III. Quantum ergodicity on graphs, perspectives

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Yesterday we focussed on the case of large regular (discrete) graphs.

Let $G = (V, E)$ be a $(q+1)$-regular graph.

Discrete laplacian : $f : V \longrightarrow \mathbb{C}$,

$$\Delta f(x) = \sum_{y \sim x} (f(y) - f(x)) = \sum_{y \sim x} f(y) - (q+1) f(x).$$

$$\Delta = \mathcal{A} - (q+1) I$$

$$Sp(\mathcal{A}) \subset [-(q+1), q+1]$$

Let $|V| = N$. We look at the limit $N \longrightarrow +\infty$.

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We assume that G_N has "few" short loops (= converges to a tree in the sense of Benjamini-Schramm).

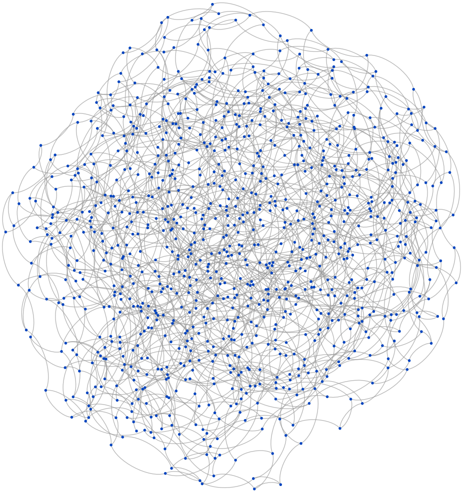
Theorem

(A-Le Masson, 2013) Assume that G_N has "few" short loops and that it forms an expander family = uniform spectral gap for $\mathcal{A}$.
Let $(\phi_i^{(N)})_{i=1}^N$ be an ONB of eigenfunctions of the laplacian on G_N.
Let $a = a_N : V_N \longrightarrow \mathbb{C}$ be such that $|a(x)| \leq 1$ for all $x \in V_N$.
Then

$$\lim_{N \longrightarrow +\infty} \frac{1}{N} \sum_{i=1}^{N} \left| \sum_{x \in V_N} a(x) |\phi_i^{(N)}(x)|^2 - \langle a \rangle \right|^2 = 0.$$

$$\langle a \rangle = \frac{1}{N} \sum_{x \in V_N} a(x)$$

Deterministic statement ; applies in particular to random regular graphs



More general version

Theorem

(A-Le Masson, 2013) Assume that G_N has "few" short loops and that it forms an expander family.
Let $(\phi_i^{(N)})_{i=1}^N$ be an ONB of eigenfunctions of the laplacian on G_N.
Let $K = K_N : V_N \times V_N \longrightarrow \mathbb{C}$ be a matrix such that $d(x,y) > D \Longrightarrow K(x,y) = 0$. Assume $|K(x,y)| \leq 1$. Then

$$\lim_{N \longrightarrow +\infty} \frac{1}{N} \sum_{i=1}^{N} \left| \langle \phi_i^{(N)}, K \phi_i^{(N)} \rangle - \langle K \rangle_{\lambda_i} \right|^2 = 0.$$

Yesterday's proof yielded the following expression for $\langle K \rangle_\lambda$: thanks to Fourier analysis on the $(q+1)$-regular tree, we wrote

$$K = \mathrm{Op}(a)$$

and we obtained

$$\langle K \rangle_\lambda = \frac{1}{N} \sum_{x \in \mathcal{D}_N} \int_{\partial \mathfrak{X}} a(x, \omega, s) d\nu_x(\omega)$$

if $\lambda = q^{1/2+is} + q^{1/2-is} = 2\sqrt{q} \cos(s \ln q)$.

Another formula for $\langle K \rangle_\lambda$

$$\langle K \rangle_\lambda = \frac{1}{N} \sum_{x \in \mathcal{D}, y \in \mathfrak{X}} K(x,y) \Phi_{sph,\lambda}(d(x,y)).$$

$\Phi_{sph,\lambda}$ is the spherical function of parameter λ on the $(q+1)$-regular tree.

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$$\Phi_\lambda(d) = q^{-d/2} \left(\frac{2}{q+1} \cos(ds \ln q) + \frac{q-1}{q+1} \frac{\sin((d+1)s \ln q)}{\sin(s \ln q)} \right)$$

if $\lambda = 2\sqrt{q} \cos(s \ln q)$.

A third formula for $\langle K \rangle_\lambda$

Write $G_N = G = (V, E) = \Gamma \backslash \mathfrak{X}$. Introduce the Hilbert space

$$\mathcal{H}_\Gamma = \{(K(x,y))_{(x,y)\in\mathfrak{X}\times\mathfrak{X}}, K(\gamma\cdot x, \gamma\cdot y) = K(x,y)\,\forall\gamma\in\Gamma,$$
$$\frac{1}{|\mathcal{D}|}\sum_{x\in\mathcal{D}, y\in\mathfrak{X}} |K(x,y)|^2 < +\infty\}$$

Consider the closed subspace

$$\mathcal{F} = \overline{Vect\{\mathcal{A}^k, k \geq 0\}} = \{K \in \mathcal{H}_\Gamma, [\mathcal{A}, K] = 0\} \subset \mathcal{H}_\Gamma.$$

A third formula for $\langle K \rangle_\lambda$

For $K \in \mathcal{H}_\Gamma$, $P_{\mathcal{F}}(K) = f_K(\mathcal{A})$ where f_K is a function on $\mathbb{R}$

(actually a polynomial of degree D if
$[d(x,y) > D \Longrightarrow K(x,y) = 0]$)

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We find $\langle K \rangle_\lambda = f_K(\lambda)$.

4 line (sketch of) proof

$$Var(K) = \frac{1}{|V|} \sum_{i=1}^{N} \left| \langle \phi_i^{(N)}, K \phi_i^{(N)} \rangle \right|^2$$

- $Var([\mathcal{A}, K]) = 0$
-

$$Var(K) \leq \frac{1}{|V|} \|K\|^2_{HS(\ell^2(V))} \leq \|K\|^2_{\mathcal{H}_{\Gamma_N}} + C(D) \sup |K|^2 \frac{1}{|V|} \sharp\{x \in V, \rho(x) < D\}.$$

- By a density argument (using expansion), $Var(K) \underset{N \longrightarrow +\infty}{\longrightarrow} 0$ if $K \perp_{\mathcal{H}_{\Gamma_N}} \mathcal{F}$.
- For general K, apply the previous line to $K - P_{\mathcal{F}}(K) = K - f_K(\mathcal{A})$.

On a regular graph, consider a “weighted” adjacency matrix,

$$\mathcal{A}_p f(x) = \sum_{y \sim x} p(x,y) f(y)$$

with “homogeneous” probability weights.

Theorem

(2015) Assume that G_N has "few" short loops and that it forms an expander family.
Let $(\phi_i^{(N)})_{i=1}^N$ be an ONB of eigenfunctions of $\mathcal{A}_p$ on G_N.
Let $K = K_N : V_N \times V_N \longrightarrow \mathbb{C}$ be a matrix such that $d(x,y) > D \Longrightarrow K(x,y) = 0$. Assume $|K(x,y)| \leq 1$. Then

$$\lim_{N \longrightarrow +\infty} \frac{1}{N} \sum_{i=1}^{N} \left| \langle \phi_i^{(N)}, K\phi_i^{(N)} \rangle - \langle K \rangle_{\lambda_i} \right|^2 = 0.$$

$$\langle K \rangle_\lambda = \frac{1}{N} \sum_{x,y} K(x,y) \frac{\Im m\, G_{\lambda+i0}(x,y)}{\Im m\, G_{\lambda+i0}(x,x)}.$$

(the Green function of the infinite $(q+1)$-regular tree)

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(the Green function of the infinite $(q+1)$-regular tree)

Write the spectral decomposition of $\mathcal{A}_p$ on $\mathfrak{X}$,

$$dP_{(-\infty,\lambda]} = P_\lambda dm(\lambda)$$

$$\langle K\rangle_\lambda = \frac{1}{|V|}\mathrm{Tr}_{\ell^2(\mathfrak{X})} \mathbb{1}_{\mathcal{D}} K P_\lambda.$$

Exploration

- $\Delta + V(x)$ in progress with Mostafa Sabri

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- regular graph with (random) weights on the edges

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- $\Delta + V(x)$ in progress with Mostafa Sabri
- regular graph with (random) weights on the edges
- some non-regular graphs ? e.g. percolation graphs based on regular graphs

Wigner matrices

Hemitian matrices of size $N \times N$, random iid entries. Law is centered, has a density and gaussian tails.
Erdös, Schlein, Yau, Yin, Bourgade (2009...), Tao-Vu : for any eigenvector ϕ,

1. $\|\phi\|_\infty \leq N^{-1/2+\epsilon}\|\phi\|_2$ ("full delocalization", 2009)
2. $\exists \eta, \nu > 0$ s.t.

$$B \subset \{1, \dots, N\}, \sum_{x \in B} |\phi(x)|^2 \geq 1 - \eta \Longrightarrow |B| \geq \nu N.$$

3. QUE (2013) : for any fixed k, $a_N : \{1, \dots, N\} \longrightarrow [-1, 1]$,

$$\left| \sum_x a_N(x) |\phi_k^{(N)}(x)|^2 - \langle a_N \rangle \right| \leq \frac{\delta |a_N|}{N}$$

with overwhelming probability.

Heavy-tailed matrices

Symmetric matrices of size $N \times N$, random iid entries. Law is centered and has "heavy tails" :

$$\mathbb{P}(|X| > t) \sim t^{-\alpha}$$

with $0 < \alpha < 2$.

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Ben Arous–Guionnet have identified the limiting distribution of eigenvalues.

Bordenave–Guionnet have shown :

– If $1 < \alpha < 2$, for all but $o(N)$ of the eigenvectors, there is delocalization in the sense that $\|\phi\|_\infty \leq n^{-\rho}\|\phi\|_2$.

– If $0 < \alpha < 2/3$ and eigenvectors with eigenvalue λ large enough, there is localization : the mass of $(|\phi(x)|^2)_{x=1,\ldots,N}$ is carried by at most $n^{1-\delta}$ entries.

Random non-symmetric matrices

Vershynin-Rudelson 2014, Tao-Vu 2012. Matrices of size $N \times N$, random iid entries. Law is centered and has subexponential tail.

Then with probability $\geq 1 - n^{1-t}$, all eigenvectors have

$$\|\phi\|_\infty \leq Ct^{3/2} \frac{\ln n^{9/2}}{\sqrt{n}} \|\phi\|_2.$$

Random band matrices

Erdös-Knowles 2010 : symmetric matrices of size $N \times N$, band-width $W \gg N^{6/7}$.
Entries are iid, centered random variables, law has sub exponential tail.

Then for most eigenvectors, the "localization length" is $\gg N$.

Erdös-Rényi graphs

Random graph with N vertices. Edges are chosen independently with probability $p = p(N)$. The adjacency matrix is then a random $N \times N$ symmetric matrix (containing only 0 and 1).

Erdös, Knowles, Yau, Yin 2013 : if $pN \gg (\ln N)^C$, then with high probability there is full delocalization of all eigenvectors :

$$\|\phi\|_\infty \leq C \frac{(\ln N)^c}{\sqrt{N}} \|\phi\|_2.$$

+QUE statement

Random regular graphs with $d_N \longrightarrow +\infty$.

Dumitriu-Pal, Tran-Vu-Wang, Geisinger.

Bauerschmidt-Knowles-Yau 2015 : if $d_N \geq (\log N)^4$, then with proba $\geq 1 - e^{-\xi \log \xi}$ all eigenvectors have

$$\|\phi\|_\infty \leq C \frac{\xi}{\sqrt{N}} \|\phi\|_2.$$

+QUE statement