Quantum Ergodicity and Benjamini-Schramm convergence of hyperbolic surfaces

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Hyperbolic Plane

- **Upper half plane model**

$$\mathbb{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$$

$$ds^2 = \frac{dx^2 + dy^2}{y^2} \qquad d\mu(z) = \frac{dx\, dy}{y^2}$$

Isometry group: $\mathrm{SL}(2, \mathbb{R})$ acting via Möbius transformations

- **Poincaré disk model**

$$\mathbb{H} = \{z = x + iy \in \mathbb{C} \mid |z| < 1\}$$

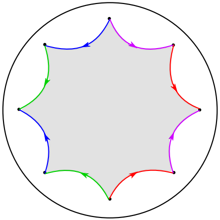
$$ds^2 = 4\frac{dx^2 + dy^2}{(1 - |z|^2)^2} \qquad d\mu(z) = \frac{4\, dx\, dy}{(1 - |z|^2)^2}$$

Isometry group: $\mathrm{SU}(1, 1)$ acting via Möbius transformations

Hyperbolic Surfaces

Hyperbolic surface:
Quotient $\Gamma\backslash\mathbb{H}$ of the plane by a discrete subgroup of isometries.
Surface of constant negative curvature -1.

Example in the disk model



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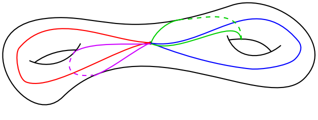

$\Gamma \backslash \mathbb{H}$

Spectrum of the Laplacian

Let $M = \Gamma\backslash\mathbb{H}$ be a compact hyperbolic surface.

Δ the Laplacian. In the upper-half plane model

$$\Delta = y^2\left(\partial_x^2 + \partial_y^2\right).$$

There exists an orthonormal basis $(\psi_j)_{j\in\mathbb{N}}$ of eigenfunctions in $L^2(M)$.

$$-\Delta\psi_j = \lambda_j\psi_j$$

$$\lambda_0 \leq \lambda_1 \leq \dots \qquad \lambda_j \to +\infty$$

Quantum ergodicity (quantitative version)

Theorem (Zelditch 94)

On compact hyperbolic surfaces, for any function $a \in C^\infty(M)$,

$$\lim_{\lambda \to +\infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \left| \langle \psi_j, a\,\psi_j \rangle - \fint a\,d\mu \right|^2 = O(\log(\lambda)^{-1}),$$

where $N(\lambda)$ is the number of eigenvalues $\leq \lambda$.

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where $N(\lambda)$ is the number of eigenvalues $\leq \lambda$.

More generally: for any $a \in C^\infty(M \times \mathbb{S}^1)$

$$\lim_{\lambda \to +\infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \left| \langle \psi_j, \mathrm{Op}(a)\, \psi_j \rangle - \fint a(z, \theta)\, d\mu(z) d\theta \right|^2 = O(\log(\lambda)^{-1}),$$

where $\mathrm{Op}(a)$ is a pseudo-differential operator of order 0.

Quantum ergodicity (an alternative approach)

$$\Delta\psi_j = -\lambda_j\psi_j, \quad 0 \le \lambda_1 \le \lambda_2 \le \ldots, \quad \langle\psi_i, \psi_j\rangle = \delta_{ij}.$$

Question Equidistribution theory of eigenfunctions for bounded eigenvalues instead of large eigenvalues $\lambda_j \to \infty$

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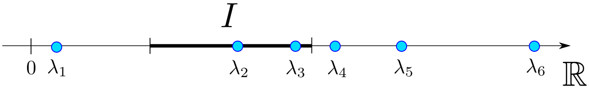

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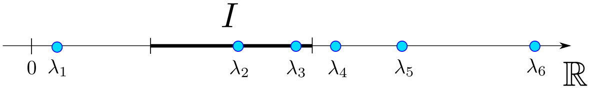

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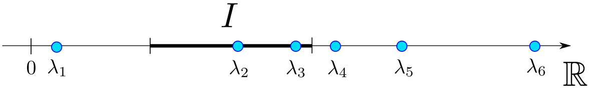

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- Now vary the surface M

  (e.g. volume, genus, injectivity radius etc.)

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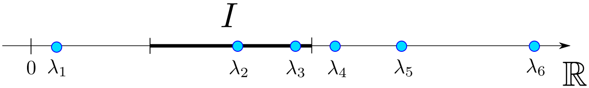

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- Consider just ψ_j with $\lambda_j \in I$.
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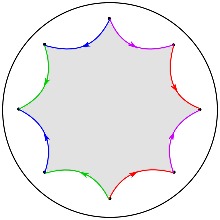
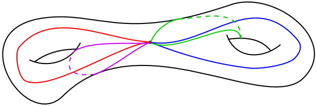
What happens to $|\psi_j|^2$ with $\lambda_j \in I$ as we vary the geometry of M?

Example: injectivity radius

- Let M be a **hyperbolic surface**, i.e. constant negative curvature -1.



$\mathbb{H}$



$\Gamma \setminus \mathbb{H}$

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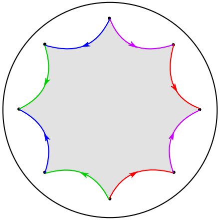

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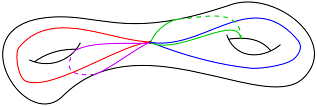

$\Gamma \backslash \mathbb{H}$

Pointwise injectivity radius of M at x:

$$\mathrm{InjRad}(M, x)$$

is the largest $R > 0$ such that $B(x, R) \subset M$ is isometric to a ball in $\mathbb{H}$.

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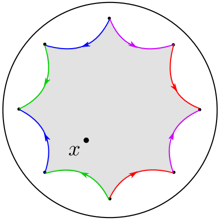

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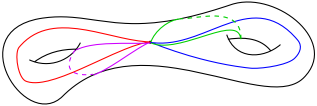

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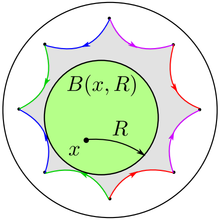

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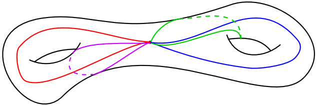

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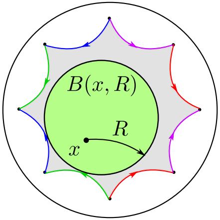

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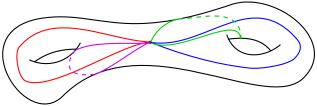

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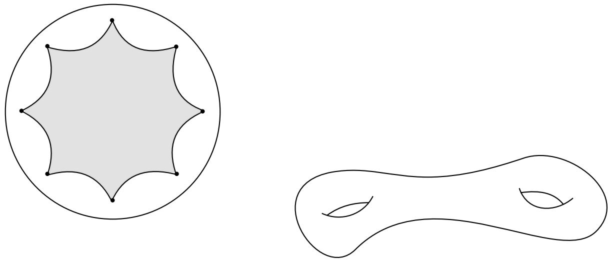
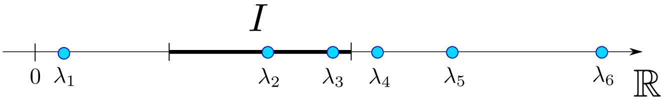
Injectivity radius of M is then:

$$\mathrm{InjRad}(M) := \inf_{x \in M} \mathrm{InjRad}(M, x)$$

Example: injectivity radius

- Increase $\mathrm{InjRad}(M) \implies$ More eigenvalues $\lambda_j \in I$.





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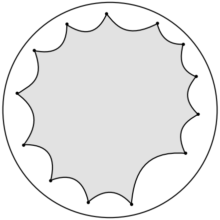

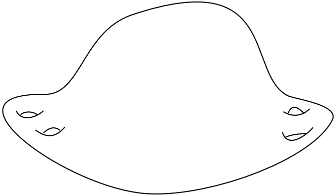

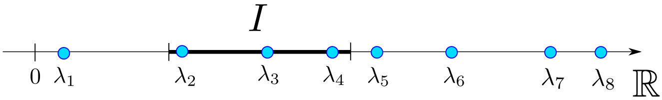

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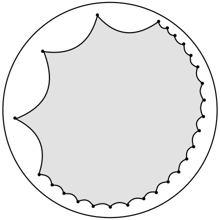

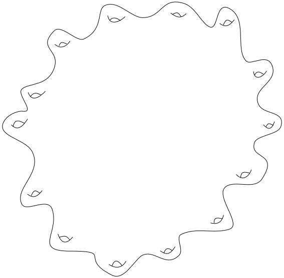

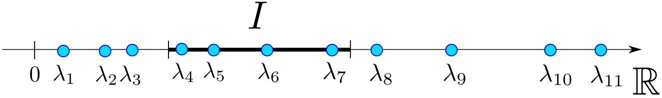

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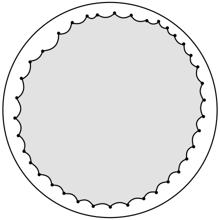
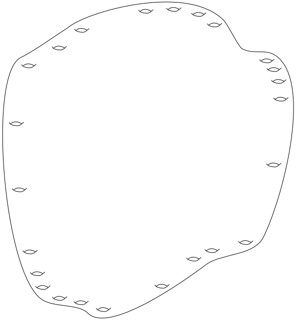
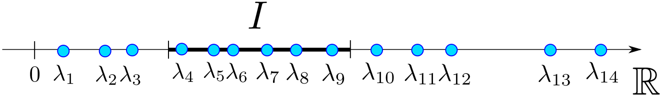
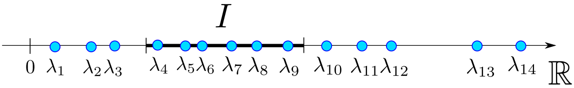






Equidistribution for large injectivity radius

$$\Delta\psi_j = -\lambda_j\psi_j, \quad 0 \le \lambda_1 \le \lambda_2 \le \dots, \quad \langle \psi_i, \psi_j \rangle = \delta_{ij}, \quad I \subset (1/4, \infty).$$



- Do "most" $|\psi_j|^2$, $\lambda_j \in I$, **equidistribute** when $\mathrm{InjRad}(M)$ is large?

Problem (Colin de Verdière 2010s)

Fix a bounded interval $I \subset (1/4, \infty)$. Is it true that

$$\frac{1}{\sharp\{\lambda_j \in I\}} \sum_{j:\lambda_j \in I} \left| \langle \psi_j, a\,\psi_j \rangle - \fint a\,d\mu \right|^2 \to 0$$

as $\mathrm{InjRad}(M) \to \infty$? (assuming a uniform spectral gap condition)

Motivation

- Quantum Ergodicity on graphs

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- QUE in the level aspect

Arithmetic example: the level aspect

Coverings X_q of the modular surface $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$

$$X_q = \Gamma_0(q)\backslash\mathbb{H}$$

where

$$\Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \mod q \right\}$$

Theorem (Nelson 2012, Nelson-Pitale-Saha 2014)

Holomorphic forms of fixed weight and increasing level q equidistribute. (QUE)

The question is open for **Maass forms**, i.e. eigenfunctions of the Laplacian.

Benjamini-Schramm convergence

A sequence of hyperbolic surfaces (M_n) **Benjamini-Schramm converges to** $\mathbb{H}$ if for any $R > 0$,

$$\lim_{n\to\infty} \frac{\mathrm{Vol}(\{x \in M_n : \mathrm{InjRad}(M_n, x) < R\})}{\mathrm{Vol}(M_n)} = 0$$

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Abert, Bergeron, Biringer, Gelander, Nikolov, Raimbault, Samet:
On the growth of L^2-invariants for sequences of lattices in Lie groups.
Preprint. arXiv:1210.2961, 98 pp

To simplify, we state the theorem for large injectivity radius, but the ideas adapt to Benjamini-Schramm convergence

Theorem (LM-Sahlsten 2016)

Fix a closed interval $I \in (1/4, +\infty)$ and let $a \in L^2(M)$,

$$\sum_{\lambda_j \in I} \left| \langle \psi_j, a\, \psi_j \rangle - \fint a\, d\mu \right|^2 \lesssim_I \frac{\|a\|_2^2}{\rho(\beta)\, \mathrm{InjRad}(M)},$$

where $\rho(\beta)$ is a function of the spectral gap β.

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where $\rho(\beta)$ is a function of the spectral gap β.

For a sequence M_n with uniform spectral gap and such that $\mathrm{InjRad}(M_n) \to +\infty$ we have

$$\frac{1}{\sharp\{\lambda_j \in I\}} \sum_{\lambda_j \in I} \left| \langle \psi_j, a_n\, \psi_j \rangle - \fint a_n\, d\mu \right|^2 \to 0$$

for any sequence of test functions $a_n \in L^2(M_n)$.

Examples

- **Compact arithmetic surfaces** of congruence type when the level tends to infinity. (**Abert et al.** On the growth of L^2-invariants for sequences of lattices in Lie groups. *Preprint*. arXiv:1210.2961)
- **Large random Riemann surfaces** for a natural model satisfy the conditions almost surely (**R. Brooks and E. Makover**: Random Construction of Riemann Surfaces, *J. Differential Geom.* 68(1), 2004.)

Reduction to a Hilbert-Schmidt estimate

Replace the wave propagator with renormalized **averages over discs**

$$P_t\, u(z) = e^{-t/2} \int_{D(z,t)} u(w) d\mu(w)$$

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Proposition (Spectral side)

If $\int a\, d\mu = 0$ then by estimating the **spherical transform** of the kernel of P_t we have for any T

$$\sum_{\lambda_j \in I} |\langle \psi_j, a\, \psi_j \rangle|^2 \lesssim_I \left\| \frac{1}{T} \int_0^T P_t\, a\, P_t\, dt \right\|_{\mathrm{HS}}^2$$

Geometric side

Kernel of $P_t \, a \, P_t$:

$$[P_t \, a \, P_t](z, z') = e^{-t} \int_{D(z,t) \cap D(z',t)} a(w) \, d\mu(w)$$

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We want to compute

$$\left\| \frac{1}{T} \int_0^T P_t \, a \, P_t \, dt \right\|_{\mathrm{HS}}^2 = \iint \left| \frac{1}{T} \int_0^T [P_t \, a \, P_t](z, z') \right|^2 d\mu(z) \, d\mu(z')$$

Nevo's mean ergodic theorem

Gorodnik, Nevo: The ergodic theory of lattice subgroups, *Annals of Mathematics Studies* 172, Princeton University Press, 2010

Theorem (Nevo)

Let (F_t) *be a family of* **measurable sets of positive measure** *and* $\pi(F_t)$ *the* **averaging operator over** F_t *(convolution with the characteristic function of the set), then*

$$\left\| \pi(F_t)\, f - \fint_M f \, d\mu \right\|_2 \lesssim \frac{1}{\mu(F_t)^{\rho(\beta)}} \|f\|_2.$$

Sketch of proof

We propagate only up to the radius of injectivity $T \leq \mathrm{InjRad}(M)$

- **"Almost" orthogonality**

$$\left\| \frac{1}{T} \int_0^T P_t \, a \, P_t \, dt \right\|_{\mathrm{HS}}^2 = \frac{1}{T^2} \int_0^T \| P_t \, a \, P_t \|_{\mathrm{HS}}^2 \, dt$$

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- Using Nevo's theorem

$$\| P_t \, a \, P_t \|_{\mathrm{HS}}^2 \lesssim \| a \|_2^2$$

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- Finally $$\left\| \frac{1}{T} \int_0^T P_t \, a \, P_t \, dt \right\|_{\mathrm{HS}}^2 \lesssim \frac{1}{T^2} \int_0^T \| a \|_2^2 \, dt \lesssim \frac{\| a \|_2^2}{T}$$

Perspectives

- Phase space version

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- More general manifolds, variable curvature, Maass forms

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- More general manifolds, variable curvature, Maass forms
- Quantum Unique Ergodicity: arithmetic QUE, entropy results