# Three point function at weak coupling and the spin vertex

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Y. Jiang, I. Kostov, A. Petrovskii, D.S., arXiv:1410.8660

related work:
Y. Kazama, S. Komatsu, T. Nishimura arXiv:1410:8533; 1506:03203
Y. Jiang, A. Petrovskii, arXiv:1412:2256

Hidden symmetries,
CRM Montreal
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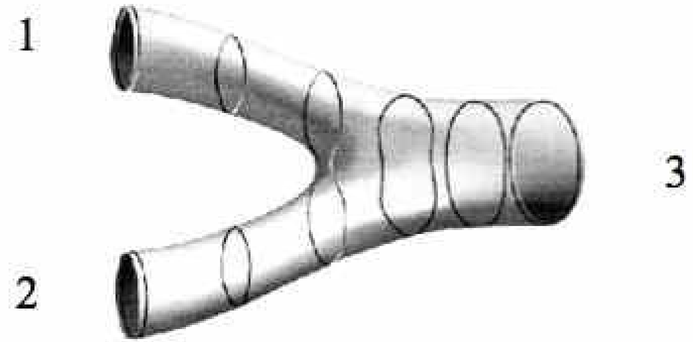
# The spin vertex approach

spin vertex = weak coupling equivalent of the string vertex




$$C_{123} \sim \langle V_{123}|\mathcal{O}_1\rangle|\mathcal{O}_2\rangle|\mathcal{O}_3\rangle$$

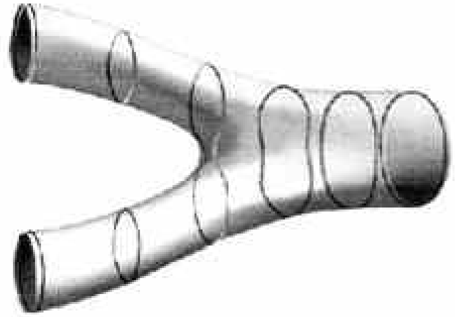
**[Spradlin, Volovich, 02-03; Dobashi, Yoneya, Shimada, 04;... Alday, David, Gava, Narain, 05;... Bajnok, Janik 15,....]**

motivations:

- compare computations in the BMN regime at weak and strong coupling
- implement symmetry/integrability constraints at higher loop to shortcut the gauge theory perturbative computations; make contact with algebraic Bethe Ansatz results
- weak coupling alternative to the bootstrap approach **[Basso, Komatsu, Vieira, 15]** and form factor approach
- how can one compute the three point function having as an input only the Baxter polynomials (in the QSC spirit)?

# The three point function in N=4 SYM

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(y)\mathcal{O}_3(z)\rangle = \frac{C_{123}}{|x-y|^{\Delta_1+\Delta_2-\Delta_3}|x-z|^{\Delta_1+\Delta_3-\Delta_2}|y-z|^{\Delta_2+\Delta_3-\Delta_1}}$$




**initial data:** three states with definite conformal dimensions and psu(2,2|4) charges

$$\mathcal{O}_\alpha(x)\,, \quad \alpha = 1,2,3$$

- each characterized by a set of rapidities $\{u_{i,\alpha} = \frac{1}{2}\cot\frac{p_{i,\alpha}}{2}\sqrt{1+16g^2\sin^2\frac{p_{i,\alpha}}{2}}\}$

$$\Delta_\alpha = L_\alpha - M_\alpha + \sum_{i=1}^{M_\alpha}\sqrt{1+16g^2\sin^2\frac{p_{i,\alpha}}{2}}$$

efficiently encoded in the zeros of the Baxter functions $\{Q_a^\alpha(u;u_{i,\alpha})\,, \quad a=1,\dots 8\}$

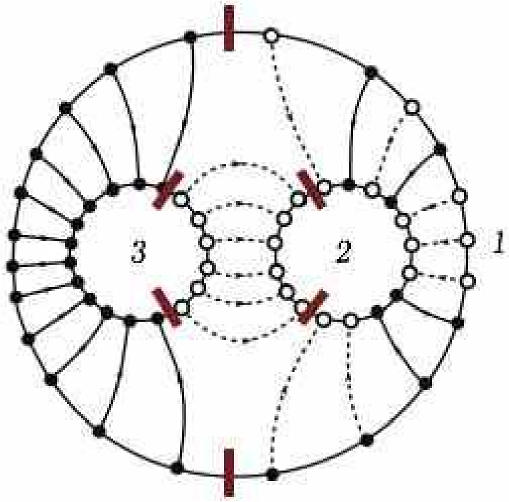
- and polarizations (or global rotations with respect to some reference BPS state, e.g Tr $Z^L$)

$$g_\alpha = e^{\zeta_\alpha^A J^A}$$

# The three point function at weak coupling

At tree level the three point function can be computed using gaussian contraction

$\longrightarrow$ pure combinatorics




$$\langle \bar{Z}(x) Z(y) \rangle \sim \frac{1}{|x-y|^2}$$

$$\langle \bar{X}(x) X(y) \rangle \sim \frac{1}{|x-y|^2}$$

**Spin chain language**: the combinatorics can be expressed in terms of scalar products of states of (pieces of) spin chains **[Roiban, Volovich, 04]**

- use Algebraic Bethe Ansatz (ABA) to build and cut the chains into pieces

$\longrightarrow$ "tailoring" of spin chains **[Escobedo, Gromov, Sever, Vieira, 10]**

# The three point function at weak coupling

Cutting the chains into pieces generates sums over partitions of magnons similar (but not always identical to) the sums over hexagons **[Basso, Komatsu, Vieira, 15]**

Resumming the contribution of magnons and taking the limit of large number of magnons are among the open problems.

In some special cases these sums can explicity taken, and obtain determinant representations **[Foda, 11]** whose semiclassical limit is rather straightforward **[Escobedo, Sever, Vieira, 11; Kostov, 12; Kostov, Bettelheim, 14]**

*e.g.* in some of the su(2) sector at tree level and one loop **[Jiang, Kostov, Loebbert, DS, 14]** the semiclassical limit is

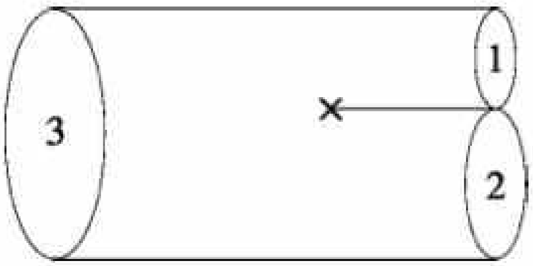
$$\log C_{123}(g) \simeq \oint\limits_{\mathcal{C}^{(12|3)}} \frac{du}{2\pi}\, \mathrm{Li}_2\left(e^{ip^{(1)}(u)+ip^{(2)}(u)-ip^{(3)}(u)}\right) + \oint\limits_{\mathcal{C}^{(13|2)}} \frac{du}{2\pi}\, \mathrm{Li}_2\left(e^{ip^{(3)}(u)+ip^{(1)}(u)-ip^{(2)}(u)}\right) - \frac{1}{2}\sum_{a=1}^{3} \int\limits_{\mathcal{C}^{(a)}} \frac{dz}{2\pi}\, \mathrm{Li}_2\left(e^{2ip^{(a)}(z)}\right)$$

in agreement with **[Kazama, Komatsu, 13 & unpublished]**

# The string vertex (strings in the pp-wave limit)

Simple configuration at strong coupling: near-extremal configuration with one string of length $J_3$ spliting into two strings of length $J_1$ and $J_2$ with $J_3 = J_1 + J_2$

with transverse excitations with polarizations $j = 1, \ldots, 8$




**[Spradlin, Volovich, 02-03; Dobashi, Yoneya, Shimada, 04;... ]**

**BMN excitations:** (dilute gas of magnons with momentum ~1/L)

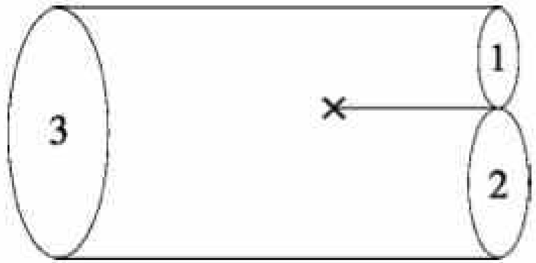
modes (massive bosons/fermions): $$E = J + \sum_k \sqrt{1 + \frac{\pi \lambda n_k^2}{L^2}}$$

the S matrix in the BMN limit is trivial: $S(p_1, p_2) = \pm 1$

$a_n^{(s)j\dagger}$ creates excitation with momentum $p = \frac{2\pi n}{J_s}$ in the $s$-th string

for configurations beyond near-extremality **[Bajnok, Janik, 15]** *cf.* Zoli's talk

# The string vertex (strings in the pp-wave limit)




$$C_{123} = f(\Delta_1, \Delta_2\Delta_3)\langle 1|\langle 2|\langle 3|V_3\rangle$$

string vertex state:

$$|V_3\rangle = \mathcal{P}|E_a\rangle,$$

where

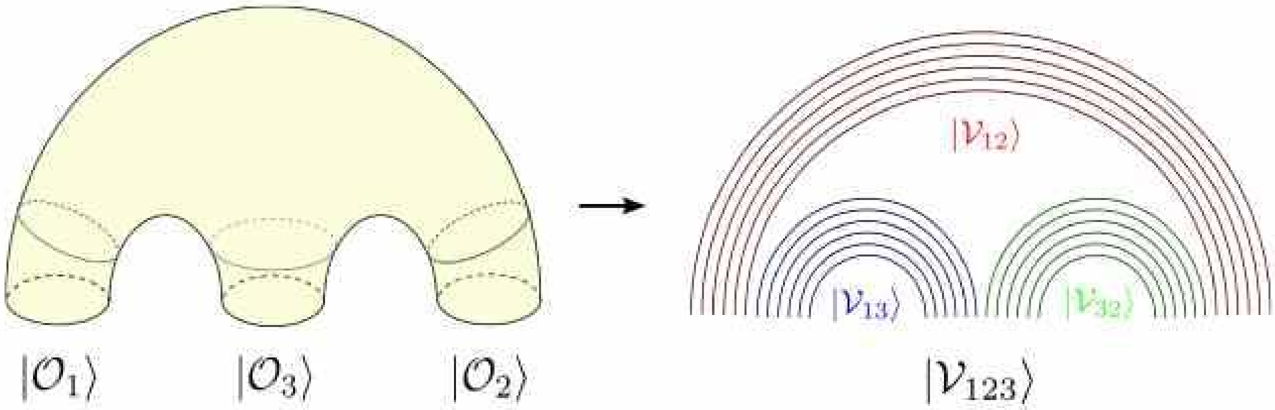
$$|E_a\rangle = \exp\left( -\frac{1}{2}\sum_{i=1}^{8}\sum_{r,s=1}^{3}\sum_{m,n=-\infty}^{\infty} a_m^{(r)i\dagger}\widetilde{N}_{mn}^{rs}a_n^{(s)i\dagger}\right)|0\rangle,$$

and $\mathcal{P}$ is a polynomial in the creation/annihilation operators.

comparison with the computation at weak coupling in the BMN limit: agreement at the leading order; disagreement at one loop [Schulgin, Zayakin, 13]

# The spin vertex

A structure similar to the string vertex can be built at weak coupling, too
**[ e.g. Alday, David, Gava, Narain, 05]**




At tree level the spin vertex mimics the planar Wick contractions

$\longrightarrow$ combining incoming states into singlets


all the three states are treated equally

$$\sum_a |a\rangle \otimes |\bar{a}\rangle$$

# Constructing the singlet states


$$= \quad \sum_a |a\rangle \otimes |\bar{a}\rangle$$

$a$ is a state in a particular lowest weight module $V_+$ of psu(2,2|4)

$$a = Z, X, Y, \bar{Z}, \bar{X}, \bar{Y} + \text{fermions, derivatives, etc}$$

since psu(2,2|4) is non-compact $\bar{a}$ should be in the highest weight module $\mathbf{V}_-$ of dual to $\mathbf{V}_+$

- Build $\mathbf{V}_+$ and $\mathbf{V}_-$ via the oscillator representation (spin chain language) **[Bars, Gunaydin, 83,...]**

# The oscillator representation

emphasizing the maximally compact subalgebra su(2) x su(2) x u(1) x su(4)

$$[a_i, a_j^\dagger] = \delta_{ij} \ , \quad [b_i, b_j^\dagger] = \delta_{ij} \ , \quad \{c_k, c_l^\dagger\} = \delta_{kl} \ , \quad i,j = 1,2 \ , \quad k,l = 1,\dots,4 \ .$$

optional particle-hole transformation $\qquad d_i = c_{i+2}^\dagger \ , \qquad d_i^\dagger = c_{i+2} \quad i = 1,2$

u(2,2|4) generators (spin chain) $\qquad E^{AB} = \bar{\psi}^A \psi^B$

$$\psi = (\ a_i \quad -b_i^\dagger \quad c_i \quad d_i^\dagger\ ), \qquad \bar{\psi} = (\ a_i^\dagger \quad b_i \quad c_i^\dagger \quad d_i\ )$$

psu(2,2|4): vanishing central central charge condition:

$$\sum_A E^{AA} = \sum_{i=1,2} (N_{a_i} - N_{b_i} + N_{c_i} - N_{d_i}) = 0$$

# The spin vertex

there exists a non-unitary (Wick-like) rotation $U$ which transforms a direction of positive signature (5) into one of negative (0) signature and viceversa

$$\eta^{\mathrm{D}}_{PQ} = \mathrm{diag}(- + + + | + -)$$

$$\updownarrow \qquad \updownarrow \qquad U = \exp -\frac{\pi}{2} M_{05} = \exp -\frac{\pi}{4}(P_0 - K_0)$$

$$\eta^{\mathrm{E}}_{PQ} = \mathrm{diag}(+ + + + | - -)$$

**[Alday, David, Gava, Narain, 05; Govil, Günaydin, 13]**

at tree level:

$$U = \exp -\frac{\pi}{4} \sum_{i=1,2} (a_i^\dagger b_i^\dagger + a_i b_i)$$

transformed oscillators
(Bogoliubov-like transformation):

$$\lambda_\alpha \equiv U\, a_\alpha\, U^{-1} = a_\alpha - b^\dagger_\alpha, \quad \widetilde{\lambda}_{\dot\alpha} \equiv U\, b_\alpha\, U^{-1} = b_\alpha - a^\dagger_\alpha,$$
$$\mu_\alpha \equiv U\, a^\dagger_\alpha\, U^{-1} = a^\dagger_\alpha + b_\alpha, \quad \widetilde{\mu}_{\dot\alpha} \equiv U\, b^\dagger_\alpha\, U^{-1} = b^\dagger_\alpha + a_\alpha$$

- "D-scheme" **[Kazama, Komatsu, Nishimura, 15]:** the action of the conformal group is manifest

# The $U^2$ transformation

the operator $U^2$ realizes a PT transformation (changes the sign of $x_0$ and $x_5$)

transforms positive energy state into negative energy states

$$U^{-2} D U^2 = -D$$

all loop property

- positive energy (lowest weight) module $\mathbf{V}_+$ : built on the oscillator vacuum

$$|0\rangle = |0\rangle_B \otimes |0\rangle_F \qquad (a_i, b_i, c_i, d_i)|0\rangle = 0$$

- negative energy (highest weight) module $\mathbf{V}_-$ : built on the dual vacuum

$$|\bar{0}\rangle = |\bar{0}\rangle_B \otimes |\bar{0}\rangle_F \qquad (a_i^\dagger, b_i^\dagger, c_i^\dagger, d_i^\dagger)|\bar{0}\rangle = 0$$

bosonic particle-hole transformation implemented by $U^2$

$$|\bar{0}\rangle_B = U^2 |0\rangle_B$$

necessary to construct the psu(2,2|4) singlets

# The $U^2$ transformation

There exists a fermionic analog of the $U$ transformation,

$$U_F = \exp -\frac{\pi}{4} \sum_{i=1,2} (c_i^\dagger d_i^\dagger + c_i d_i) \qquad |\bar{0}\rangle_F = U_F^2 |0\rangle_F$$

$U_F$ is unitary, so it maps the modules to themselves $\qquad V_\pm \overset{U_F^2}{\longleftrightarrow} V_\pm$

$U_F{}^2$ implements particle-hole exchange on fermionic oscillators; su(2) x su(2) rotation

# Two-point function and the "vertex"

Operator-state correspondence (E-scheme):

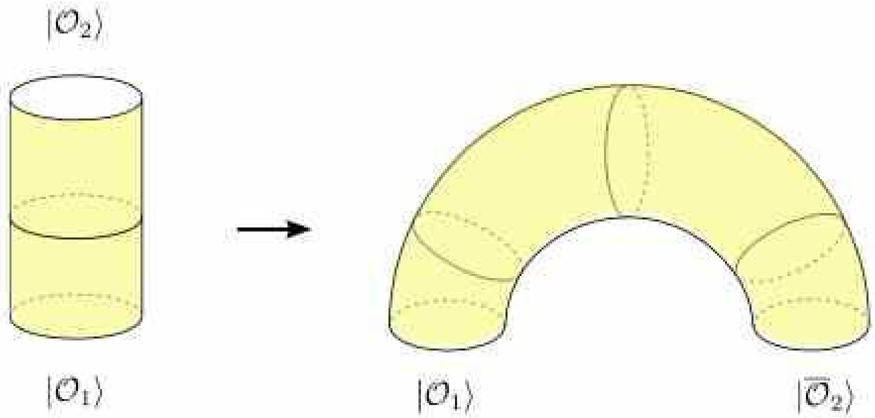
$$\mathcal{O}(x) = e^{iPx}\mathcal{O}(0)e^{-iPx} \qquad \longrightarrow \qquad e^{iPx}U|\mathcal{O}\rangle$$

$$\langle \mathcal{O}_2^\dagger(y)\mathcal{O}_1(x)\rangle = \langle \mathcal{O}_2|U^\dagger e^{iP(x-y)}U|\mathcal{O}_1\rangle$$

Flip the outgoing state into an incoming state

and pair the two states into the singlet $\langle \mathcal{V}_{12}|$ :

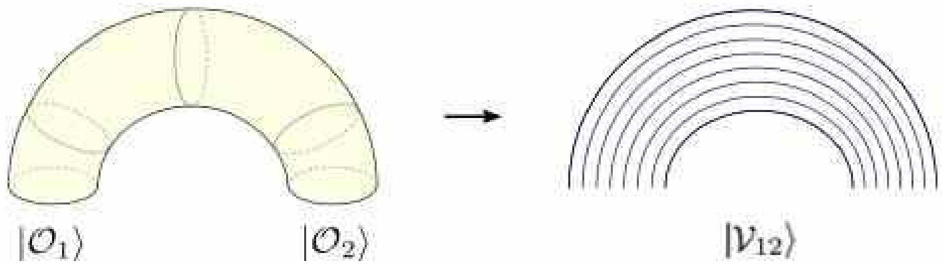
à la **[EGSV, 10]**




$$\langle \mathcal{O}_2^\dagger(y)\mathcal{O}_1(x)\rangle = \langle \mathcal{V}_{12}|\, e^{i[L^+_{(1)}x + L^+_{(2)}y]}\, |\bar{\mathcal{O}}_2\rangle^{(2)} \otimes |\mathcal{O}_1\rangle^{(1)} \qquad U^{-1}P_\mu U = L^+_\mu$$

# Two-point function and the "vertex"

Tree level: Wick contractions:




$$|\mathcal{V}_{12}\rangle = \exp - \sum_{s=1}^{L} \sum_{i=1,2} \left( a_{i,s}^{(1)} a_{i,s}^{(2)\dagger} - b_{i,s}^{(1)} b_{i,s}^{(2)\dagger} + d_{i,s}^{(1)} d_{i,s}^{(2)\dagger} - c_{i,s}^{(1)} c_{i,s}^{(2)\dagger} \right) |0\rangle^{(2)} \otimes |\bar{0}\rangle^{(1)}$$

$$|0\rangle^{(2)} \otimes |0\rangle^{(1)} = \left( |0\rangle_L^{(2)} \otimes \cdots \otimes |0\rangle_1^{(2)} \right) \otimes \left( |0\rangle_1^{(1)} \otimes \ldots \otimes |0\rangle_L^{(1)} \right)$$

[Alday, David, Gava, Narain, 05]

delta function-like expression

$$|\mathcal{V}_{12}\rangle = \sum_{N_a, N_b, N_c, N_d} |N_a, N_b, N_c, N_d\rangle^{(2)} \otimes |\bar{N}_a, \bar{N}_b, \bar{N}_c, \bar{N}_d\rangle^{(1)}$$

$$|N_a, N_b, N_c, N_d\rangle = \frac{1}{\sqrt{N_a!\, N_b!}} \prod_{k=1,2} (d_k^\dagger)^{N_{d_k}} (c_k^\dagger)^{N_{c_k}} (b_k^\dagger)^{N_{b_k}} (a_k^\dagger)^{N_{a_k}} |0\rangle \ ,$$

$$|\bar{N}_a, \bar{N}_b, \bar{N}_c, \bar{N}_d\rangle = \frac{(-1)^{N_a + N_c}}{\sqrt{N_a!\, N_b!}} \prod_{k=1,2} a_k^{N_{a_k}} b_k^{N_{b_k}} c_k^{N_{c_k}} d_k^{N_{d_k}} |\bar{0}\rangle \ ,$$

# Two-point function and the "vertex"

$$|\mathcal{V}_{12}\rangle = \exp - \sum_{s=1}^{L} \sum_{i=1,2} \left( a_{i,s}^{(1)} a_{i,s}^{(2)\dagger} - b_{i,s}^{(1)} b_{i,s}^{(2)\dagger} + d_{i,s}^{(1)} d_{i,s}^{(2)\dagger} - c_{i,s}^{(1)} c_{i,s}^{(2)\dagger} \right) |0\rangle^{(2)} \otimes |\bar{0}\rangle^{(1)}$$

$$|\mathcal{V}_{12}\rangle = \sum_{N_a, N_b, N_c, N_d} |N_a, N_b, N_c, N_d\rangle^{(2)} \otimes |\bar{N}_a, \bar{N}_b, \bar{N}_c, \bar{N}_d\rangle^{(1)}$$

the exponential form includes states with arbitrary (integer) central charge $C_s$ at each site $s$

the simple form is introduces at the expenses of enlarging the Hilbert space

one can easily project on the $C_s = 0$ modules; these are automatically selected when projected on the incoming states respecting this condition

**main property: local psu(2,2|4) symmetry** $\quad E^{AB} = \bar{\psi}^A \psi^B$

$$\left( E_s^{AB(1)} + E_s^{AB(2)} + (-1)^{|B|} \delta^{AB} \right) |\mathcal{V}_{12}\rangle = 0 \,, \qquad s = 1, \ldots L \,.$$

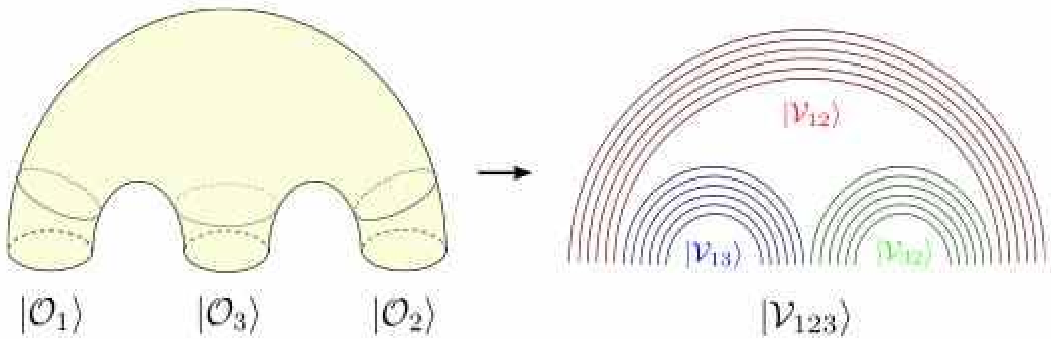
proven using the action of the oscillators on the vertex

# Three-point function and the vertex

straightforward generalization to the three point function at tree level
(one singlet for every "bridge" ij)

$$|\mathcal{V}_{123}\rangle = |\mathcal{V}_{12}\rangle \otimes |\mathcal{V}_{13}\rangle \otimes |\mathcal{V}_{32}\rangle$$

$$\begin{aligned} |\mathcal{O}_1\rangle &\simeq |\mathcal{O}_{13}\rangle \otimes |\mathcal{O}_{12}\rangle \ , \\ |\mathcal{O}_2\rangle &\simeq |\mathcal{O}_{21}\rangle \otimes |\mathcal{O}_{23}\rangle \ , \\ |\mathcal{O}_3\rangle &\simeq |\mathcal{O}_{32}\rangle \otimes |\mathcal{O}_{31}\rangle \ . \end{aligned}$$




$$\langle \mathcal{O}_2(y)\mathcal{O}_3(z)\mathcal{O}_1(x)\rangle = \langle \mathcal{V}_{123}|\, e^{i[L^+_{(1)}x + L^+_{(2)}y + L^+_{(3)}z]} |\mathcal{O}_2\rangle \otimes |\mathcal{O}_3\rangle \otimes |\mathcal{O}_1\rangle$$

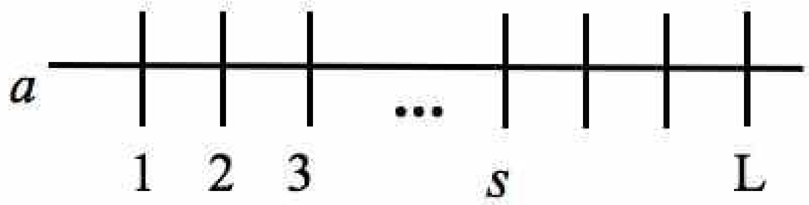
# Local symmetry vs. Yangian symmetry

promote the local symmetry to Yangian symmetry

$$\left(E_s^{AB(1)} + E_s^{AB(2)} + (-1)^{|B|}\delta^{AB}\right)|\mathcal{V}_{12}\rangle = 0\,, \qquad s = 1, \ldots L\,. \qquad \textbf{(*)}$$

start by defining the monodromy matrix (which generates the Yangian)

$$T(u) = L_1(u)\ldots L_L(u)$$




Lax matrix at site $s$: $$L_s(u) = u - i/2 - i(-1)^{|A|}E_0^{AB}E_s^{BA}$$

auxiliary space in defining, (4|4) representation $$(E_0^{AB})_{CD} = \delta_C^A\delta_D^B$$

quantum space in the oscillator representation $$E_s^{AB} = \bar{\psi}_s^A\psi_s^B$$

# Local symmetry vs. Yangian symmetry

transfer Lax matrices from one chain to another using the local symmetry

$$\left(E_s^{AB(1)} + E_s^{AB(2)} + (-1)^{|B|}\delta^{AB}\right)|\mathcal{V}_{12}\rangle = 0\,, \qquad s = 1, \ldots L\,.$$

$$L_s^{(1)}(u)|\mathcal{V}_{12}\rangle = -L_s^{(2)}(-u)|\mathcal{V}_{12}\rangle$$

$$L_1^{(1)}(u)\ldots L_L^{(1)}(u)|\mathcal{V}_{12}\rangle = L_1^{(2)}(-u)\ldots L_L^{(2)}(-u)|\mathcal{V}_{12}\rangle$$

compare with the monodromy matrices on the two chains:

$$T^{(1)}(u) = L_1^{(1)}(u)\ldots L_L^{(1)}(u)\,, \qquad T^{(2)}(u) = L_L^{(2)}(u)\ldots L_1^{(2)}(u)$$

wrong rapidity sign
wrong order of sites on chain (2)

# Local symmetry vs. Yangian symmetry

the wrong order can be cured by taking the transpose in the auxiliary space $t_0$

and using the invariance of Lax matrix by transposition in both spaces $\qquad L(u) = L^{t_0,t}(u)$

in some sectors like su(2) the traceless generators obey $\qquad E^{ab} = -\sigma E^{ab,t} \sigma^{-1}$

with $\sigma = i\sigma_2$ the charge conjugation matrix in the quantum space

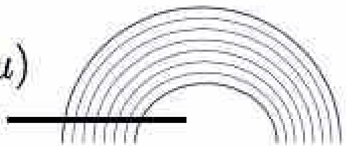
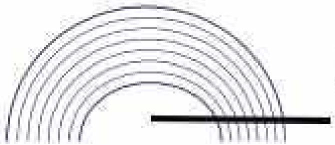
this helps proving

$$T^{(1)}(u)|\mathcal{V}_{12}\rangle = \sigma_0 T^{(2),t_0}(u)\sigma_0^{-1}|\mathcal{V}_{12}\rangle$$

or

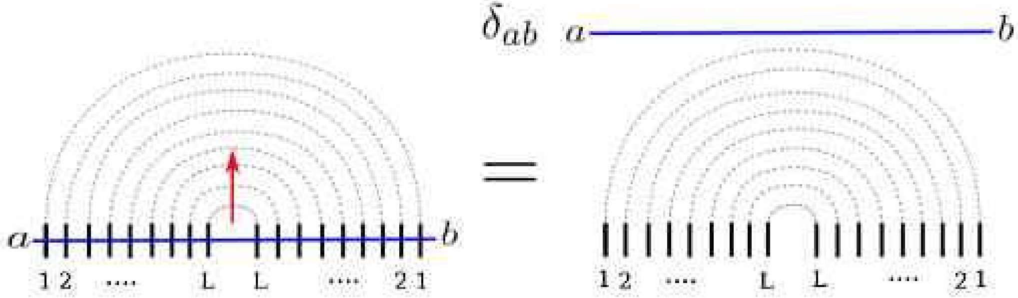
$$\begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}^{(1)} |\mathcal{V}_{12}\rangle = \begin{pmatrix} D(u) & -B(u) \\ -C(u) & A(u) \end{pmatrix}^{(2)} |\mathcal{V}_{12}\rangle$$

useful relation which helps transfer magnons from one (piece of) chain to another **[see *e.g.* Ivan's talk]**

$B^{(1)}(u)$  $\longrightarrow$  $B^{(2)}(u)$

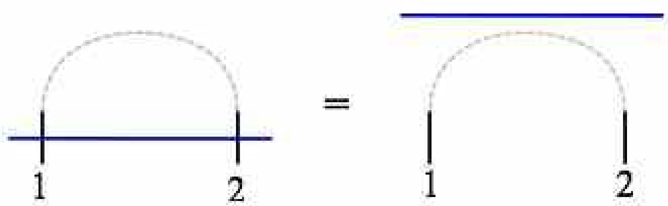
# Monodromy condition for the vertex

the vertex (singlet) is also an Yangian invariant for the monodromy matrix $T_{12}(u)$




$$T_{12}(u)|\mathcal{V}_{12}\rangle = f(u)|\mathcal{V}_{12}\rangle$$

it is sufficient to prove the above relation for two chains of one site



# Monodromy condition for the vertex

proof of the monodromy relation for two sites 1 and 2
(with the auxiliary space 0 in the defining representation)

R matrix $R_{01}(u) = u - i\Pi_{01} \equiv L_1(u + i/2)$

with the graded "permutation" $\Pi_{01} = (-1)^{|A|} E_0^{AB} E_1^{BA}$

$$\Pi_{01}^2 = c + (c-1)\Pi_{01}$$

$c = E_1^{BB}$

$c = 0$ for psu(2,2|4)

$c = 1$ for su(2)

1) unitarity condition $R_{01}(u) R_{01}(i(c-1) - u) \sim 1$

2) symmetry of the vertex $R_{02}(u)|\mathcal{V}_{12}\rangle = -R_{01}(-i-u)|\mathcal{V}_{12}\rangle$

$$R_{01}(u) R_{02}(u - ic)|\mathcal{V}_{12}\rangle \sim |\mathcal{V}_{12}\rangle$$

# Monodromy condition for the vertex

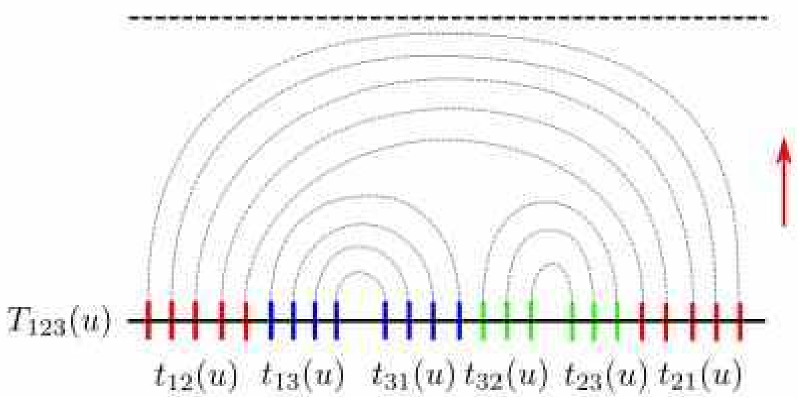
$$R_{01}(u)R_{02}(u-ic)|\mathcal{V}_{12}\rangle \sim |\mathcal{V}_{12}\rangle$$

the monodromy condition depends on the sector through the value of $c$

in the full psu(2,2|4) there is no relative shift in the rapidity $c = 0$

the significance of this fact not fully understood (relation to crossing?)

monodromy relation for three chains:




strong coupling semiclassical equivalent:

$$\Omega_1(u)\Omega_2(u)\Omega_3(u) = 1$$

**[Kazama, Komatsu, 13]**

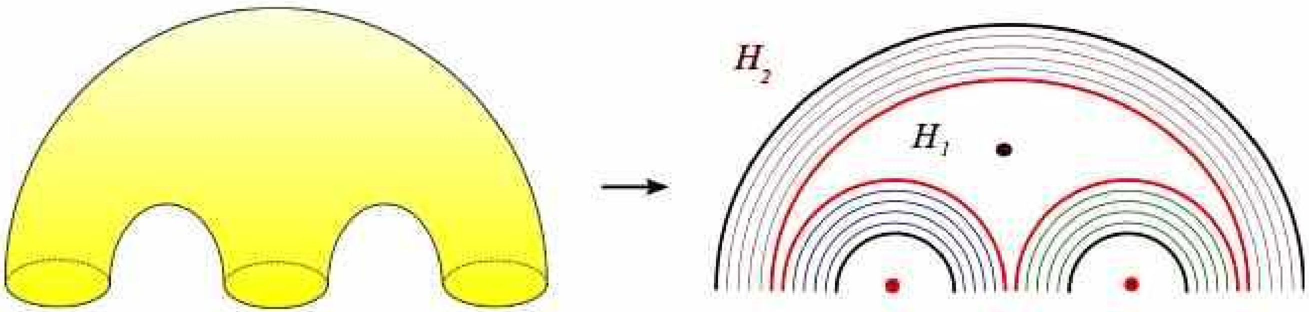
monodromy in physical space **[Kazama, Komatsu, Nishimura, 15]** $\longrightarrow$ conserved charges

# Higher loops and the hexagon bootstap

going to higher loop in the spin chain picture  requires:

controlling the length-changing processes (avoided in the scattering picture)

considering the insertions at the splitting points (become twist insertions in **[Basso, Komatsu, Vieira, 15]**)



# Conclusion

- we have built a weak coupling version of the string vertex
- based on the oscillator representation of psu(2,2|4)
- adapted for the spin chain language and perturbative computations
- implementation of the symmetries at tree level/monodromy condition

- in the BMN regime coincides with the string vertex result at leading order **[Jiang, Petrovskii, 14]**
- instrumental in computing efficiently correlation functions in su(2) sectors with the ABA tools

- check the hexagon axioms in the weak coupling?