École
« L'abécédaire de SIDE (ASIDE) »
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School
"Abecedarian of SIDE (ASIDE)"
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SESSION—Continuous, discrete and ultradiscrete Painlevé equations

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We will give an introductory lecture on the theory of Painlevé equations and their discrete counterparts. The lecture is divided into three parts: (I) Painlevé equations, (II) discrete Painlevé equations, and (III) ultra-discrete Painlevé equations.

— *Overview by Masataka Kanki*

Part 1 : "Painlevé equations" by *Nobutaka Nakazono*

In the early 20th-century, for the sake of finding "new class of special functions" Painlevé and Gambier classified all differential equations in the form of $y'' = F(y', y, t)$, where $y = y(t)$, $' = d/dt$ and F is a rational function, by imposing the condition that the solutions should admit only poles as movable singular points, which is now referred to as the Painlevé property. As a result, they obtained new six equations, which are now called the Painlevé I through VI equations. In this sense, the solutions of the Painlevé equations are called Painlevé transcendents. In 2008, it was shown that all solutions of the Painlevé equations with the exception of special cases are indeed transcendental. On the other hand, we are also interested in the special cases. We call such solutions as special solutions. As the famous examples of special solutions, rational solution and hypergeometric solution are known. It is also well known that Painlevé equations have Bäcklund transformations, which generate an infinite number of special solutions starting from one special solution. In this part, we will deal with Painlevé IV equation and study its special solutions and Bäcklund transformations. For this part, we would recommend the following book as a reference: Masatoshi Noumi, Painlevé equations through symmetry, American Mathematical Society, Vol. 223, 2004.

Part 2 : "Discrete Painlevé equations" by *Yang Shi*

In the past decades, there has been growing interests in the study of discrete or difference equations due to their increasing relevance in mathematics, physics and many other areas of sciences. Discrete analogues of the six continuous Painlevé equations have been derived through different perspectives: as non-autonomous QRT mappings by singularity confinement technique; from the Bäcklund transformations of the continuous Painlevé equations; as reductions of partial difference equations; or as isomonodromy deformation problems. In 2001, a classification was obtained by Sakai by formulating the discrete Painlevé equations as the discrete dynamical systems described by the Cremona transformation over rational surfaces. The surfaces are constructed by blowing up the projective space P^2 at 9 points. The configuration of these 9 points is related to the affine Weyl group symmetry of the discrete Painlevé equations and we can draw a degeneration diagram of the blown-up surfaces, which includes the six continuous Painlevé equations as limiting cases. Many of the remarkable properties found in the continuous Painlevé equation remain true in the discrete setting. We will illustrate some of these properties through examples, with an emphasis on the special solutions and the symmetry of the discrete Painlevé equations.

Part 3 : "Ultradiscrete Painlevé equations" by *Masakata Kanki*

In 1996, Tokihiro, Takahashi, Matsukidaira and Satsuma invented the limiting procedure called the ultra-discretization. The procedure was used to correlate the discrete KdV equation with a cellular automaton of Takahashi-Satsuma. It is a powerful tool to discretize the dependent variables of discrete dynamical systems, by which various "integrable" cellular automata and their solitary wave solutions have been obtained. These cellular automata are called the ultra-discrete equations and are defined over the max-plus semi-field. One of the weak points of ultra-discretization is that it is not usually applicable to equations with minus signs. Later, the method of "ultra-discretization with parity variables" was constructed, which widens the application of ultra-discretization to equations with arbitrary signs. In this part, after introducing the technique of ultra-discretization through examples, we present several ultra-discrete analogues of the discrete Painlevé equations: i.e., second nonlinear difference equations over the max-plus semi-field. Many of the properties in previous parts remain true even in this simplified setting. Ultra-discrete analogues of the solutions described by the special functions such as the "ultra-discrete" Airy functions are presented. We will also discuss the relation of our topics to other closely related areas such as the tropical geometry and the integrability detection of ultra-discrete equations.