Quantum ergodicity

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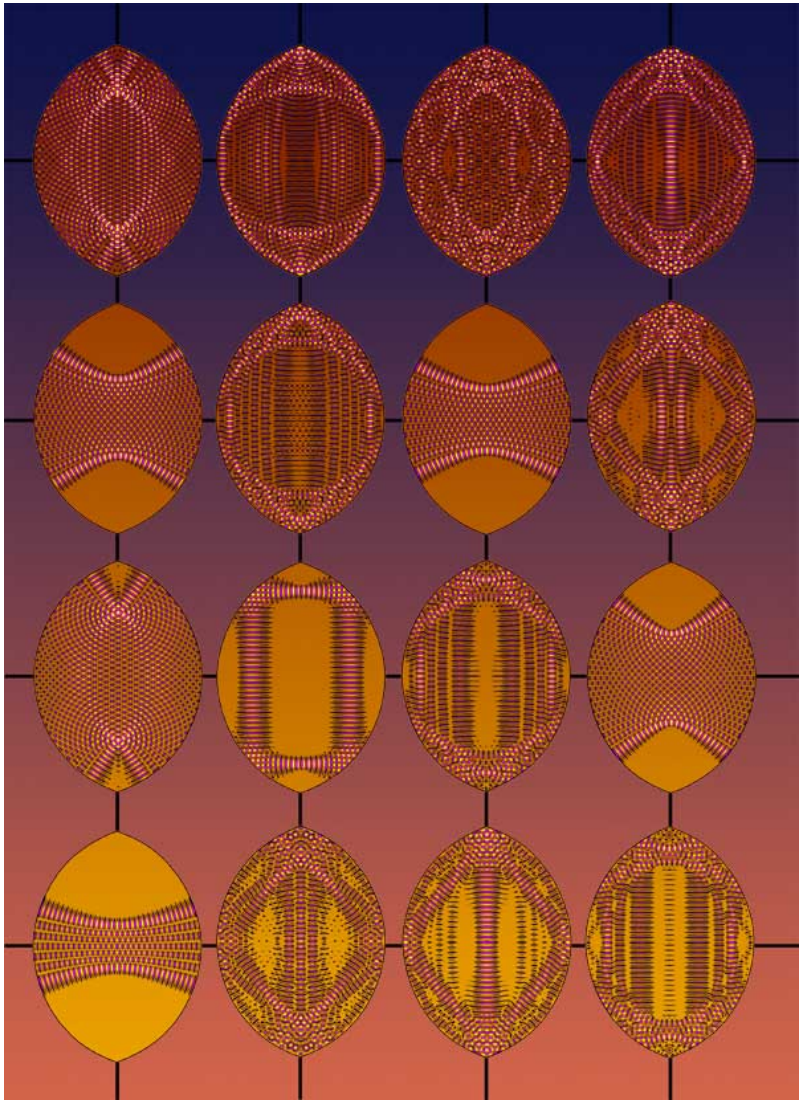
- I. Quantum ergodicity on manifolds.
- II. QE on (discrete) graphs : regular graphs.
- III. QE on graphs : other models, perspectives.



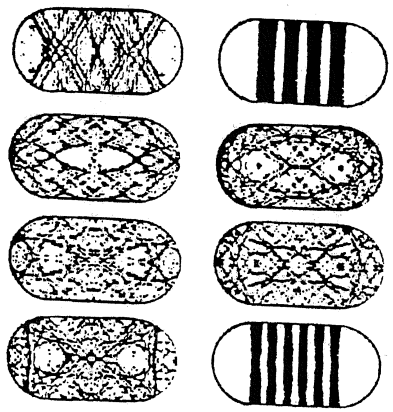


FIGURE: A few eigenfunctions of the Bunimovich billiard (Heller, 89).

A survey of QE on manifolds

M a compact riemannian manifold, of dimension d.

$$\Delta\psi_k = -\lambda_k\psi_k$$

$$\|\psi_k\|_{L^2(M)} = 1$$

in the limit $\lambda_k \longrightarrow +\infty$.

We study the weak limits of the probability measures on M,

$$|\psi_k(x)|^2 d\mathrm{Vol}(x)$$

$\lambda_k \longrightarrow +\infty$.

Most results could be generalized to more general Schrödinger equations

$$\left(-\hbar^2\frac{\Delta}{2}+V(x)\right)\psi_\hbar(x)=E_\hbar\psi_\hbar(x),$$

in the "semiclassical limit" $\hbar\longrightarrow 0$, $E_\hbar\longrightarrow E_0\in\mathbb{R}$.

This question is linked with the ergodic theory for the geodesic flow / billiard flow / Hamiltonian flow.

Hence the name quantum ergodicity.

Let $(\psi_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $L^2(M)$, with

$$-\Delta\psi_k = \lambda_k\psi_k, \qquad \lambda_k \leq \lambda_{k+1}.$$

QE theorem (simplified) :

Theorem (Shnirelman 74, Zelditch 85, Colin de Verdière 85)

Assume that the action of the geodesic flow is **ergodic** *for the Liouville measure. Let $a \in C^0(M)$. Then*

$$\frac{1}{N(\lambda)} \sum_{\lambda_k \leq \lambda} \left| \int_M a(x)|\psi_k(x)|^2 d\mathrm{Vol}(x) - \int_M a(x) d\mathrm{Vol}(x) \right|^2 \underset{\lambda\longrightarrow\infty}{\longrightarrow} 0.$$

Equivalently, there exists a subset $\mathcal{S} \subset \mathbb{N}$ of density 1, such that

$$\int_M a(x)|\psi_k(x)|^2 d\mathrm{Vol}(x) \xrightarrow[k\longrightarrow+\infty]{k\in\mathcal{S}} \int_M a(x) d\mathrm{Vol}(x).$$

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Equivalently,

$$|\psi_k(x)|^2 d\mathrm{Vol}(x) \xrightarrow[k\longrightarrow+\infty]{k\in\mathcal{S}} d\mathrm{Vol}(x)$$

in the weak topology.

The full statement uses analysis on phase space, i.e. $T^*M = \{(x, \xi), x \in M, \xi \in T^*_x M\}$.

For $a = a(x, \xi)$ a "reasonable" function on T^*M, we can define an operator on $L^2(M)$,

$$a(x, D_x) \qquad (D_x = \frac{1}{i}\partial_x)$$

Say $a \in \mathcal{S}^0(T^*M)$ if a is smooth and 0-homogeneous in ξ (i.e. a is a smooth fn on the sphere bundle SM).

A few rules of pseudodifferential calculus

$M = \mathbb{R}^d$, $a = a(x,\xi) : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{C}$. Say $a \in \mathcal{S}^k(T^*\mathbb{R}^d)$ if a is smooth and k-homogeneous in ξ.
Define

$$a(x, D_x)u = \frac{1}{(2\pi)^d} \int a(x,\xi)\, e^{i\xi\cdot x} \hat{u}(\xi) d\xi.$$

Proposition

(i) If $k = 0$, $a(x, D_x)$ extends to a bounded operator on $L^2(\mathbb{R}^d)$; more generally, $a(x, D_x)$ sends H^s to H^{s-k}.

(ii) $[a(x, D_x), b(x, D_x)] = \{a, b\}(x, D_x) + I.o.t.$

In particular for $b(x, D_x) = \sqrt{-\Delta}$,

$$\left[\sqrt{-\Delta}, a(x, D_x)\right] = \left(\frac{\xi}{|\xi|} \cdot \partial_x a\right)(x, D_x) + I.o.t.$$

$$-\Delta\psi_k = \lambda_k\psi_k, \qquad \lambda_k \leq \lambda_{k+1}.$$

For $a \in \mathcal{S}^0(T^*M)$, we consider

$$\langle \psi_k, a(x, D_x)\psi_k \rangle_{L^2(M)}.$$

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For $a \in \mathcal{S}^0(T^*M)$, we consider

$$\langle \psi_k, a(x, D_x)\psi_k \rangle_{L^2(M)}.$$

This amounts to $\int_M a(x)|\psi_k(x)|^2 d\mathrm{Vol}(x)$ if $a = a(x)$.

Let $(\psi_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $L^2(M)$, with

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QE theorem :

Theorem (Shnirelman, Zelditch, Colin de Verdière)

Assume that the action of the geodesic flow is **ergodic** *for the Liouville measure. Let $a(x,\xi) \in \mathcal{S}^0(T^*M)$. Then*

$$\frac{1}{N(\lambda)} \sum_{\lambda_k \leq \lambda} \left| \langle \psi_k, a(x, D_x)\psi_k \rangle_{L^2(M)} - \int_{|\xi|=1} a(x,\xi) dx d\xi \right|^2 \longrightarrow 0.$$

Recall : the geodesic flow $\phi^t : SM \longrightarrow SM$ $(t \in \mathbb{R})$ is the flow generated by the vector field $\frac{\xi}{|\xi|} \cdot \partial_x$.

Ergodicity of the geodesic flow means that the only ϕ^t-invariant L^2 functions on SM are the constant functions.

Equivalently : the only L^2-functions a on SM such that $\frac{\xi}{|\xi|} \cdot \partial_x a = 0$ are the constant functions.

Idea of proof.

For any bounded operator K on $L^2(M)$, define the quantum variance

$$Var_\lambda(K) = \frac{1}{N(\lambda)} \sum_{\lambda_k \leq \lambda} \left| \langle \psi_k, K\psi_k \rangle_{L^2(M)} \right|^2 .$$

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The proof start from the trivial observation that

$$Var_\lambda([f(\Delta), K]) = 0$$

for any K, for any real function f.

$$[f(\Delta), K] = f(\Delta)K - Kf(\Delta).$$

- The proof start from the trivial observation that

$$Var_\lambda([\sqrt{-\Delta}, K]) = 0$$

for any K.

In addition, if $K = a(x, D_x)$ is a pseudodifferential operator with $a \in \mathcal{S}^0(T^*M)$, then

$$[\sqrt{-\Delta}, a(x, D_x)] = b(x, D_x)$$

where

$$b(x, \xi) = \left(\frac{\xi}{|\xi|} \cdot \partial_x a \right)(x, \xi) + r(x, \xi)$$

where r is -1-homogeneous in ξ and $\frac{\xi}{|\xi|} \cdot \partial_x$ is the derivative along the geodesic flow.

- This implies that

$$Var_\lambda \left((\frac{\xi}{|\xi|} \cdot \partial_x a)(x, D_x) \right) \underset{\lambda \longrightarrow +\infty}{\longrightarrow} 0.$$

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$$Var_\lambda \left(a(x, D_x) \right) \underset{\lambda \longrightarrow +\infty}{\longrightarrow} 0$$

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- In addition,

$$Var_\lambda (a(x, D_x)) \leq C \int_{|\xi|=1} |a(x, \xi)|^2 dx d\xi.$$

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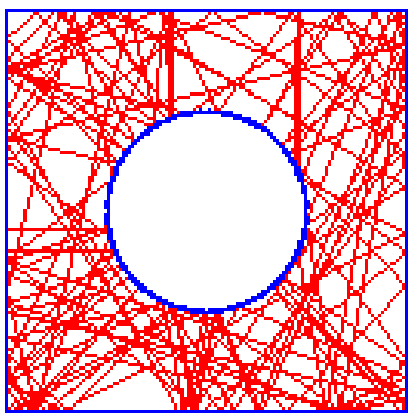
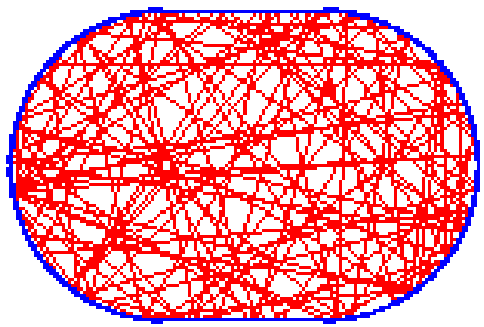
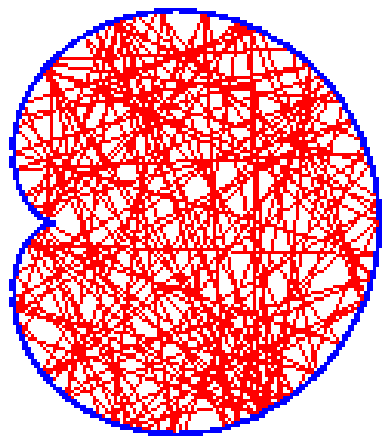
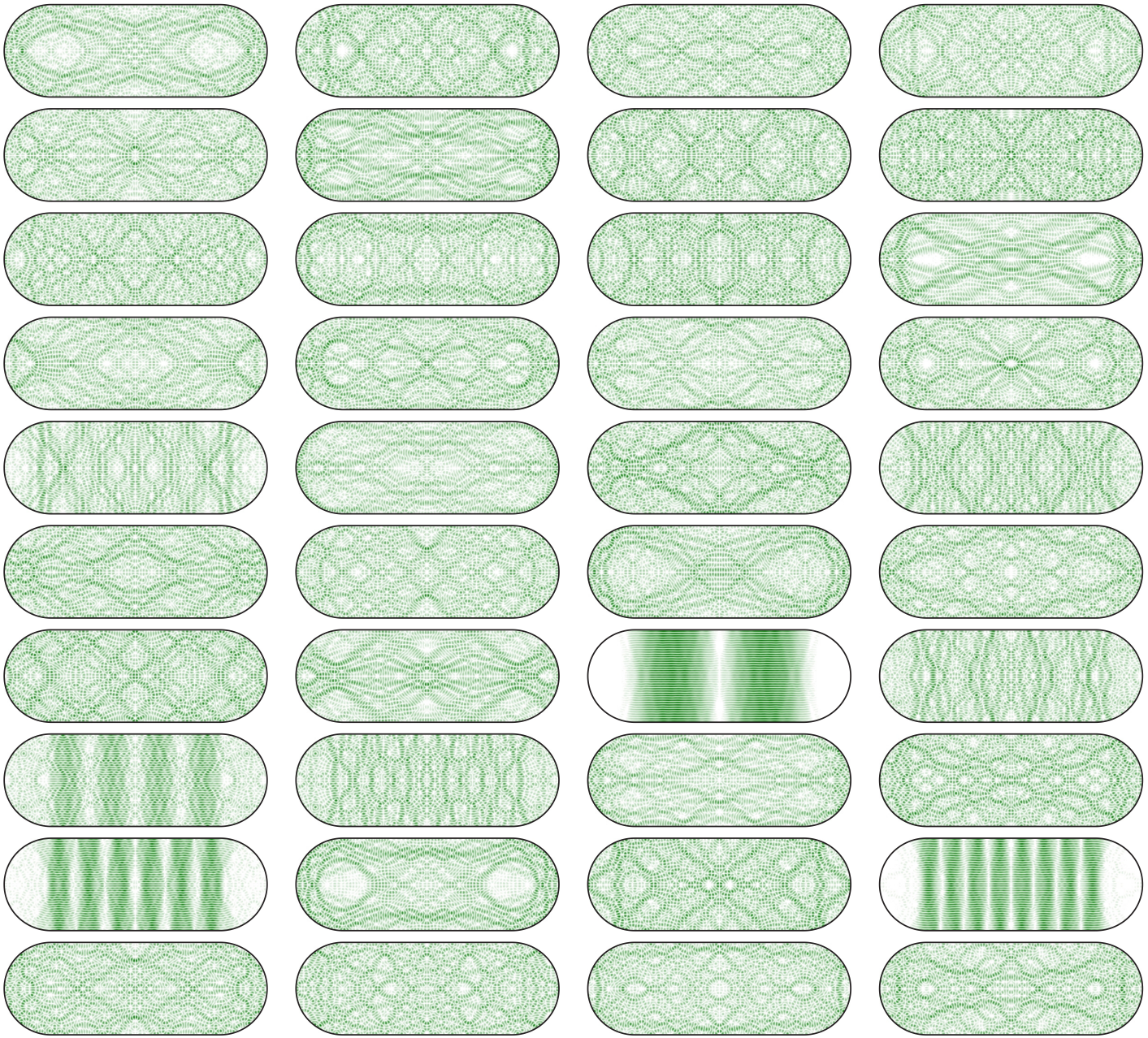






[Arnd Bäcker]



Hassell's result on stadium billiards (2008)

For almost every stadium, there exists a sequence of eigenfunctions that does not equidistribute over phase space.

The flat torus $\mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$

Consider the special example

$$\phi_k(x) = e^{ik.x},$$

$k \in \mathbb{Z}^d$,

$$\Delta\phi_k = -|k|^2\phi_k.$$

These are joint eigenfunctions of $\Delta, D_{x_1}, \ldots, D_{x_d}$.

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Consider the limit $|k| \longrightarrow +\infty$, $k/|k| \longrightarrow \xi_0$.
One shows easily that

$$\langle \phi_k, a(x, D_x)\phi_k \rangle_{L^2(\mathbb{T}^d)} \longrightarrow \int_{x\in\mathbb{T}^d} a(x, \xi_0)dx.$$

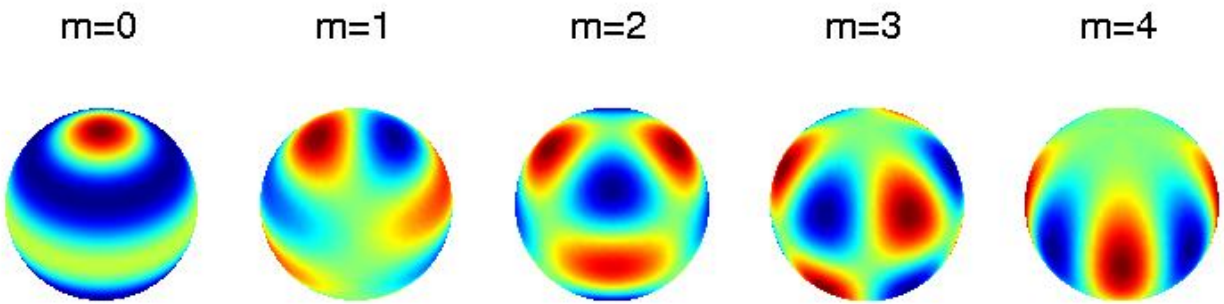
QE does not hold for the eigenfunctions (ϕ_k).

The sphere.

The eigenfunctions of $\Delta_{\mathbb{S}^2}$ are spherical harmonics, i.e. restrictions to $\mathbb{S}^2 \subset \mathbb{R}^3$ of harmonic homogeneous polynomials in 3 variables.

Harmonic homogeneous polynomials of degree ℓ give rise to eigenfunctions of $\Delta_{\mathbb{S}^2}$ for the eigenvalue $-\ell(\ell+1)$ (the dimension of the eigenspace is $2\ell+1$).

$\Delta_{\mathbb{S}^2}$ commutes with the infinitesimal rotations
$\mathbf{J}_{12} = \frac{1}{i}\left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}\right)$.



The basis $(Y_\ell^m)_{\ell \geq 0, |m| \leq \ell}$ of joint eigenfunctions of $\Delta_{\mathbb{S}^d}$ and J_{12} canNOT satisfy QE.

Random eigenbases (Zelditch)

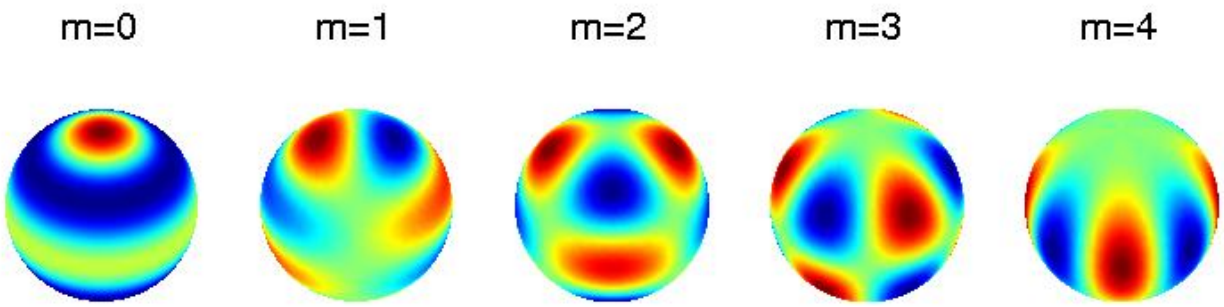
The set of bases of eigenfunctions of $\Delta_{\mathbb{S}^2}$ is in bijection with

$$U(1) \times U(3) \times U(5) \times U(7) \times \dots .$$

Theorem

(Zelditch 92, VanDerKam 97).
Almost every choice of eigenbasis satisfies QE (even QUE).

Efns concentrating on a closed geodesic



One can easily exhibit sequences of eigenfunctions concentrating on an equator.

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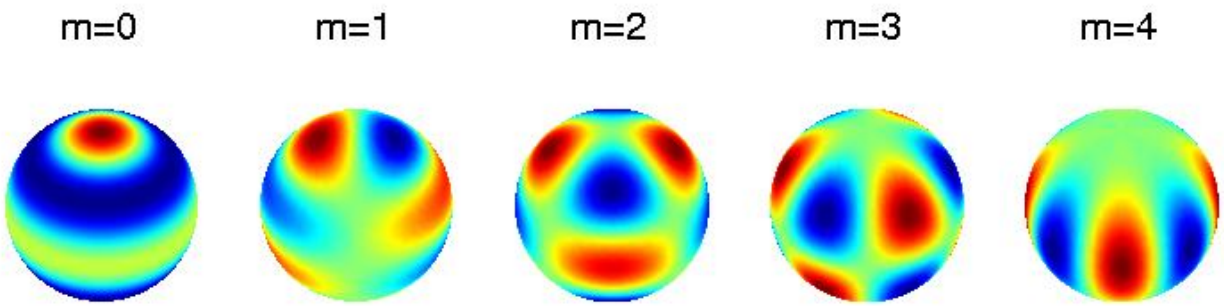

One can easily exhibit sequences of eigenfunctions concentrating on an equator.

From that, Jakobson and Zelditch deduced that any convex combination of equators can be achieved as a limit of eigenfunctions and any invariant measure for the geodesic flow.

The flat torus, $\mathbb{R}^d/2\pi\mathbb{Z}^d$ revisited

$$\phi_\lambda(x) = \sum_{k\in\mathbb{Z}^d, |k|^2=\lambda} c_k e^{ik.x}.$$

(Jakobson, Bourgain 97, Jaffard 90, A-Fermanian-Macià, A-Macià 2012)
It's not possible for a sequence of eigenfunctions to concentrate on a closed geodesic.

Assume

$$|\phi_\lambda(x)|^2 dx \underset{\lambda\longrightarrow+\infty}{\longrightarrow} \nu(dx)$$

Then ν is absolutely continuous, i.e. it has a density $\nu(dx) = \rho(x)dx$.
Moreover, for any non-empty open $\Omega \subset \mathbb{T}^d$, there exists $c(\Omega) > 0$ such that

$$\nu(\Omega) \geq c(\Omega).$$

The flat torus, $\mathbb{R}^d/2\pi\mathbb{Z}^d$ revisited

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Assume

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Remark

A full characterization of the set of limit measures is still missing.

The Quantum Unique Ergodicity conjecture

QUE conjecture :

Conjecture (Rudnick, Sarnak 94)

On a negatively curved manifold, we have convergence of the whole sequence : $\langle \psi_k, a(x, D_x)\psi_k \rangle_{L^2(M)} \longrightarrow \int_{(x,\xi)\in SM} a(x,\xi)dxd\xi.$

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Proven by E. Lindenstrauss in the special case of arithmetic congruence surfaces, for joint eigenfunctions of the Laplacian and the Hecke operators.

Entropy of eigenfunctions

Assume

$$\langle \psi_k, a(x, D_x)\psi_k \rangle_{L^2(M)} \longrightarrow \int_{(x,\xi)\in SM} a(x,\xi) d\mu(x,\xi).$$

If M has negative curvature, A-Nonnenmacher (06) proved that μ must have positive Kolmogorov-Sinai entropy.

$\mu \mapsto h_{KS}(\mu) \geq 0$ has the following properties :

- A measure supported by a periodic trajectory has zero entropy.
- According to the Ruelle-Pesin inequality,

$$h_{KS}(\mu) \leq \int \sum_{k=1}^{d-1} \lambda_k^+ d\mu,$$

  where the functions $\lambda_1^+(\rho) \geq \lambda_2^+(\rho) \geq \cdots \geq \lambda_{d-1}^+(\rho) > 0$ are the positive Lyapunov exponents of the flow. Equality iff μ is the Liouville measure (Ledrappier-Young).
- On a manifold of constant curvature -1, the inequality reads $h_{KS}(\mu) \leq d - 1$.
- The functional h_{KS} is affine on the convex set of invariant probability measures.

Assume for all a

$$\langle \psi_k, a(x, D_x)\psi_k \rangle_{L^2(M)} \longrightarrow \int_{(x,\xi)\in SM} a(x,\xi) d\mu(x,\xi).$$

A (06) proved that $h_{KS}(\mu) > 0$.

In constant curvature -1, A-Nonnenmacher (06) proved the explicit bound $h_{KS}(\mu) \geq \frac{d-1}{2}$.

As a corollary, the support of μ has Hausdorff dimension $\geq d$.

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As a corollary, the support of μ has Hausdorff dimension $\geq d$.
Conjectured bound in variable curvature :

$$h_{KS}(\mu) \geq \frac{1}{2} \int \sum_{k=1}^{d-1} \lambda_k^+ d\mu.$$

(proven by G. Rivière for surfaces of non-positive curvature).

Brin and Katok's definition of entropy.

For any time $T > 0$, introduce a distance on SM,

$$d_T(\rho, \rho') = \max_{t \in [-T/2, T/2]} d(\phi^t \rho, \phi^t \rho'),$$

Denote by $B_T(\rho, \epsilon)$ the ball of centre ρ and radius ϵ for the distance d_T.
Let μ be a ϕ^t–invariant probability measure on SM. Then, for μ-almost every ρ, the limit

$$\lim_{\epsilon \longrightarrow 0} \liminf_{T \longrightarrow +\infty} -\frac{1}{T} \log \mu\big(B_T(\rho, \epsilon)\big) \stackrel{\text{def}}{=} h_{KS}(\mu, \rho)$$

exists (local entropy).
The Kolmogorov-Sinai entropy is the average of the local entropies :

$$h_{KS}(\mu) = \int h_{KS}(\mu, \rho) d\mu(\rho).$$

Open questions for negatively curved manifolds :

Assume for all a, b,

$$\langle \psi_k, a(x, D_x)\psi_k \rangle_{L^2(M)} \longrightarrow \int_{(x,\xi)\in SM} a(x,\xi) d\mu(x,\xi).$$

$$\int_M b(x)|\psi_k(x)|^2 d\,\mathrm{Vol}(x) \longrightarrow \int_M b(x) d\nu(x).$$

$$\nu = \pi_* \mu$$

- Is μ ergodic ?
- can μ have ergodic components of zero entropy ? for instance, can μ have a component carried on a closed geodesic ?
- Is ν absolutely continuous ? does it put positive mass on any open set ?

Recent interest in "local equidistribution"

M. Berry's "semiclassical eigenfunction hypothesis".

Q : let (ψ_k) be an ONB of eigenfunctions of Δ. Let $r_k = r(\lambda_k) \longrightarrow 0$. Do we have

$$\sup_{x_0} \left| \frac{1}{\mathrm{Vol}(B(x_0, r_k))} \int_{B(x_0, r_k)} |\psi_k(x)|^2 d\,\mathrm{Vol}(x) - 1 \right| \underset{k \longrightarrow \infty}{\longrightarrow} 0?$$

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Hezari-Rivière, Han 2015 : negatively curved manyfolds, $r_k = \ln \lambda_k^{-\alpha}$, along a density 1 subsequence.

Lester-Rudnick 2015, $\mathbb{T}^d$, $d \geq 2$, $r_k = \lambda_k^{-\alpha}$ $(\alpha < \frac{1}{2(d-1)})$, along a density 1 subsequence.

Han 2015, $\mathbb{T}^d$, $d \geq 5$, $r_k = \lambda_k^{-\alpha}$ $(\alpha < \frac{d-2}{4d})$, a.s. for random eigenbases.