The three point function through separation of variables

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Why SoV?

— Analytical results for 3pf available only in exceptional cases
when it is evaluated by a determinant.
We need more universal formalism.

— Alternative to the hexagon bootstrap at weak coupling [last Pedro's lecture].

— A possible approach to explore the quasiclassical limit as a starting point to study the hypothetical Quantum Spectral Curve for the 3pf.

I. 3-point function of su(2) fields in the SO(4) sector

The "classical" EGSV configuration [Escobedo-Gromov-Sever-Vieira 2010]

$$C_{123} \sim \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle \qquad \mathcal{O}_1 \in \{X, Z\}\,, \quad \mathcal{O}_2 \in \{\bar{X}, Z\}\,, \quad \mathcal{O}_1 \in \{X, \bar{Z}\}\,,$$

is expressed in terms of the **DWBPF** (6-vertex partition functions on **rectangles**) [Foda 2012] and is essentially an Izergin-Korepin determinant [Korepin 1982].

This is however an exceptional case.

Example of a 3pf of su(2) fields which is **not** a determinant:
[see Shota's talk this morning and at IGST 2015]

$$\mathcal{O}_1 \in \{X, Z\}\,, \quad \mathcal{O}_2 \in \{\bar{X}, \bar{Z}\}\,, \quad \mathcal{O}_1 \in \{\tilde{X}, \tilde{Z}\}\,, \qquad \tilde{X} = \frac{X + \bar{Z}}{\sqrt{2}} \;\;, \tilde{Z} = \frac{Z + \bar{Z} + X + \bar{X}}{2}$$

A class of such 3pf (I-I-I type) is evaluated by a triple sum over partitions.

[Kazama-Komatsu-Nishimura'2014]

$$SU(2|2)_L \times SU(2|2)_R \quad \to \quad SU(2)_L \times SU(2)_R$$

"Double arrow" formalism:
$$|Z\rangle = |\uparrow\rangle_{\rm L} \otimes |\uparrow\rangle_{\rm R} \equiv |\uparrow\uparrow\rangle\,, \qquad |\bar{Z}\rangle = |\downarrow\rangle_{\rm L} \otimes |\downarrow\rangle_{\rm R} \equiv |\downarrow\downarrow\rangle\,,$$
$$|X\rangle = |\uparrow\rangle_{\rm L} \otimes |\downarrow\rangle_{\rm R} \equiv |\uparrow\downarrow\rangle\,, \qquad |\bar{X}\rangle = -|\downarrow\rangle_{\rm L} \otimes |\uparrow\rangle_{\rm R} \equiv -|\downarrow\uparrow\rangle$$

I-I-I type:

L: primary, non-BPS Bethe states: $|\psi_a\rangle_{\rm L} = {\rm g}_a |\mathbf{u}_a\rangle_{\rm L}\,, \quad \mathbf{u}_a = \{u_{a,j}\}_{j=1}^{M_a}$

R: BPS $|\hat{\psi}_a\rangle_{\rm L} = \hat{\rm g}_a | \uparrow^{L_a}\rangle_{\rm R}$

global SU(2) rotations

(The EGSV configuration is I-I-II type.)

Factorization: $|\Psi_a\rangle = |\psi_a\rangle_{\rm L} \otimes |\hat{\psi}_a\rangle_R \qquad (a = 1, 2, 3)$

$$\langle {\rm V}_3|\; |\Psi_1\rangle\,|\Psi_2\rangle\,|\Psi_3\rangle = C_{123}^{\rm L} C_{123}^{\rm R}$$

[Kazama-Komatsu-Nishimura'2014]

all life is in the left sector

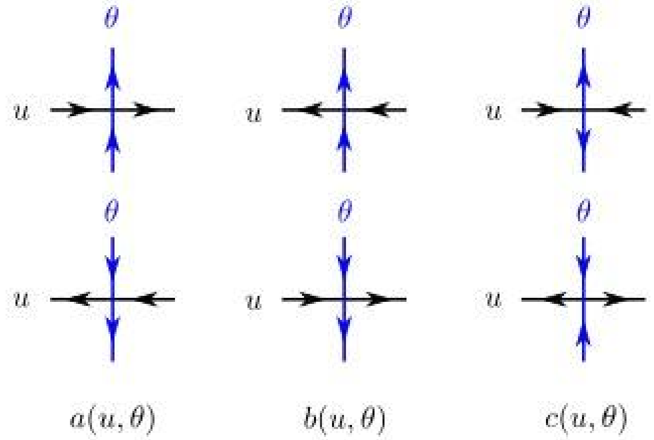
The map to the 6v model exists for all these functions, but the lattice is a **hexagon**, not a rectangle

* the 3pf has been computed for the I-I-II type (EGSV) configuration, where it reduces to a determinant. The I-I-I function is NOT a determinant.

I-I-I type 3pf as 6-vertex model on a hexagon

The six-vertex model [Baxter, 72] gives a statistical interpretation of the XXX spin chain.

Graphical representation of the XXX R-matrix: Scattering of a "fundamental particle" with rapidity u and a "mirror" particle with rapidity θ:



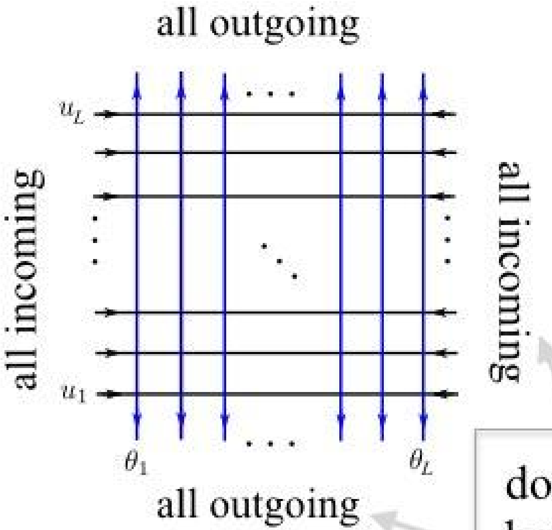
$$a(u,\theta) = u - \theta + i/2\,,$$
$$b(u,\theta) = u - \theta - i/2$$
$$c(u,\theta) = i$$

For N fundamental and N mirror particles:
DWPF~ **Izergin-Korepin** determinant [Korepin, 1982]

$\boldsymbol{\theta} = \{\theta_1, \cdots, \theta_L\}$ — inhomogeneities
$\mathbf{u} = \{u_1, \cdots, u_L\}$ — rapidities

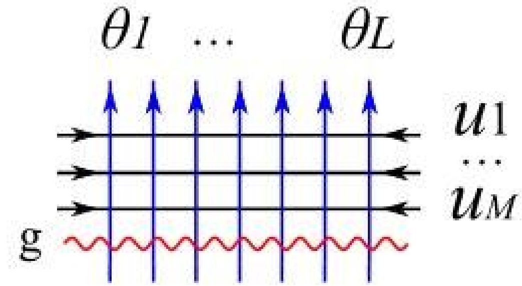
$$\mathcal{Z}_L(\mathbf{u}|\boldsymbol{\theta}) \equiv \langle \downarrow^L | \prod_{k=1}^{L} B(u_k) | \uparrow^L \rangle \sim \det \frac{1}{(u_j - \theta_k^+)(u_j - \theta_k^-)}$$

$$\theta^{\pm} = \theta \pm i/2$$



domain-wall boundary conditions

Wave function $|\psi\rangle = \mathrm{g}|\mathbf{u}\rangle$



$$|\mathbf{u}\rangle = B(u_1)\ldots B(u_M)|\uparrow^L\rangle$$

$$\mathrm{g} \;=\; e^{z\,S^-}e^{i\varphi\,S_3}e^{-\bar{z}\,S^+}$$

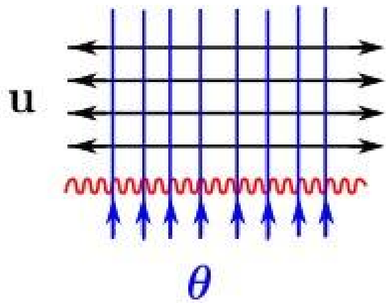
$$S_3|\mathbf{u}\rangle = (\tfrac{1}{2}L - M)|\mathbf{u}\rangle$$

$$S^+|\mathbf{u}\rangle = 0$$

$$\mathrm{g}_z = e^{\,z\,S^-}$$

commutes with magnon-creation operators

$$\mathrm{g}_z\,|\mathbf{u}\rangle \sim e^{zS^-}B(u_1)\ldots B(u_M)\,|\uparrow^L\rangle$$
$$= B(u_1)\ldots B(u_M)\;e^{zS^-}|\uparrow^L\rangle$$



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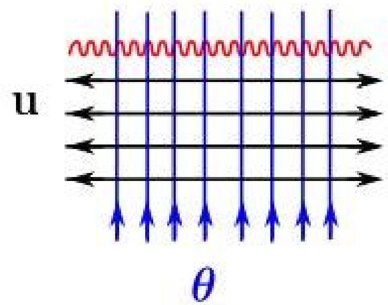


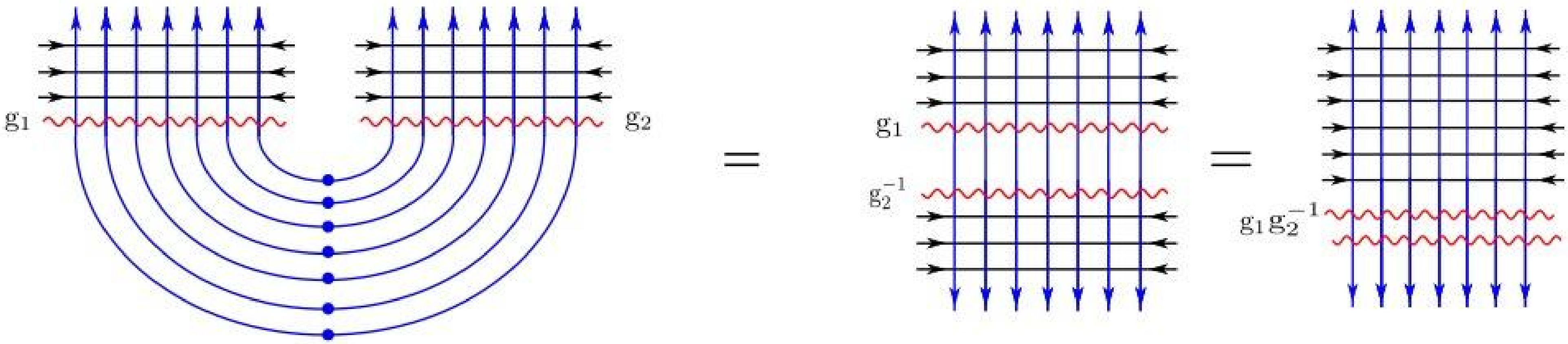
2-point function: 6v partition function on a **rectangle** (a.k.a. partial DWPF)

[I.K.-Y. Matsuo 2012]
[Kazama-Komatsu-Nishimura'2014]

$$\langle \mathrm{V}_2 || \psi_1 \rangle | \psi_2 \rangle \equiv \langle \mathrm{V}_2 | \, \mathrm{g}_2 \, | \mathbf{u}_2 \rangle \; \mathrm{g}_1 \, | \mathbf{u}_1 \rangle$$

spin vertex

$$= (-1)^{M_2} \langle \downarrow^L | \prod_{j=1}^{M_2} B(u_{2,j}) \, \mathrm{g}_2^{-1} \mathrm{g}_1 \prod_{i=1}^{M_1} B(u_{1,i}) | \uparrow^L \rangle$$

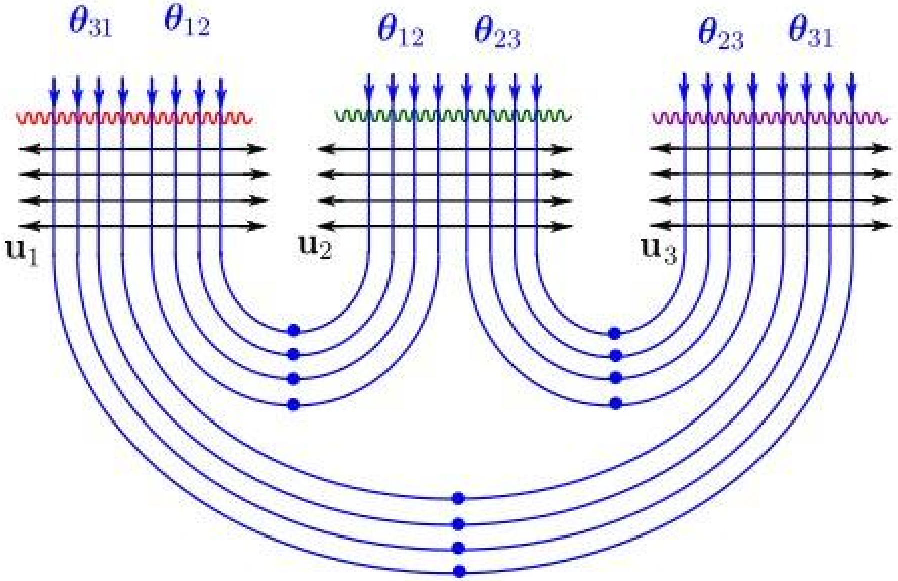


$$\Sigma_2 = i^L \sigma_2^{\otimes L} = e^{i \pi S_2}$$

$$\mathrm{T}(u) = \Sigma_2 \sigma^2 \, \mathrm{T}(u) \, (\Sigma_2 \sigma^2)^{-1}$$

3-point function: 6v partition function on a hexagon:

$$\langle \mathrm{V}_3| \, |\Psi_1\rangle \, |\Psi_2\rangle \, |\Psi_3\rangle \sim \left(\langle \mathrm{V}_3| \, |\psi_1\rangle \, |\psi_2\rangle \, |\psi_3\rangle\right)_{\mathrm{L}} \qquad |\psi_a\rangle = e^{z_a S_a^-} |\mathbf{u}_a\rangle \qquad a = 1, 2, 3.$$



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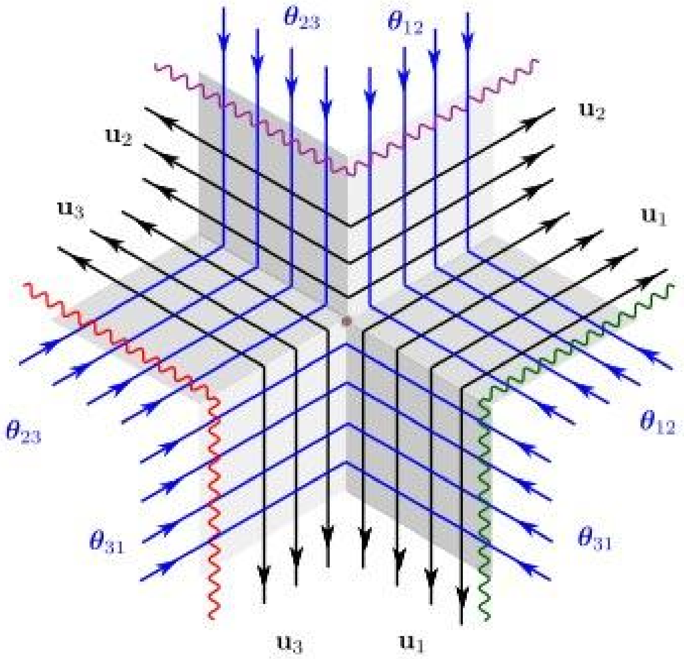
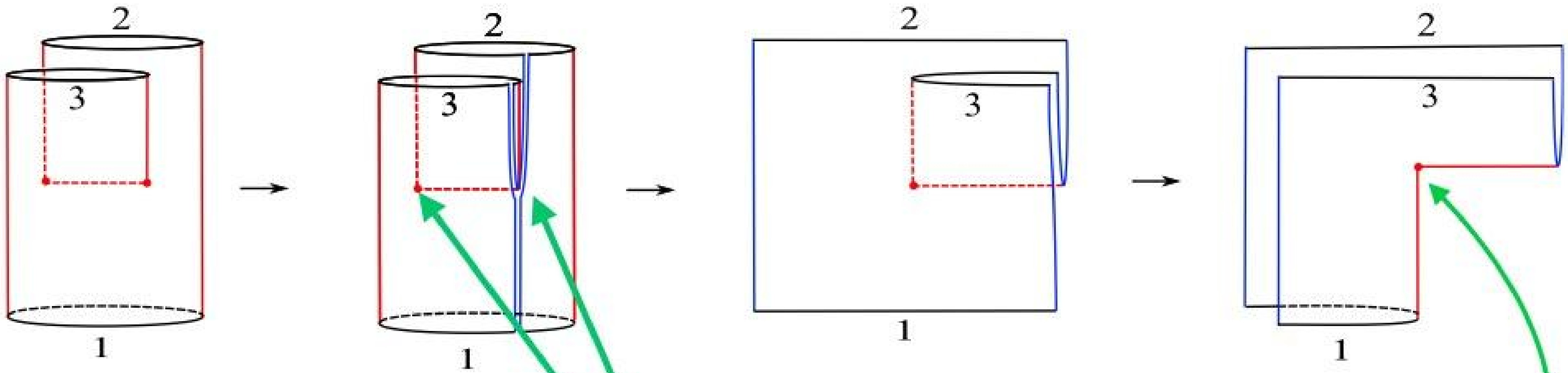
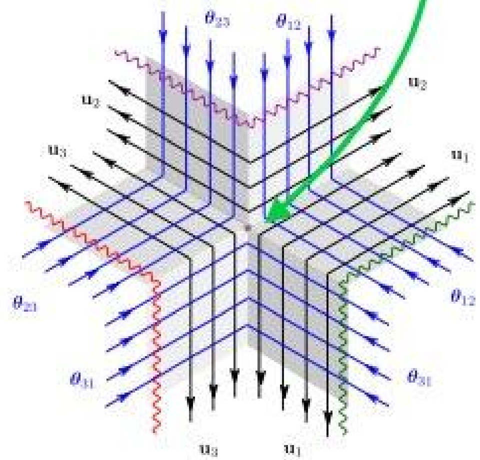


Lattice with a conical singularity with excess angle π if all the angles are π/2.

There is a second conical singularity which has been undone after cutting open the three spin chains. The hexagon as obtained by cutting a discretized (both in spatial and temporal directions) world sheet:

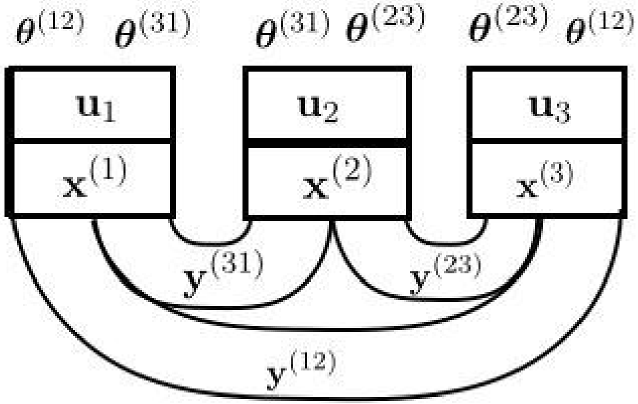


2 negative curvature defects,
excess angle π



II. The 3pf by Separation of Variables (SoV)

$$C^{\mathrm{L}}_{123} =$$



0) Separation of variables for su(2)

In the SoV basis the wave function factorizes
to one-particle wave functions:

$$\Psi(\mathbf{x}) = \langle \mathbf{x} | \psi \rangle = b^L \prod_{j=1}^{L} Q_{\mathbf{u}}(x_j)$$

$$Q_{\mathbf{u}}(x) = (x - \mathbf{u}) \equiv \prod_{j=1}^{M} (x - u_j)$$
Baxter polynomial

Sklyanin's recipe: diagonalize the magnon-creation operators B(u):

$$B_K(u) = b(u - \hat{x}_1) \dots (u - \hat{x}_L)$$

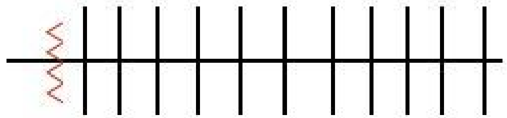
SoV basis: $\hat{x}_k |\mathbf{x}\rangle = x_k |\mathbf{x}\rangle$

Subtlety: B(u) is nilpotent => Compute with **twisted** monodromy matrix, then remove the twist
[Niccoli 2013, Kazama-Komatsu-Nishimura'2013]

e.g. left twist

$$\mathrm{K} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\mathrm{T}_{\mathrm{K}}(u) = \mathrm{K}\,\mathrm{L}_1(u - \theta_1) \cdots \mathrm{L}_L(u - \theta_L) \equiv \begin{pmatrix} A_{\mathrm{K}}(u) & B_{\mathrm{K}}(u) \\ C_{\mathrm{K}}(u) & D_{\mathrm{K}}(u) \end{pmatrix}$$



$$B_{\mathrm{K}}(u) = a\, B(u) + b\, D(u)$$

1) Explicit Construction of the SoV basis

[Niccoli; Komatsu-Jiang-IK-Serban 2015]

$$\mathrm{K}_z = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} = e^{z\,\sigma^+} \qquad\qquad B_z(u) = B(u) + zD(u), \quad C_z(u) = C(u)$$

1) Start with the simplest SoV states

$$|B_z(u)|\downarrow^L\rangle = zQ^+_{\boldsymbol{\theta}}(u))|\downarrow^L\rangle \quad ==> \quad B_z(\theta_j^-)|\downarrow^L\rangle = 0 \qquad ==> \qquad |\downarrow^L\rangle = |\mathbf{x}\rangle, \quad x_k = \theta_k^-$$

2) Generate the rest of the basis by applying creation operators:

$$|\mathbf{x}\rangle = |\prod_{k\in\alpha} A_z(\theta_k^-)|\downarrow^L\rangle \qquad \alpha \subset \{1,\ldots,L\} \qquad\qquad x_j = \begin{cases} \theta_j^+ & \text{if } \; j \in \alpha, \\ \theta_j^- & \text{if } \; j \bar{\in} \alpha. \end{cases}$$

$$\boxed{A(v)B(u) = \tfrac{u-v+i}{u-v}\, B(u)A(v) - \tfrac{i}{u-v} B(v)A(u)\,, \quad v = \theta_k^-}$$

Similarly, for the dual basis: $\quad \langle\uparrow^L| = \langle\mathbf{x}|, \quad x_k = \theta_k^+ \qquad \langle\mathbf{x}| = \langle\uparrow^L|\prod_{k\in\alpha} A_z(\theta_k^+)$

2) The measure:

$$\mu(\mathbf{x};\boldsymbol{\theta}) \sim \frac{1}{z^L} \operatorname*{Res}_{\mathbf{y}\to\mathbf{x}} \left[\frac{\Delta(\mathbf{y})}{\prod_{j=1}^L Q_{\boldsymbol{\theta}}(y_j^+) Q_{\boldsymbol{\theta}}(y_j^-)} \right]$$

$$\mathbb{I} = \sum_{\mathbf{x}} |\mathbf{x}\rangle \; \mu(\mathbf{x}) \; \langle \mathbf{x}| \qquad x_j = \theta_j^{s_j}, \quad s_j \in \{+,-\}$$

$$\langle \mathbf{x}' | \mathbf{x} \rangle = \mu^{-1}(\mathbf{x}) \; \delta_{\boldsymbol{x}',\mathbf{x}} \qquad \delta_{\boldsymbol{x}',\mathbf{x}} = \delta_{x'_1, x_1} \ldots \delta_{x'_L, x_L}$$

From the explicit form of the eigenvectors,

$$A_z(u) = A(u) + zC(u)$$

$$\mu^{-1}\left(\theta_1^{s_1}, \ldots, \theta_L^{s_L}\right) = z^L \langle \uparrow^L | \prod_{k=1}^L C(\theta_k^{-s_k}) | \downarrow^L \rangle$$

This is a special case of the Izergin determinant:

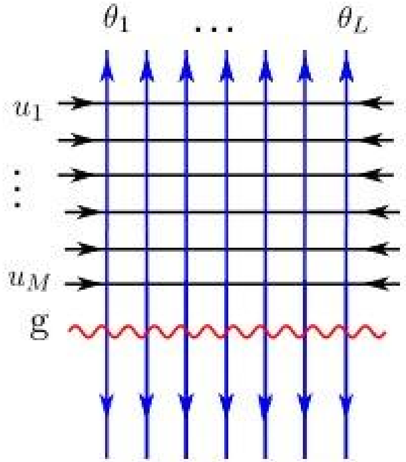
$$\mu(\mathbf{x};\boldsymbol{\theta}) \sim \prod_{j<k}^L (x_j - x_k) \prod_k e^{\pi(x_k - \theta_k)} \prod_{j \neq k} \frac{1}{(x_j - \theta_k^+)(x_j - \theta_k^-)}$$

[Kazama-Komatsu-Nishimura'2013]

3) The two-point function (DWPF on a rectangle)

The sum over the discrete values $x_k = \theta_k^{s_k}$ can be transformed into a contour integral.

E.g. the two-point function (the rectangle) is evaluated as



$$\langle \downarrow^L | B(u_1) \cdots B(u_M)\, \mathrm{g}\, | \uparrow^L \rangle$$

$$\sim \oint_{\mathcal{C}_{\boldsymbol{\theta}}} \prod_{j=1}^{L} \frac{dx_j}{2\pi i} \, \frac{Q_{\mathbf{u}}(x_j)\; e^{2\pi (j-1) x_k}}{Q_{\boldsymbol{\theta}}^{+}(x_j) Q_{\boldsymbol{\theta}}^{-}(x_j)} \; \prod_{j<k}^{L} (x_j - x_k) \; \frac{\zeta^{L-M}\, (x_1 + \cdots + x_L)^{L-M}}{(L-M)!}$$

$$\mathrm{g} = e^{\zeta\, S^-} = \sum_{n=0}^{\infty} \frac{\zeta^n (S^-)^n}{n!}$$

The dependence of the twist cancels completely!

4) The splitting coefficient:

$$\overset{\text{left}}{\boldsymbol{\theta} = \boldsymbol{\theta}_1} \cup \overset{\text{right}}{\boldsymbol{\theta}_2}.$$

$$|\mathbf{x}\rangle_{\mathrm{K}_1|\mathrm{K}_2} \longrightarrow \overset{\text{left}}{|\mathbf{y}_1\rangle_{\mathrm{K}_1|0}} \otimes \overset{\text{right}}{|\mathbf{y}_2\rangle_{0|\mathrm{K}_2}}$$

$$B_{\mathrm{K}_1|\mathrm{K}_2}(u) = \overset{\text{left}}{A_{\mathrm{K}_1|0}(u)}\overset{\text{right}}{B_{0|\mathrm{K}_2}(u)} + \overset{\text{left}}{B_{\mathrm{K}_1|0}(u)}\overset{\text{right}}{D_{0|\mathrm{K}_2}(u)}$$

==> Difference equations+ initial data for

$$\Phi(\mathbf{y}_1;\mathbf{y}_2|\mathbf{x}) \equiv {}_{\mathrm{K}_1|0}\langle\mathbf{y}_1| \otimes {}_{0|\mathrm{K}_2}\langle\mathbf{y}_2|\ |\mathbf{x}\rangle_{\mathrm{K}_1|\mathrm{K}_2}$$

==> solution

$$\Phi(\mathbf{y}_1;\mathbf{y}_2|\mathbf{x}) = \mathtt{twist} \times \mathtt{Gamma},$$

$$\mathtt{twist} = (ib_{12}/b_1)^{N_+^{\mathbf{y}_2}}\,(-ib_{12}/b_2)^{N_-^{\mathbf{y}_1}}\,(-b_1)^{N_+^{\mathbf{x}}}\,(b_2)^{N_-^{\mathbf{x}}}$$

$$\mathtt{Gamma} = \frac{\Gamma\left(i(\boldsymbol{\theta}_1^+ - \boldsymbol{\theta}_2^-)\right)}{\Gamma\left(i(\mathbf{y}_1 - \mathbf{y}_2)\right)}\,\frac{\Gamma\left(1 - i(\mathbf{y}_1 - \boldsymbol{\theta}_1^-)\right)}{\Gamma\left(1 - i(\mathbf{y}_2 - \boldsymbol{\theta}_2^-)\right)}\,\frac{\Gamma\left(1 + i(\mathbf{x} - \boldsymbol{\theta}_1^+)\right)}{\Gamma\left(1 + i(\mathbf{x} - \mathbf{y}_1)\right)}\,\frac{\Gamma\left(1 - i(\mathbf{x} - \boldsymbol{\theta}_2^-)\right)}{\Gamma\left(1 - i(\mathbf{x} - \mathbf{y}_2)\right)}$$

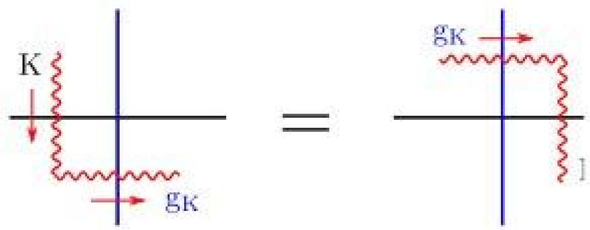
$$f(\mathbf{x} - \mathbf{y}) \equiv \prod_{x_i \in \mathbf{x}, y_j \in \mathbf{y}} f(x_i - y_j)$$

Remark:

Left/right twisted states are related to left (or right) twisted states by global rotation

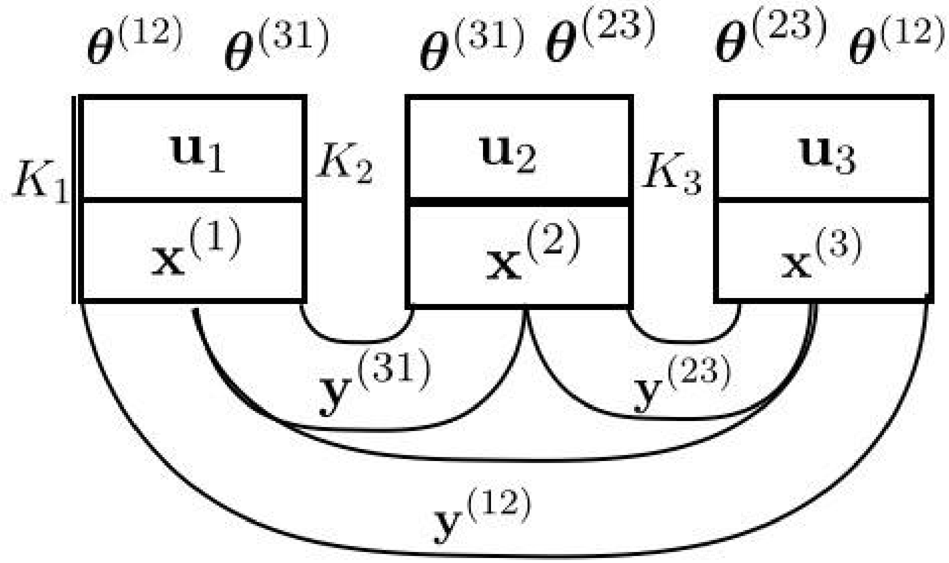
$$|\mathbf{x}\rangle_{\mathrm{K}_1|\mathrm{K}_2} = \mathrm{g}_{\mathrm{K}_2}\,|\mathbf{x}\rangle_{\mathrm{K}_{12}|0}\,, \qquad {}_{\mathrm{K}_1|\mathrm{K}_2}\langle\mathbf{x}| = {}_{\mathrm{K}_{12}|0}\langle\mathbf{x}|\;\mathrm{g}_{\mathrm{K}_2}^{-1}$$

Proof:
YBE for twists ==> $\mathrm{K}\;\mathrm{T}(u)\,\mathrm{K}^{-1} = \mathrm{g}_{\mathrm{K}}^{-1}\,\mathrm{T}(u)\,\mathrm{g}_{\mathrm{K}}$



5) The 3pt function as a multiple contour integral

$$C^{\mathrm{L}}_{123} =$$

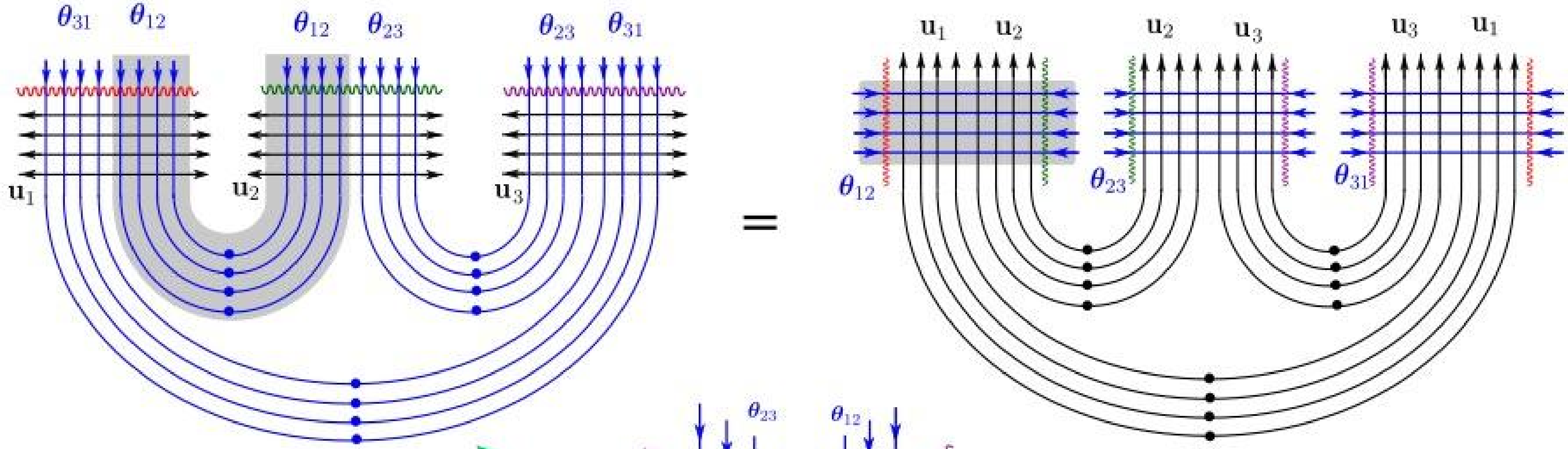


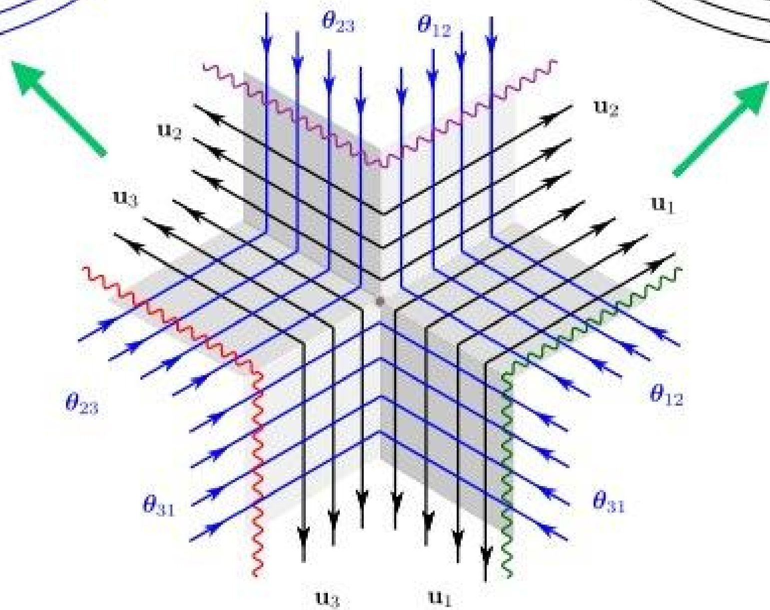
Problem: Difficult (perhaps impossible) to eliminate the twists $K_1, \; K_2, \; K_3$

Solution: apply the "mirror" (space <=> time) symmetry of the hexagon
and use the three global rotations g1, g2, g3 as twists.
No need to introduce the twists by hand!

Mirror transformation of the 3pf:

Bethe roots <===> inhomogeneities
Twists <====> global rotations





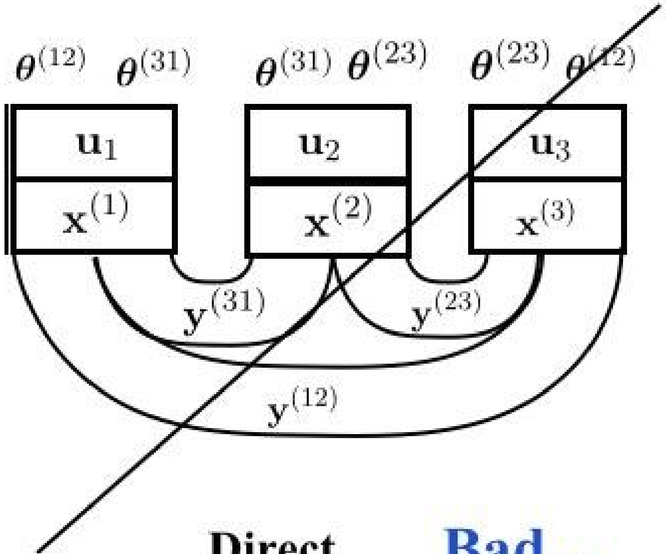
$$a(u,\theta) \to b(\theta,u) = -a(u,\theta)\,,$$

$$b(u,\theta) \to a(\theta,u) = -b(u,\theta)\,,$$

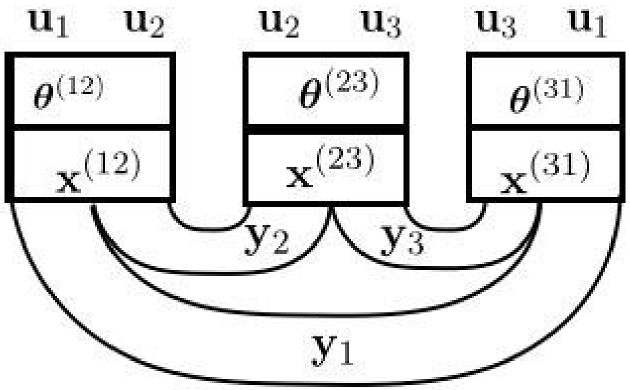
$$c(u,\theta) \to c(\theta,u) = c(u,\theta)\,.$$

3pf in terms of SoV

$C^{\mathrm{L}}_{123} =$

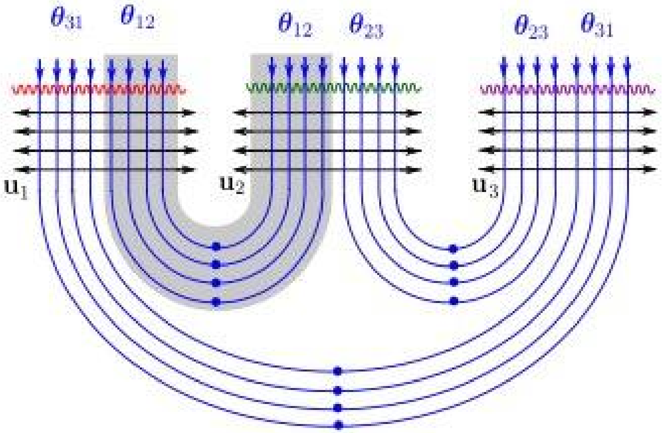


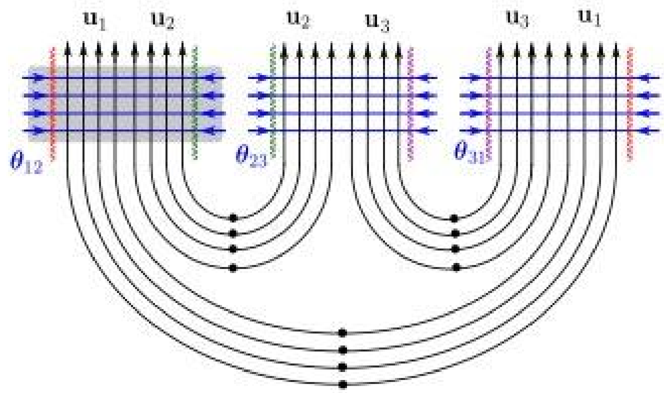
$C^{\mathrm{L}}_{123} =$



Direct **Bad…**

Mirror **Good!**





Contour integral representation of the 3pf

$$C^{\mathrm{L}}_{123} = \texttt{factor} \times \oint_{\mathbf{u}_a \pm i/2} d\mu(\mathbf{x}^{(12)})\, d\mu(\mathbf{x}^{(23)})\, d\mu(\mathbf{x}^{(31)})\, d\omega(\mathbf{y}_1)\, d\omega(\mathbf{y}_2)\, \omega(\mathbf{y}_3)$$

$$\times \prod_{(ab)} \frac{\Gamma\left(i(\mathbf{u}_a^+ - \mathbf{u}_b^-)\right)}{\Gamma\left(i(\mathbf{y}_a - \mathbf{y}_b)\right)} \frac{\Gamma\left(1 - i(\mathbf{u}_a^+ - \mathbf{x}^{(ab)})\right)\Gamma\left(1 + i(\mathbf{u}_b^- - \mathbf{x}^{(ab)})\right)}{\Gamma\left(1 - i(\mathbf{y}_a - \mathbf{x}^{(ab)})\right)\Gamma\left(1 + i(\mathbf{y}_b - \mathbf{x}^{(ab)})\right)}$$

$$\times \prod_{k=1}^{M_a+M_b} Q_{\boldsymbol{\theta}^{(ab)}}(x_k^{(ab)})\;\; T(z_1, z_2, z_3),$$

$$d\mu(\mathbf{x}^{(ab)}) = \prod_{k=1}^{M_a+M_b} \frac{dx_k^{(ab)}}{2\pi i} \frac{\Delta(\mathbf{x}^{(ab)})\Delta(e^{2\pi \mathbf{x}^{(ab)}})}{(\mathbf{x}^{(ab)} - \mathbf{u}_a^+)(\mathbf{x}^{(ab)} - \mathbf{u}_a^-)(\mathbf{x}^{(ab)} - \mathbf{u}_b^+)(\mathbf{x}^{(ab)} - \mathbf{u}_b^-)}$$

$$d\omega(\mathbf{y}_a) = \prod_{j=1}^{M_a} \frac{dy_{a,j}}{2\pi i} \frac{\Delta(\mathbf{y}_a)\Delta(e^{2\pi \mathbf{y}_a})}{\prod_{j=1}^{M_a}\prod_{k=1}^{M_a} \cosh\pi\,(\mathbf{y}_a - \mathbf{u}_a)}$$

$$f(\mathbf{x}, \mathbf{y}) \stackrel{\text{def}}{=} \prod_{x \in \mathbf{x}, y \in \mathbf{y}} f(x, y)$$

$$u^{\pm} \equiv u \pm i/2$$

$$T(z_1, z_2, z_3) = \prod_{(ab)} (z_{ab})^{i(\mathbf{y}_a - \mathbf{y}_b - \mathbf{u}_a + \mathbf{u}_b)}\, (z_{ab})^{\frac{1}{2}(M_a + M_b)}$$

$$\times (z_1)^{i(\mathbf{x}^{(31)} - \mathbf{x}^{(12)} + \mathbf{y}_2 - \mathbf{y}_3)} (z_2)^{i(\mathbf{x}^{(23)} - \mathbf{x}^{(31)} + \mathbf{y}_3 - \mathbf{y}_1)} (z_3)^{i(\mathbf{x}^{(23)} - \mathbf{x}^{(31)} + \mathbf{y}_1 - \mathbf{y}_2)}.$$

Conclusion

— Explicit expression for the SoV basis with general twists

— Splitting factor in the SoV basis

— Integral representation of the I-I-I type three-point function

Remains to be done:

— Try to compute the quasiclassical limit and compare with the strong coupling result
  * Non-trivial! - even for the rectangle (isolated saddle points).

— Generalize to higher loops and other sectors (c.f. [Sobko 2013] for the sl(2) sector)

— Is the mirror rotation here related to the mirror rotation in the all-coupling BKV bootstrap?

Thank You!