On the spatial distribution of critical points of Random Plane Waves

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Random Plane Wave

- A 1-dim version of the Random Wave was used by Rice to investigate the likelihood of a given signal to exceed a level.

- In 1950 Longuet-Higgins generalised Rice's model to 2-dim plane to describe ocean waves.

- In 1977 M. Berry conjectured that *high energy behaviour of eigenfunctions* in the chaotic case is universal and have statistically the same behaviour as random plane wave.

Random Plane Wave

- Space of spherical harmonics of degree l is of dimension $2l + 1$, let $\{\phi_i\}_{i=1,\dots 2\ell+1}$ be an arbitrary L^2-orthonormal basis. Define random Gaussian spherical harmonic

$$f_l = \sum_{i=1}^{2l+1} c_i \phi_i$$

  c_i are i.i.d. standard Gaussian variables. *RPW is the scaling limit of the Gaussian spherical harmonic*.

- We can relate high energy Gaussian spherical harmonics with to the the behaviour of RPW inside a big ball.

- Here we focus on critical points in *small balls and repulsion*.

Random Plane Wave

Definition

The **Random Plane Wave** with energy $E = k^2$ is the centred Gaussian field on $\mathbb{R}^2$ with covariance kernel

$$K(x, y) = J_0(k|x - y|).$$

One may think of RPW as a *random Gaussian solution* of

$$\Delta f + k^2 f = 0$$

i.e.

$$F(x) = \sum_{n=-\infty}^{\infty} c_n J_{|n|}(kr) e^{in\theta}$$

c_n are standard Gaussian independent save to $c_{-n} = \bar{c}_n$.

Critical Points

We study the **critical point set**

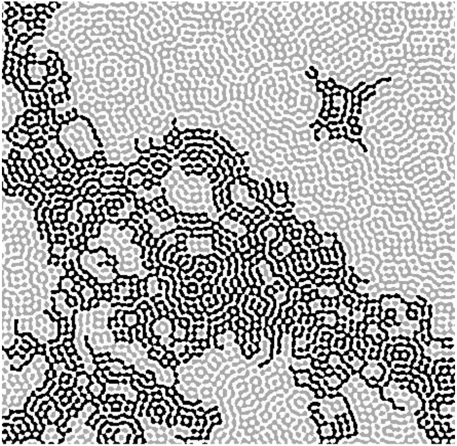
$$\{x \in \mathbb{R}^2 : \nabla F(x) = 0\}.$$

We are interested in the **spatial distribution** of critical points across the surface.

We fix $k = 1$ since RPW with different values of k differ by the scaling.

Motivations

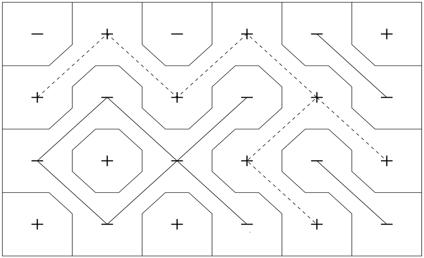
Nodal lines $\{x \in \mathbb{R}^2 : F(x) = 0\}$
Nodal domains $\mathbb{R}^2 \backslash \{x \in \mathbb{R}^2 : F(x) = 0\}$



Nodal domains for k = 100.
Two nodal domains are highlighted [Bogomolny-Schmit 2002]

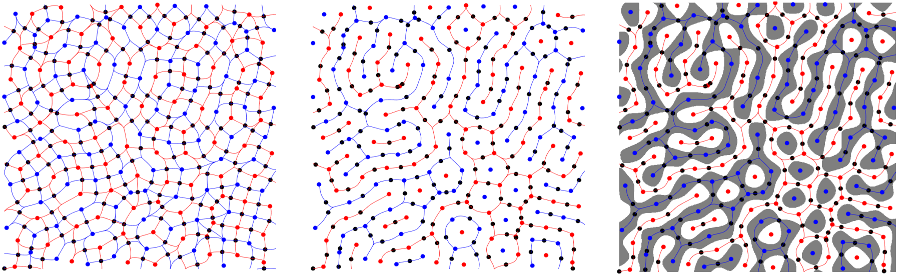
Motivations

[Bogomolny-Schmit 2002] introduce a bond percolation model on the square lattice to describe nodal domains.



Motivations

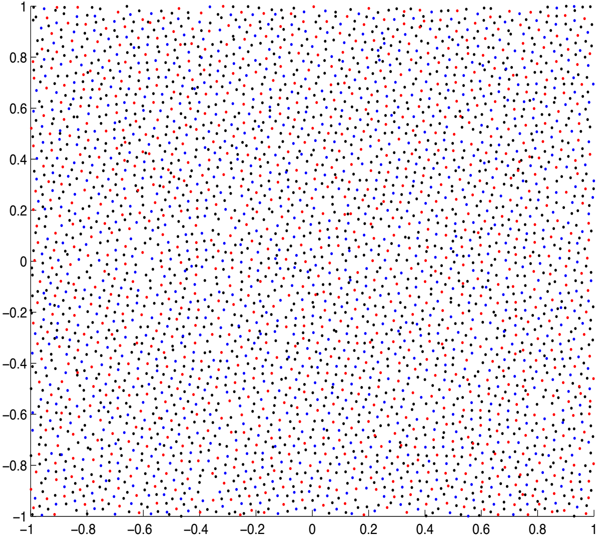
[Beliaev-Kereta 2013] introduce a bond percolation on a random graph where nodes are local maxima and edges are gradient streamlines passing through saddles.



Pictures by D. Beliaev and T. Sharpe.

Motivations

Critical points of RPW. Picture by D. Beliaev.



Motivations

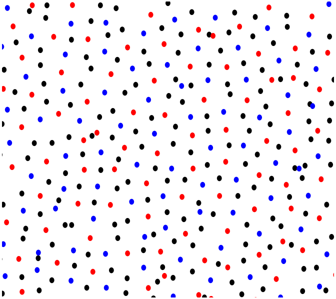


Figure : Critical points

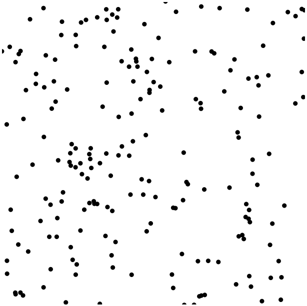


Figure : Poisson

Repulsion?

Expectation and Variance

$$\mathcal{N}_\rho^c = \#\{x \in \mathcal{B}(\rho) : \nabla F(x) = 0\}$$

Theorem (Beliaev-C.-Wigman)

For every $\rho > 0$

$$\mathbb{E}[\mathcal{N}_\rho^c] = \frac{1}{2\sqrt{3}}\rho^2.$$

As $\rho \to 0$

$$\mathbb{E}[\mathcal{N}_\rho^c\,(\mathcal{N}_\rho^c - 1)] = \frac{1}{2^5\,3\,\sqrt{3}}\,\rho^4 + O(\rho^6).$$

Proof

- Critical points density $K_1 : \mathbb{R}^2 \to \mathbb{R}$

$$K_1(x) = \phi_{\nabla F(x)}(0,0) \cdot \mathbb{E}[|\mathrm{det} H_F(x)| \big| \nabla F(x) = 0]$$

- Two-point correlation function $K_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

$$\begin{aligned} K_2(x,y) = {} & \phi_{(\nabla F(x), \nabla F(y))}(\mathbf{0}, \mathbf{0}) \\ & \cdot \mathbb{E}[|\mathrm{det} H_F(x)| \cdot |\mathrm{det} H_F(y)| \big| \nabla F(x) = \nabla F(y) = 0]. \end{aligned}$$

Proof

We apply a suitably modified version of Kac-Rice formula for counting the number of zeros of the gradient of F:

$$\mathbb{E}[\mathcal{N}_\rho^c] = \int_{\mathcal{B}(0,\rho)} K_1(x)dx,$$

and, for $x \neq y$,

$$\mathbb{E}[\mathcal{N}_\rho^c \cdot (\mathcal{N}_\rho^c - 1)] = \iint_{\mathcal{B}(0,\rho)\times\mathcal{B}(0,\rho)} K_2(x,y)dxdy;$$

provided that the Gaussian distribution of $(\nabla F(x), \nabla F(y)) \in \mathbb{R}^4$ is non-degenerate for all $(x,y) \in \mathcal{B}(0,\rho) \times \mathcal{B}(0,\rho)$.

Proof

Lemma

For every $x \in \mathbb{R}^2$

$$K_1(x) = \frac{1}{2\sqrt{3}\pi}.$$

As $r = d(x,y) \to 0$

$$K_2(r) = \frac{1}{96\sqrt{3}\pi^2} + O(r^2).$$

No repulsion!

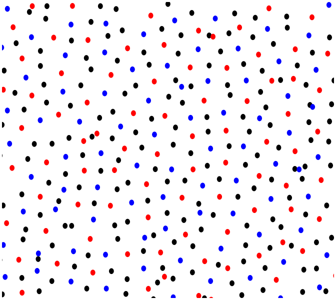


Figure : Critical points

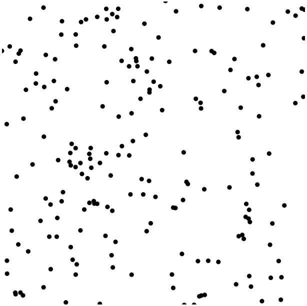


Figure : Poisson

As $r \to 0$

$$\frac{K_2(r)}{(K_1(r))^2} = \frac{1}{2^3\sqrt{3}} + O(r^2)$$

$r > 0$

$$\frac{K_2(r)}{(K_1(r))^2} = 1$$

Probability of $0, 1$ pts

$$\mathcal{N}_\rho^c = \#\{x \in \mathcal{B}(\rho) : \nabla F(x) = 0\}$$

Corollary

As $\rho \to 0$

$$\mathbb{P}(\mathcal{N}_\rho^c = 0) = 1 - \frac{1}{2\sqrt{3}}\rho^2 + O(\rho^4),$$

$$\mathbb{P}(\mathcal{N}_\rho^c = 1) = \frac{1}{2\sqrt{3}}\rho^2 + O(\rho^4).$$

Proof

$$\mathbb{P}(\mathcal{N}_\rho^c = 1) \leq \mathbb{P}(\mathcal{N}_\rho^c \geq 1) \leq \sum_{k=1}^{\infty} k\,\mathbb{P}(\mathcal{N}_\rho^c = k) = \mathbb{E}[\mathcal{N}_\rho^c] = \frac{1}{2\sqrt{3}}\rho^2.$$

We note that

$$\mathbb{E}[\mathcal{N}] = \mathbb{P}(\mathcal{N} = 1) + \sum_{k=2}^{\infty} k\,\mathbb{P}(\mathcal{N} = k)$$

and

$$\begin{aligned}\mathbb{P}(\mathcal{N} = 1) = \mathbb{E}[\mathcal{N}] - \sum_{k=2}^{\infty} k\,\mathbb{P}(\mathcal{N} = k) &\geq \mathbb{E}[\mathcal{N}] - \sum_{k=2}^{\infty} k(k-1)\,\mathbb{P}(\mathcal{N} = k) \\ &= \mathbb{E}[\mathcal{N}] - \mathbb{E}[\mathcal{N}(\mathcal{N}-1)]\end{aligned}$$

then

$$\mathbb{P}(\mathcal{N}_\rho^c = 1) \geq \frac{1}{2\sqrt{3}}\rho^2 - \frac{1}{2^5\,3\,\sqrt{3}}\,\rho^4 + O(\rho^6).$$

Proof - Lemma

- Critical point density K_1

$$K_1(x) = \phi_{\nabla F(x)}(0,0) \cdot \mathbb{E}[|\det H_F(x)| \big| \nabla F(x) = 0],$$

where

$$\phi_{\nabla F(x)}(0,0) = \frac{1}{\pi},$$

and

$$\mathbb{E}[|\det H_F(x)| \big| \nabla F(x) = 0] = \mathbb{E}[|\det H_F(x)|] = \frac{1}{2\sqrt{3}}.$$

- Two-point correlation function K_2 around the origin $\Longrightarrow$ **perturbation theory.**

Proof - Lemma

- Write the two-point correlation K_2 function as a function of the perturbing elements of the covariance matrix $\Delta(r)$ of $(\nabla^2 F(x), \nabla^2 F(y) | \nabla F(x) = \nabla F(y) = 0)$.

- The Gaussian expectations K_2 is analytic functions of the of the perturbing elements of $\Delta(r)$.

- K_2 is a smooth function, defined on some neighbourhood of the origin.

- We can Taylor expand K_2 around the origin.

Proof - Lemma

$$K_2(r) = \frac{1}{(2\pi)^5\sqrt{\mathsf{det}A(r)}}\int_{\mathbb{R}^6}|x_1x_3 - x_2^2|\cdot|x_4x_6 - x_5^2|$$
$$\times\frac{1}{\sqrt{\mathsf{det}\Delta(r)}}\exp\left\{-\frac{1}{2}\mathbf{x}^t\Delta(r)^{-1}\mathbf{x}\right\}d\mathbf{x}.$$

- For every $r > 0$ $\Delta(r)$ is symmetric $\implies$ we diagonalise $\Delta(r)$: $\Delta(r) = P(r)^t\Lambda(r)P(r)$
- Compute the eigenvalues and eigenvectors of the perturbed matrix $\Delta(r)$.
- Note that

$$\frac{1}{\sqrt{\mathsf{det}\Delta(r)}}\exp\{-\frac{1}{2}\mathbf{x}^t\Delta(r)^{-1}\mathbf{x}\} = \frac{1}{\sqrt{\prod\lambda_i(r)}}\exp\{-\frac{1}{2}(P(r)\mathbf{x})^t\Lambda(r)^{-1}P(r)\mathbf{x}\}.$$

- Change variable: $\mathbf{z} = P(r)\mathbf{x}$ i.e. $\mathbf{x} = P(r)^{-1}\mathbf{z}$.

Proof - Lemma

- Finally

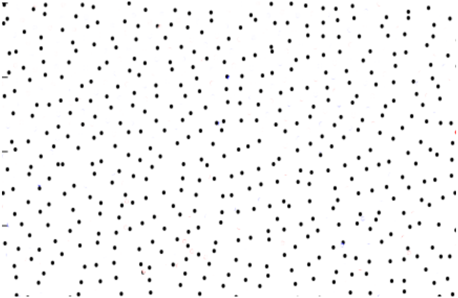
$$
\begin{aligned}
&K_2(r)\\
&= \frac{1}{(2\pi)^5\sqrt{\mathrm{det}A(r)}}\int_{\mathbb{R}^6} \big|f(r,\mathbf{z})\big| \exp\left\{-\frac{1}{2}\sum_{i=1}^{6} z_i^2\right\} d\mathbf{z}\\
&= \frac{1}{\pi^5 2\sqrt{3}r^2 + O(r^4)}\Big[\frac{r^2}{384}\int_{\mathbb{R}^6} z_4^2 z_6^2 \exp\left\{-\frac{1}{2}\sum_{i=1}^{6} z_i^2\right\} d\mathbf{z} + O(r^4)\Big]\\
&= \frac{1}{96\sqrt{3}\,\pi^2} + O(r^2).
\end{aligned}
$$

And...

- $\mathbb{E}[\mathcal{N}_\rho^c\,(\mathcal{N}_\rho^c - 1)\,(\mathcal{N}_\rho^c - 2)]* = *O(\rho^6)$

  $\implies \mathbb{P}(\mathcal{N}_\rho^c = 2)* = *\frac{1}{2^6\,3\,\sqrt{3}}\,\rho^4 + O(\rho^6)$

- Repulsion between saddles?



Thank You!