field Theories: the **correlation functions**

- they stand for observables of the system
- they are in some cases exactly computable

In Liouville QFT, they are defined for distinct $z_i \in \mathbb{S}^2$ and $\alpha_i \in \mathbb{R}$ by

$$\langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle_g := \int \left(\prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \right) e^{-S_L(X,g)} DX$$

where ϕ is the Liouville field

$$\phi(z) = X(z) + \frac{Q}{2} \ln g(z).$$

Link with classical Liouville theory

Formally

$$\langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle_g := C \int e^{-S_L^{(z_i,\alpha_i)_i}(X,g)} DX$$

where

$$S_L^{(z_i,\alpha_i)_i}(X,g) = \frac{1}{4\pi} \int_\Sigma \left(|\partial_g X|^2 + QR_g X + 4\pi\mu e^{\gamma X} \right) d\mathrm{V}_g - \sum_i \alpha_i X(z_i)$$

Classically, the minimum X of this functional solves the Liouville equation with sources

$$\gamma \triangle_g X - R_g = 2\pi\mu\gamma^2 e^{\gamma X} - 2\pi\gamma \sum_i \alpha_i \delta_{z_i}.$$

The metric $e^{\gamma X} g$ has uniformized curvature on $\mathbb{S}^2 \setminus \{z_1, \ldots, z_n\}$ and conical singularities at the points $(z_i)_i$.

Existence of the correlation functions

Theorem (DKRV, 2014)

The correlation $\langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle_g$ *exists and is non trivial if and only if:*

$$\forall i,\ \alpha_i < Q \quad \text{and} \quad \sum_{i=1}^{n} \alpha_i > 2Q \qquad (\textit{Seiberg bounds})$$

In particular, existence implies $n \geq 3$*!*

Axiomatic of Conformal Field Theories

Let $(\alpha_i)_i$ satisfy the Seiberg bounds then

Theorem (DKRV, 2014)

1) **Conformal covariance:** *If ψ is a Mobius transform, we have*

$$\langle \prod_{i=1}^{n} e^{\alpha_i \phi(\psi(z_i))} \rangle_g = \prod_{i=1}^{n} |\psi'(z_i)|^{-2\Delta_{\alpha_i}} \langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle_g$$

where $\Delta_{\alpha_i} = \frac{\alpha_i}{2}(Q - \frac{\alpha_i}{2})$ is called conformal weight of $e^{\alpha_i X(z)}$.

2) **Conformal anomaly:** *If $g' = e^{\varphi} g$ then*

$$\ln \frac{\langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle_{g'}}{\langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle_g} = -\frac{1+6Q^2}{96\pi} \int_{\mathbb{C}} \left(|\partial_g \varphi|^2 + 2R_g \varphi \right) dV_g$$

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Context: Quantify how the quantum theory is dominated by the classical one

Setup: consider the following probability law $\tilde{\mathbb{P}}$ on $H^{-s}(\mathbb{S}^2, g)$ with expectation

$$\tilde{E}[F(\phi)] := C \int F(\phi) \Big(\prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \Big) e^{-S_L(X,g)} DX$$

where

- $\phi = X + \frac{Q}{2} \ln g$ is the Liouville field
- F is a continuous functional on $H^{-s}(\mathbb{S}^2, g)$
- C is a renormalization constant such that $\tilde{E}[1] = 1$

Goal: Find the limit in law of the field $\gamma\phi$ under the probability law $\tilde{\mathbb{P}}$ when

$$\gamma \to 0 \text{ and } \mu = \frac{\Lambda}{\gamma^2}, \quad \text{and} \quad \forall i,\ \alpha_i = \frac{\chi_i}{\gamma}$$

for fixed $\chi_i < 2$.

Semiclassical limit (LRV, 2014)

As $\gamma \to 0$, the field $\gamma\phi$ converges in probability under $\tilde{\mathbb{P}}$ towards the solution $\phi_\star$ to the equation

$$\triangle\phi_\star = 2\pi\Lambda e^{\phi_\star} - 2\pi \sum_i \chi_i \delta_{z_i}$$

Fluctuations (LRV, 2014)

As $\gamma \to 0$, the field $\phi - \frac{1}{\gamma}\phi_\star$ converges in law under $\tilde{\mathbb{P}}$ towards a Massive Free Field in the metric $e^{\phi_\star(z)}dz^2$

One can also prove a large deviation principle.

Remark: This result combined with the BPZ equations provides a proof of Polyakov's conjecture for the accessory parameters (see V. Vargas' talk)

Make the change of variables $\gamma X -> X'$ in the correlation functions

$$\int \Big(\prod_{i=1}^{n} e^{\alpha_i X(z_i)}\Big) e^{-S_L(X,g)} DX$$

to get for small γ

$$\int \Big(\prod_{i=1}^{n} e^{\alpha_i X(z_i)}\Big) e^{-S_L(X,g)} DX \sim \int e^{-\frac{1}{\gamma^2} S(X')} DX'$$

with

$$S(h) := \frac{1}{4\pi} \int \Big(|\partial_g h|^2 + 2R_g h + 4\pi\Lambda e^h\Big)\, d\mathrm{V_g} - \sum_i \chi_i h(z_i).$$

Hence (Laplace's method)

$$\int \Big(\prod_{i=1}^{n} e^{\alpha_i X(z_i)}\Big) e^{-S_L(X,g)} DX \sim e^{-\frac{1}{\gamma^2} S(h_\star)}$$

where

$$S(h_\star) = \min_h S(h)$$

Take a linear functional L on H^{-s}. Similarly ($\gamma X -> X'$)

$$\int e^{L(X/\gamma)} \Big(\prod_{i=1}^{n} e^{\alpha_i X(z_i)} \Big) e^{-S_L(X,g)} DX \sim \int e^{-\frac{1}{\gamma^2}\left(S(X') - L(X')\right)} DX'$$

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Otherwise stated

$$\gamma^2 \ln \tilde{E}[e^{\gamma^{-2} L(\gamma X)}] \sim -\min_h \big(S(h) - S(h_\star) - L(h)\big).$$

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By standard large deviation results

$$\tilde{P}[\gamma X - h_\star \in A] \sim e^{-\frac{1}{\gamma^2}\min_{h\in A}(S(h+h_\star)-S(h_\star))}$$

for $A \subset H^{-s}$.

Thanks!