# The XYZ spin chain/8-vertex model with quasi-periodic boundary conditions

## Exact solution by Separation of Variables

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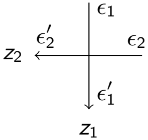
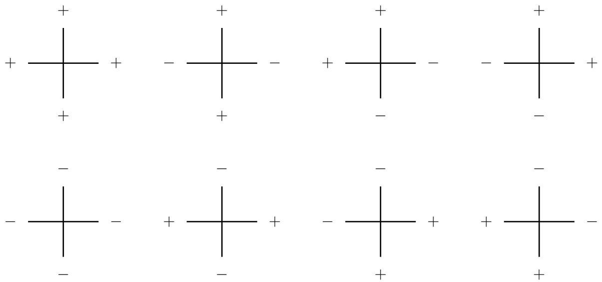
In collaboration with *G. Niccoli*.

# 8-vertex model

2-d square lattice model
link $\longrightarrow \epsilon_j = \pm$
vertex $\longrightarrow$ Boltzmann weight

$$\mathbf{R}^{8V}(z_1 - z_2)^{\epsilon_1,\epsilon_2}_{\epsilon'_1,\epsilon'_2} = $$




$a^{8V}$ $b^{8V}$ $c^{8V}$ $d^{8V}$

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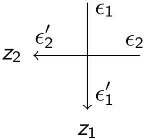



$$\mathbf{R}^{8V}(z) = \begin{pmatrix} a^{8V}(z) & 0 & 0 & d^{8V}(z) \\ 0 & b^{8V}(z) & c^{8V}(z) & 0 \\ 0 & c^{8V}(z) & b^{8V}(z) & 0 \\ d^{8V}(z) & 0 & 0 & a^{8V}(z) \end{pmatrix}$$

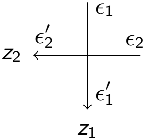
$z$ : spectral parameter
$p = e^{i\pi\omega}$: elliptic parameter

$$a^{8V}(z) = \rho \frac{\theta_4(z|2\omega)\,\theta_4(\eta|2\omega)}{\theta_4(z+\eta|2\omega)\,\theta_4(0|2\omega)}, \qquad b^{8V}(z) = \rho \frac{\theta_1(z|2\omega)\,\theta_4(\eta|2\omega)}{\theta_1(z+\eta|2\omega)\,\theta_4(0|2\omega)},$$

$$c^{8V}(z) = \rho \frac{\theta_4(z|2\omega)\,\theta_1(\eta|2\omega)}{\theta_1(z+\eta|2\omega)\,\theta_4(0|2\omega)}, \qquad d^{8V}(z) = \rho \frac{\theta_1(z|2\omega)\,\theta_1(\eta|2\omega)}{\theta_4(z+\eta|2\omega)\,\theta_4(0|2\omega)}.$$

2-d square lattice model
link $\longrightarrow \epsilon_j = \pm$
vertex $\longrightarrow$ Boltzmann weight

$$\mathbf{R}_{12}^{8V}(z_1 - z_2)_{\epsilon_1', \epsilon_2'}^{\epsilon_1, \epsilon_2} =$$




$$\mathbf{R}_{12}^{8V}(z) = \begin{pmatrix} a^{8V}(z) & 0 & 0 & d^{8V}(z) \\ 0 & b^{8V}(z) & c^{8V}(z) & 0 \\ 0 & c^{8V}(z) & b^{8V}(z) & 0 \\ d^{8V}(z) & 0 & 0 & a^{8V}(z) \end{pmatrix} \quad \in \mathrm{End}(V_1 \otimes V_2), \quad V_i \simeq \mathbb{C}^2$$

satisfying the Quantum Yang-Baxter Equation (QYBE) on $V_1 \otimes V_2 \otimes V_3$, $V_i \simeq \mathbb{C}^2$:

$$\mathbf{R}_{12}^{8V}(z_1 - z_2)\,\mathbf{R}_{13}^{8V}(z_1)\,\mathbf{R}_{23}^{8V}(z_2) = \mathbf{R}_{23}^{8V}(z_2)\,\mathbf{R}_{13}^{8V}(z_1)\,\mathbf{R}_{12}^{8V}(z_1 - z_2)$$

# Commuting transfer matrices for the 8-vertex model

- **Monodromy Matrix:**
  (on $V_0 \otimes \mathbb{V}_N, \quad \mathbb{V}_N = V_1 \otimes V_2 \otimes \ldots \otimes V_N, \quad V_i \simeq \mathbb{C}^2$)

$$\mathrm{M}_0^{(8V)}(\lambda) = R_{0N}^{(8V)}(\lambda - \xi_N) \cdots R_{02}^{(8V)}(\lambda - \xi_2)\, R_{01}^{(8V)}(\lambda - \xi_1)$$
$$= \begin{pmatrix} A^{(8V)}(\lambda) & B^{(8V)}(\lambda) \\ C^{(8V)}(\lambda) & D^{(8V)}(\lambda) \end{pmatrix}_{[0]}$$

  satisfying

$$R_{00'}^{(8V)}(\lambda_1 - \lambda_2)\, \mathrm{M}_0^{(8V)}(\lambda_1)\, \mathrm{M}_{0'}^{(8V)}(\lambda_2) = \mathrm{M}_{0'}^{(8V)}(\lambda_2)\, \mathrm{M}_0^{(8V)}(\lambda_1)\, R_{00'}^{(8V)}(\lambda_1 - \lambda_2)$$

  $\rightsquigarrow$ commutation relations for $A^{(8V)}, \; B^{(8V)}, \; C^{(8V)}, \; D^{(8V)}$

- **Transfer Matrix:** $T^{(8V)}(\lambda) = \mathrm{tr}_0 \left\{ \mathrm{M}_0^{(8V)}(\lambda) \right\} \quad \rightsquigarrow \quad [T^{(8V)}(u), T^{(8V)}(v)] = 0$

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  $\rightsquigarrow$ commutation relations for $A^{(8\mathrm{V})},\ B^{(8\mathrm{V})},\ C^{(8\mathrm{V})},\ D^{(8\mathrm{V})}$

- **Transfer Matrix:** (periodic boundary conditions)

$$T^{(8\mathrm{V})}(\lambda) = \mathrm{tr}_0 \left\{ \mathrm{M}_0^{(8\mathrm{V})}(\lambda) \right\} \quad \rightsquigarrow \quad [T^{(8\mathrm{V})}(u), T^{(8\mathrm{V})}(v)] = 0$$

# Commuting transfer matrices for the 8-vertex model

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- **Transfer Matrix:** $T^{(8V)}(\lambda) = \mathrm{tr}_0 \left\{ \mathrm{M}_0^{(8V)}(\lambda) \right\}$ $\rightsquigarrow$ $[T^{(8V)}(u), T^{(8V)}(v)] = 0$

*Remark.* $[R^{(8V)}(\lambda), \sigma^\alpha \otimes \sigma^\alpha] = 0$ for $\alpha = x, y, z$

$\rightsquigarrow$ for any fixed $\alpha$, $T_\alpha^{(8V)}(\lambda) = \mathrm{tr}_0 \left\{ \sigma_0^\alpha\, \mathrm{M}_0^{(8V)}(\lambda) \right\}$ defines a one-parameter family of commuting quasi-periodic transfer matrices

$$\left. \frac{\partial \log \mathrm{T}_\alpha^{(8V)}(\lambda)}{\partial \lambda} \right|_{\substack{\lambda=0 \\ \xi_n=0}} = H_{\mathrm{XYZ}} = \frac{1}{2} \sum_{n=1}^{N} \left\{ J_x \sigma_n^x \sigma_{n+1}^x + J_y \sigma_n^y \sigma_{n+1}^y + J_z \sigma_n^z \sigma_{n+1}^z \right\} + \frac{1}{2} J_0,$$

with $\sigma_{N+1}^\beta = \sigma_1^\alpha\, \sigma_1^\beta\, \sigma_1^\alpha$

**Goal:** find the (complete set of) eigenvalues and eigenstates of $T_\alpha^{(8V)}(\lambda)$

# Commuting transfer matrices for the 8-vertex model

- **Monodromy Matrix:**
  (on $V_0 \otimes \mathbb{V}_N, \quad \mathbb{V}_N = V_1 \otimes V_2 \otimes \ldots \otimes V_N, \quad V_i \simeq \mathbb{C}^2$)

$$\mathrm{M}_0^{(8V)}(\lambda) = R_{0N}^{(8V)}(\lambda - \xi_N) \cdots R_{02}^{(8V)}(\lambda - \xi_2)\, R_{01}^{(8V)}(\lambda - \xi_1)$$
$$= \begin{pmatrix} A^{(8V)}(\lambda) & B^{(8V)}(\lambda) \\ C^{(8V)}(\lambda) & D^{(8V)}(\lambda) \end{pmatrix}_{[0]}$$

  satisfying

$$R_{00'}^{(8V)}(\lambda_1 - \lambda_2)\, \mathrm{M}_0^{(8V)}(\lambda_1)\, \mathrm{M}_{0'}^{(8V)}(\lambda_2) = \mathrm{M}_{0'}^{(8V)}(\lambda_2)\, \mathrm{M}_0^{(8V)}(\lambda_1)\, R_{00'}^{(8V)}(\lambda_1 - \lambda_2)$$

- **Transfer Matrix:** $T^{(8V)}(\lambda) = \mathrm{tr}_0 \left\{ \mathrm{M}_0^{(8V)}(\lambda) \right\} \quad \rightsquigarrow \quad [T^{(8V)}(u), T^{(8V)}(v)] = 0$

  $T_\alpha^{(8V)}(\lambda) = \mathrm{tr}_0 \left\{ \sigma_0^\alpha\, \mathrm{M}_0^{(8V)}(\lambda) \right\} \quad \rightsquigarrow \quad [T_\alpha^{(8V)}(u), T_\alpha^{(8V)}(v)] = 0$

**Goal:** find the (complete set of) eigenvalues and eigenstates of $T_\alpha^{(8V)}(\lambda)$

However:

- no simple reference state
- $[X^{(8V)}(\lambda), X^{(8V)}(\mu)] \neq 0$ for $X = A, B, C, D$

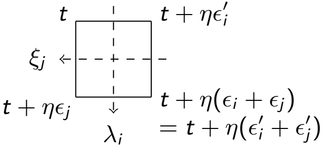
$\longrightarrow$ not directly solvable by Bethe ansatz nor by separation of variables

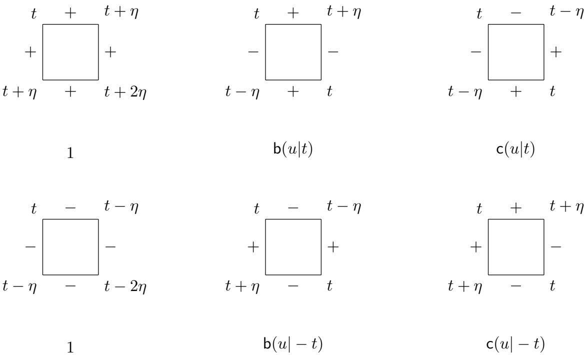
Baxter's solution (Ann.Phys.73) $\longrightarrow$ map onto an IRF model (SOS model)

# (8V)SOS model (dynamical 6-vertex model)

2-d square lattice model
vertex $\longrightarrow$ local height $t_j$
  $t_j - t_k = \pm\eta$ (adjacent)
face $\longrightarrow$ Boltzmann weight

$$\mathbf{R}(\lambda_i - \xi_j | t)^{\epsilon_i, \epsilon_j}_{\epsilon'_i, \epsilon'_j} =$$



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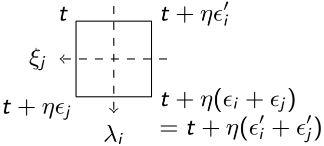


$$\mathbf{R}(\lambda|t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(\lambda|t) & c(\lambda|t) & 0 \\ 0 & c(\lambda|-t) & b(\lambda|-t) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\lambda$ : spectral parameter
$t \in t_0 + \eta\mathbb{Z}$ : dynamical parameter

$$b(\lambda|t) = \frac{\theta(t+\eta)\,\theta(\lambda)}{\theta(t)\,\theta(\lambda+\eta)} \qquad c(\lambda|t) = \frac{\theta(\lambda+t)\,\theta(\eta)}{\theta(t)\,\theta(\lambda+\eta)} \qquad \theta(u) = \theta_1(u|\omega)$$

satisfying the **Dynamical** Quantum Yang-Baxter Equation:

$$\mathbf{R}_{12}(\lambda_1 - \lambda_2 | t + \eta\sigma_3^z)\,\mathbf{R}_{13}(\lambda_1 - \lambda_3 | t)\,\mathbf{R}_{23}(\lambda_2 - \lambda_3 | t + \eta\sigma_1^z)$$
$$= \mathbf{R}_{23}(\lambda_2 - \lambda_3 | t)\,\mathbf{R}_{13}(\lambda_1 - \lambda_3 | t + \eta\sigma_2^z)\,\mathbf{R}_{12}(\lambda_1 - \lambda_2 | t)$$

Spin conservation, solvable by Bethe Ansatz

# Baxter's Vertex-IRF Transformation

It is equivalent to the following Dynamical Gauge Equivalence:

$$R_{12}^{(8V)}(\lambda_{12})\, S_1(\lambda_1|t)\, S_2(\lambda_2|t+\eta\sigma_1^z) = S_2(\lambda_2|t)\, S_1(\lambda_1|t+\eta\sigma_2^z)\, \mathbf{R}_{12}^{(SOS)}(\lambda_{12}|t)$$

with $S(\lambda|t) = \begin{pmatrix} \theta_2(-\lambda+t|2\omega) & \theta_2(\lambda+t|2\omega) \\ \theta_3(-\lambda+t|2\omega) & \theta_3(\lambda+t|2\omega) \end{pmatrix}$ $\qquad \lambda_{12} \equiv \lambda_1 - \lambda_2$

$\rightsquigarrow$ relation between monodromy matrices:

$$\mathsf{M}_0^{(8V)}(\lambda)\, S_0(\lambda|t)\, S_q(t+\eta\sigma_0^z) = S_q(t)\, S_0(\lambda|t+\eta S^z)\, \mathbf{M}_0^{(SOS)}(\lambda|t)$$

where $\mathsf{M}_0^{(8V)}(\lambda) = R_{0N}^{(8V)}(\lambda-\xi_N)\cdots R_{02}^{(8V)}(\lambda-\xi_2)\, R_{01}^{(8V)}(\lambda-\xi_1)$

$$\mathbf{M}_0^{(SOS)}(\lambda|t) = \mathbf{R}_{0N}(\lambda-\xi_N|t+\eta\sum_{a=1}^{N-1}\sigma_a^z)\cdots \mathbf{R}_{02}(\lambda-\xi_2|t+\eta\sigma_1^z)\, \mathbf{R}_{01}(\lambda-\xi_1|t)$$

$$S_q(t) = S_1(\xi_1|t)\, S_2(\xi_2|t+\eta\sigma_1^z)\ldots S_N(\xi_N|t+\eta\sum_{a=1}^{N-1}\sigma_a^z) \quad \text{and} \quad S^z = \sum_{n=1}^{N}\sigma_n^z$$

$\longrightarrow$ Eigenvalues and eigenvectors of the (periodic) 8-vertex transfer matrix (Baxter, 1973)

Restrictions: Lattice size $N = 2n$ even

$L\eta = m_1\pi + m_2\pi\omega \qquad L\in\mathbb{N},\ m_1, m_2 \in \mathbb{Z}$ (cyclic SOS model)

# The SOS model: Dynamical Yang-Baxter algebra

Felder, Varchenko (1996) : representations of $E_{\tau,\eta}(sl_2)$

$$\mathbf{M}_0(\lambda|t) = \mathbf{R}_{0\mathrm{N}}(\lambda - \xi_\mathrm{N}|t + \eta \sum_{a=1}^{\mathrm{N}-1} \sigma_a^z) \cdots \mathbf{R}_{02}(\lambda - \xi_2|t + \eta\sigma_1^z)\, \mathbf{R}_{01}(\lambda - \xi_1|t)$$
$$= \begin{pmatrix} A(u|t) & B(u|t) \\ C(u|t) & D(u|t) \end{pmatrix}_{[0]}$$

satisfy

$$\mathbf{R}_{00'}(\lambda_{00'}|t + \eta S^z)\, \mathbf{M}_0(\lambda_0|t)\, \mathbf{M}_{0'}(\lambda_{0'}|t + \eta\sigma_0^z)$$
$$= \mathbf{M}_{0'}(\lambda_{0'}|t)\, \mathbf{M}_0(\lambda_0|t + \eta\sigma_{0'}^z)\, \mathbf{R}_{00'}(\lambda_{00'}|t) \qquad \text{with } S^z = \sum_{n=1}^N \sigma_n^z$$

$\rightsquigarrow$ introduce dynamical operators $\widehat{\tau}$ and $\widehat{T}_\tau$ such that $\widehat{T}_\tau\, \widehat{\tau} = (\widehat{\tau} + \eta)\, \widehat{T}_\tau$ so as to simplify the commutation relations:

$$\mathcal{M}_0(u) = \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u) \\ \mathcal{C}(u) & \mathcal{D}(u) \end{pmatrix}_{[0]} = \mathbf{M}_0(u|\widehat{\tau}) \begin{pmatrix} \widehat{T}_\tau & 0 \\ 0 & \widehat{T}_\tau^{-1} \end{pmatrix}_{[0]}$$

satisfy

$$\mathbf{R}_{00'}(\lambda_{00'}|\widehat{\tau} + \eta S^z)\, \mathcal{M}_0(\lambda_0)\, \mathcal{M}_{0'}(\lambda_{0'}) = \mathcal{M}_{0'}(\lambda_{0'})\, \mathcal{M}_0(\lambda_0)\, \mathbf{R}_{00'}(\lambda_{00'}|\widehat{\tau})$$

$\rightsquigarrow$ **operator algebra:** $\mathcal{A}(u)$, $\mathcal{B}(u)$, $\mathcal{C}(u)$, $\mathcal{D}(u)$ act on $\mathbb{H} = \mathbb{V}_N \otimes \mathbb{D}$

$\mathbb{V}_N = \otimes_{n=1}^N V_n$ (with $V_n \simeq \mathbb{C}^2$) : quantum spin space

$\mathbb{D}$ : representation space of dynamical operators algebra

# The SOS model: Representation spaces of the operator algebra

$\mathcal{A}(u)$, $\mathcal{B}(u)$, $\mathcal{C}(u)$, $\mathcal{D}(u)$ act on $\mathbb{H} = \mathbb{V}_N \otimes \mathbb{D}$

- $\mathbb{V}_N = \otimes_{n=1}^{N} V_n$ (with $V_n \simeq \mathbb{C}^2$) : ($2^N$-dimensional) quantum spin space
- $\mathbb{D}$ : representation space of dynamical operators algebra

basis: $|\, t(j)\, \rangle$, $j \in \mathbb{Z}$ (or $j \in \mathbb{Z}/L\mathbb{Z}$ if $L\eta = m_1\pi + m_2\pi\omega$: cyclic case)
with $\hat{\tau}\,|\, t(j)\,\rangle = t(j)\,|\, t(j)\,\rangle$, $t(j) = t_0 - \eta j$
$\hat{T}_\tau\,|\, t(j)\,\rangle = |\, t(j+1)\,\rangle$

- **Periodic transfer matrix:** $\mathcal{T}(u) = \mathrm{tr}_0\{\mathcal{M}_0(u)\} = \mathcal{A}(u) + \mathcal{D}(u)$
  $\rightsquigarrow$ commuting family on the subspace $\mathbb{H}^{(0)} \equiv \mathbb{V}_N[0] \otimes \mathbb{D}$ of $\mathbb{H}$ associated to the zero eigenvalue of $S^z = \sum_{n=1}^{N} \sigma_n^z$
- **Antiperiodic transfer matrix:** $\bar{\mathcal{T}}(u) = \mathrm{tr}_0\{\sigma_0^x \mathcal{M}_0(u)\} = \mathcal{B}(u) + \mathcal{C}(u)$
  $\rightsquigarrow$ commuting family on the subspace $\bar{\mathbb{H}}^{(0)}$ of $\mathbb{H}$ associated to the zero eigenvalue of $S_\tau \equiv \eta S^z + 2\hat{\tau}$

# The SOS model: Representation spaces of the operator algebra

$\mathcal{A}(u)$, $\mathcal{B}(u)$, $\mathcal{C}(u)$, $\mathcal{D}(u)$ act on $\mathbb{H} = \mathbb{V}_N \otimes \mathbb{D}$

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# From the periodic SOS model to the periodic 8-vertex model with N even: solution by Bethe ansatz

Periodic SOS transfer matrix: $\mathcal{T}(u) = \mathrm{tr}_0\{\mathcal{M}_0(u)\} = \mathcal{A}(u) + \mathcal{D}(u)$

$\rightsquigarrow$ commuting family on the subspace $\mathbb{H}^{(0)} \equiv \mathbb{V}_N[0] \otimes \mathbb{D} \simeq \mathrm{Fun}(\mathbb{V}_N[0])$ of $\mathbb{H}$ associated to the zero eigenvalue of $S^z$

For generic values of $\eta$, the physical space of states $\mathbb{H}^{(0)}$ of the periodic SOS model is:

- infinite-dimensional (unrestricted SOS model) for $N = 2n$ even
- zero-dimensional for $N$ odd

except if $L\eta = m_1\pi + m_2\pi\omega$ (cyclic case considered by Baxter):

$$\dim \mathbb{H}^{(0)} = L \dim \mathbb{V}_N[0]$$

The eigenstates of $\mathcal{T}(u)$ can be constructed by (algebraic) Bethe ansatz (Baxter 1973, Felder and Varchenko 1996)

$\rightsquigarrow$ by means of the vertex-IRF transformation, one can construct the eigenstates of the periodic 8-vertex transfer matrix in the case $N = 2n$ even ( Baxter 1973 for $L\eta = m_1\pi + m_2\pi\omega$)
Completeness ?

# From the antiperiodic SOS model to the (quasi-periodic) 8-vertex model: solution by SOV

Antiperiodic SOS transfer matrix: $\bar{\mathcal{T}}(u) = \mathcal{B}(u) + \mathcal{C}(u)$

$\rightsquigarrow$ commuting family on the subspace $\bar{\mathbb{H}}^{(0)}$ of $\mathbb{H} = \mathbb{V}_N \otimes \mathbb{D}$ associated to the zero eigenvalue (or more generally to the eigenvalue $\mathsf{x}\pi + \mathsf{y}\pi\omega$, $\mathsf{x}, \mathsf{y} \in \{0, 1\}$) of $S_\tau \equiv \eta S^z + 2\hat{\tau}$

basis of the physical space of states $\bar{\mathbb{H}}^{(0)}$:

$$\left( \otimes_{n=1}^{\mathsf{N}} |n, h_n\rangle \right) \otimes |t_{\mathbf{h}}\rangle, \quad \mathbf{h} \equiv (h_1, \ldots, h_{\mathsf{N}}) \in \{0, 1\}^{\mathsf{N}} \qquad h_n = \begin{cases} 0 & \text{if spin } + \\ 1 & \text{if spin } - \end{cases}$$

$$\text{where} \quad t_{\mathbf{h}} = t_0 + \eta \sum_{k=1}^{N} h_k \quad \text{with} \quad t_0 = -\frac{\eta}{2} N + \mathsf{x}\frac{\pi}{2} + \mathsf{y}\frac{\pi}{2}\omega, \quad \mathsf{x}, \mathsf{y} \in \{0, 1\}$$

Remark: $\eta$ generic, $(\mathsf{x}, \mathsf{y}) \neq (0, 0)$ if N even

$\rightsquigarrow$ $\bar{\mathbb{H}}^{(0)}$ has dimension $2^N$ and is isomorphic to the space of states $\mathbb{V}_N$ of the 8-vertex model

$\rightsquigarrow$ construction of the eigenstates of the antiperiodic SOS model by separation of variables (Felder and Schorr 99, Niccoli 13, Levy-Bencheton Niccoli V.T. 15)

$\rightsquigarrow$ complete set of eigenstates of the periodic (N odd, $x, y = 0$) and quasi-periodic ($(x, y) \neq (0, 0)$) 8-vertex transfer matrices

# From the antiperiodic SOS model to the (quasi-periodic) 8-vertex model: solution by SOV

Antiperiodic SOS transfer matrix: $\bar{\mathcal{T}}(u) = \mathcal{B}(u) + \mathcal{C}(u)$

$\rightsquigarrow$ commuting family on the subspace $\bar{\mathbb{H}}^{(0)}$ of $\mathbb{H} = \mathbb{V}_N \otimes \mathbb{D}$ associated to the zero eigenvalue (or more generally to the eigenvalue $\mathsf{x}\pi + \mathsf{y}\pi\omega$, $\mathsf{x}, \mathsf{y} \in \{0, 1\}$) of $\mathsf{S}_\tau \equiv \eta \mathsf{S}^z + 2\hat{\tau}$

basis of the physical space of states $\bar{\mathbb{H}}^{(0)}$:

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where $\quad t_{\mathbf{h}} = t_0 + \eta \sum_{k=1}^{N} h_k \quad$ with $\quad t_0 = -\frac{\eta}{2} N + \mathsf{x}\frac{\pi}{2} + \mathsf{y}\frac{\pi}{2}\omega, \quad \mathsf{x}, \mathsf{y} \in \{0, 1\}$

Remark: $\eta$ generic, $(\mathsf{x}, \mathsf{y}) \neq (0, 0)$ if N even

$\rightsquigarrow$ $\bar{\mathbb{H}}^{(0)}$ has dimension $2^N$ and is isomorphic to the space of states $\mathbb{V}_N$ of the 8-vertex model:

$$\mathbf{P} : \left( \otimes_{n=1}^{\mathsf{N}} |n, h_n\rangle \right) \otimes |t_{\mathbf{h}}\rangle \mapsto \left( \otimes_{n=1}^{\mathsf{N}} |n, h_n\rangle \right)$$

$\rightsquigarrow$ construction of the eigenstates of the antiperiodic SOS model by separation of variables (Felder and Schorr 99, Niccoli 13, Levy-Bencheton Niccoli V.T. 15)

$\rightsquigarrow$ complete set of eigenstates of the periodic (N odd, $x, y$ ... 0) and

# From the antiperiodic SOS model to the (quasi-periodic) 8-vertex model: solution by SOV

Antiperiodic SOS transfer matrix: $\bar{\mathcal{T}}(u) = \mathcal{B}(u) + \mathcal{C}(u)$

$\rightsquigarrow$ commuting family on the subspace $\bar{\mathbb{H}}^{(0)}$ of $\mathbb{H} = \mathbb{V}_N \otimes \mathbb{D}$ associated to the zero eigenvalue (or more generally to the eigenvalue $\mathsf{x}\pi + \mathsf{y}\pi\omega$, $\mathsf{x}, \mathsf{y} \in \{0, 1\}$) of $\mathsf{S}_\tau \equiv \eta \mathsf{S}^z + 2\hat{\tau}$

basis of the physical space of states $\bar{\mathbb{H}}^{(0)}$:

$$\left( \otimes_{n=1}^{\mathsf{N}} |n, h_n\rangle \right) \otimes |t_{\mathbf{h}}\rangle, \quad \mathbf{h} \equiv (h_1, \ldots, h_{\mathsf{N}}) \in \{0, 1\}^{\mathsf{N}} \qquad h_n = \begin{cases} 0 & \text{if spin } + \\ 1 & \text{if spin } - \end{cases}$$

$$\text{where} \quad t_{\mathbf{h}} = t_0 + \eta \sum_{k=1}^{N} h_k \quad \text{with} \quad t_0 = -\frac{\eta}{2} N + \mathsf{x}\frac{\pi}{2} + \mathsf{y}\frac{\pi}{2}\omega, \quad \mathsf{x}, \mathsf{y} \in \{0, 1\}$$

Remark: $\eta$ generic, $(\mathsf{x}, \mathsf{y}) \neq (0, 0)$ if N even

$\rightsquigarrow$ $\bar{\mathbb{H}}^{(0)}$ has dimension $2^N$ and is isomorphic to the space of states $\mathbb{V}_N$ of the 8-vertex model

$\rightsquigarrow$ construction of the eigenstates of the antiperiodic SOS model by separation of variables (Felder and Schorr 99, Niccoli 13, Levy-Bencheton Niccoli V.T. 15)

$\rightsquigarrow$ complete set of eigenstates of the periodic (N odd, $x, y = 0$) and quasi-periodic ($(x, y) \neq (0, 0)$) 8-vertex transfer matrices

# Sklyanin's quantum Separation of Variables (SOV): idea of the method

Suppose that the monodromy matrix of the model $\widetilde{M}(\lambda) \equiv \begin{pmatrix} \widetilde{A}(\lambda) & \widetilde{B}(\lambda) \\ \widetilde{C}(\lambda) & \widetilde{D}(\lambda) \end{pmatrix}$ is such that $\widetilde{B}(\lambda)$ is a (usual, trigonometric, elliptic...) polynomial of degree N and is diagonalizable with simple spectrum, then the operator zeroes $Y_n$, $1 \leq n \leq N$, of $\widetilde{B}(\lambda)$ can be used to define a basis

$$|y_1, \dots, y_N\rangle, \qquad (y_1, \dots, y_N) \in \Lambda_1 \times \cdots \times \Lambda_N, \quad (\Lambda_i \cap \Lambda_j = \emptyset \text{ if } i \neq j)$$
$$Y_n \, |y_1, \dots, y_N\rangle = y_n \, |y_1, \dots, y_N\rangle$$

of the space of states in which the action of $\widetilde{A}(\lambda)$ and $\widetilde{D}(\lambda)$ is quasi-local, in particular

$$\widetilde{A}(Y_n) \, |y_1, \dots, y_n, \dots, y_N\rangle = a(y_n) \, |y_1, \dots, y_n + \eta, \dots, y_N\rangle$$
$$\widetilde{D}(Y_n) \, |y_1, \dots, y_n, \dots, y_N\rangle = d(y_n) \, |y_1, \dots, y_n - \eta, \dots, y_N\rangle$$

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of the space of states in which the action of $\widetilde{A}(\lambda)$ and $\widetilde{D}(\lambda)$ is quasi-local

$\rightsquigarrow$ The multi-dimensional spectral problem for the transfer matrix $\widetilde{T}(\lambda) = \widetilde{A}(\lambda) + \widetilde{D}(\lambda)$ can be reduced to a set of $N$ one-dimensional spectral problems by separation of variables: $\widetilde{T}(\lambda)| \, \Psi_t \, \rangle = t(\lambda)| \, \Psi_t \, \rangle$ iff

$$| \, \Psi_t \, \rangle = \sum_{| \, y_1, \ldots, y_N \, \rangle} \psi_t(y_1, \ldots, y_N) \, | \, y_1, \ldots, y_N \, \rangle \quad \text{with} \quad \psi_t(y_1, \ldots, y_N) = \prod_{n=1}^{N} Q_t^{(n)}(y_n)$$

where each $Q_t^{(n)}$ is solution of a discrete finite-difference equation

# Solution of the antiperiodic SOS model by SOV: Spectrum and eigenstates of the antiperiodic transfer matrix

$\rightsquigarrow$ SOV basis of $\bar{\mathbb{H}}^{(0)}$: $|\,\mathbf{h}\,\rangle$ , $\mathbf{h} = (h_1, \ldots, h_{\mathsf{N}}) \in \{0,1\}^{\mathsf{N}}$

For any fixed N-tuple of inhomogeneities $(\xi_1, \ldots, \xi_{\mathsf{N}}) \in \mathbb{C}^{\mathsf{N}}$ satisfying

$$\forall \epsilon \in \{-1,0,1\}, \quad \xi_j - \xi_k + \epsilon\eta \notin \pi\mathbb{Z} + \pi\omega\mathbb{Z} \quad \text{if } j \neq k,$$

the spectrum $\bar{\Sigma}^{(\mathsf{SOS})}$ of the antiperiodic SOS transfer matrix $\overline{\mathcal{T}}(\lambda)$ in $\bar{\mathbb{H}}^{(0)}$ is simple and coincides with the set of functions of the form

$$\bar{\mathsf{t}}(\lambda) = \sum_{k=j}^{\mathsf{N}} e^{iy(\xi_j-\lambda)} \frac{\theta(t_0 - \lambda + \xi_j)}{\theta(t_0)} \prod_{k\neq j} \frac{\theta(\lambda - \xi_k)}{\theta(\xi_j - \xi_k)} \bar{\mathsf{t}}(\xi_j),$$

which satisfy the discrete system of equations, $\forall j \in \{1, \ldots, \mathsf{N}\}$,

$$\bar{\mathsf{t}}(\xi_j)\,\bar{\mathsf{t}}(\xi_j - \eta) = (-1)^{\mathsf{x}+\mathsf{y}+\mathsf{xy}} a(\xi_j)\, d(\xi_j - \eta), \quad \text{with} \quad \begin{cases} a(\lambda) = \prod_{n=1}^{\mathsf{N}} \theta(\lambda - \xi_n + \eta) \\ d(\lambda) = a(\lambda - \eta) \end{cases}$$

The $\overline{\mathcal{T}}(\lambda)$-eigenstate associated with the eigenvalue $\bar{\mathsf{t}}(\lambda) \in \bar{\Sigma}^{(\mathsf{SOS})}$ is

$$|\,\Psi_{\bar{\mathsf{t}}}^{(\mathsf{SOS})}\,\rangle = \sum_{\mathbf{h}\in\{0,1\}^{\mathsf{N}}} \prod_{j=1}^{\mathsf{N}} Q_{\bar{\mathsf{t}}}(\xi_j - \eta h_j)\,|\,\mathbf{h}\,\rangle$$

where the coefficients $Q_{\bar{\mathsf{t}}}(\xi_j - \eta h_j)$ are (up to an overall normalization) characterized by

$$\frac{Q_{\bar{\mathsf{t}}}(\xi_j - \eta)}{Q_{\bar{\mathsf{t}}}(\xi_j)} = \frac{\bar{\mathsf{t}}(\xi_j)}{d(\xi_j - \eta)} = (-1)^{\mathsf{x}+\mathsf{y}+\mathsf{xy}} \frac{a(\xi_j)}{\bar{\mathsf{t}}(\xi_j - \eta)}.$$

# From antiperiodic SOS transfer matrix to quasi-periodic 8-vertex transfer matrix: the case $(x, y) \neq (0, 0)$

When $(x, y) \neq (0, 0)$, the vertex-IRF transformation $\mathbf{S}_q \equiv \mathbf{P} \circ S_q(\hat{\tau})$ is an isomorphism from $\bar{\mathbb{H}}^{(0)}$ (antip. SOS space of states) to $\mathbb{V}_N$ (8V space of states).

Define the (x, y)-twisted 8-vertex transfer matrix:

$$T^{(8V)}_{(x,y)}(\lambda) = \mathrm{tr}_0 \left[ (\sigma_0^x)^y \, (\sigma_0^z)^x \, M_0^{(8V)}(\lambda) \right]$$

It has the following action on the states $|\, \mathbf{v} \,\rangle$ of $\bar{\mathbb{H}}^{(0)}$:

$$T^{(8V)}_{(x,y)}(\lambda) \, \mathbf{S}_q \, |\, \mathbf{v} \,\rangle = (-1)^x \, i^{xy} \, \mathbf{S}_q \, \overline{\mathcal{T}}^{(SOS)}(\lambda) \, |\, \mathbf{v} \,\rangle.$$

Complete characterization of the (x, y)-twisted 8-vertex transfer matrix spectrum and eigenstates:

Let $(x, y) \neq (0, 0)$. If $|\, \Psi^{(SOS)}_{\bar{t}} \,\rangle \in \bar{\mathbb{H}}^{(0)}$ is an eigenvector of the antiperiodic SOS transfer matrix $\overline{\mathcal{T}}^{(SOS)}(\lambda)$ with eigenvalue $\bar{t}(\lambda)$, then

$$\mathbf{S}_q \, |\, \Psi^{(SOS)}_{\bar{t}} \,\rangle \in \mathbb{V}_N$$

is an eigenvector of the quasi-periodic 8-vertex transfer matrix $T^{(8V)}_{(x,y)}(\lambda)$ with eigenvalue

$$t^{(8V)}_{(x,y)}(\lambda) \equiv (-1)^x \, i^{xy} \, \bar{t}(\lambda),$$

and conversely.

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and conversely.

# From antiperiodic SOS transfer matrix to quasi-periodic 8-vertex transfer matrix: the case $(x, y) \neq (0, 0)$

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Define the (x, y)-twisted 8-vertex transfer matrix:

$$\mathrm{T}^{(8V)}_{(\mathrm{x},\mathrm{y})}(\lambda) = \mathrm{tr}_0 \left[ (\sigma_0^x)^{\mathrm{y}} \, (\sigma_0^z)^{\mathrm{x}} \, \mathrm{M}_0^{(8V)}(\lambda) \right]$$

It has the following action on the states $|\, \mathbf{v} \,\rangle$ of $\bar{\mathbb{H}}^{(0)}$:

$$\mathrm{T}^{(8V)}_{(\mathrm{x},\mathrm{y})}(\lambda) \, \mathbf{S}_q \,|\, \mathbf{v} \,\rangle = (-1)^{\mathrm{x}} \, i^{\mathrm{xy}} \, \mathbf{S}_q \, \overline{\mathcal{T}}^{(SOS)}(\lambda) \,|\, \mathbf{v} \,\rangle.$$

**Complete characterization of the (x, y)-twisted 8-vertex transfer matrix spectrum and eigenstates:**

Let $(\mathrm{x}, \mathrm{y}) \neq (0, 0)$. If $|\, \Psi_{\bar{\mathrm{t}}}^{(SOS)} \,\rangle \in \bar{\mathbb{H}}^{(0)}$ is an eigenvector of the antiperiodic SOS transfer matrix $\overline{\mathcal{T}}^{(SOS)}(\lambda)$ with eigenvalue $\bar{\mathrm{t}}(\lambda)$ , then

$$\mathbf{S}_q \,|\, \Psi_{\bar{\mathrm{t}}}^{(SOS)} \,\rangle \in \mathbb{V}_\mathrm{N}$$

is an eigenvector of the quasi-periodic 8-vertex transfer matrix $\mathrm{T}^{(8V)}_{(\mathrm{x},\mathrm{y})}(\lambda)$ with eigenvalue

$$\mathrm{t}^{(8V)}_{(\mathrm{x},\mathrm{y})}(\lambda) \equiv (-1)^{\mathrm{x}} \, i^{\mathrm{xy}} \, \bar{\mathrm{t}}(\lambda),$$

and conversely.

# From the antiperiodic SOS transfer matrix to the periodic 8-vertex transfer matrix in the case $N$ odd

When $(\mathrm{x},\mathrm{y})=(0,0)$, $\mathbf{S}_q$ is not bijective from $\bar{\mathbb{H}}^{(0)}$ to $\mathbb{V}_N$

but, from the symmetry of the periodic (resp. antiper.) transfer matrices:

$$\Gamma_z\, \mathrm{T}^{(8V)}_{(0,0)}(\lambda)\, \Gamma_z = \mathrm{T}^{(8V)}_{(0,0)}(\lambda), \quad \Gamma_z\, \overline{\mathcal{T}}^{(SOS)}(\lambda)\, \Gamma_z = -\overline{\mathcal{T}}^{(SOS)}(\lambda), \quad \text{with } \Gamma_z = \otimes_{n=1}^{N} \sigma_n^z$$

it is possible to define a second vertex-IRF transformation $\bar{\mathbf{S}}_q = \Gamma_z\, \mathbf{S}_q\, \Gamma_z$, such that

$$\mathrm{T}^{(8V)}_{(0,0)}(\lambda)\, \mathbf{S}_q\, |\, \mathbf{v}\,\rangle = \mathbf{S}_q\, \overline{\mathcal{T}}^{(SOS)}(\lambda)\, |\, \mathbf{v}\,\rangle$$

$$\mathrm{T}^{(8V)}_{(0,0)}(\lambda)\, \bar{\mathbf{S}}_q\, |\, \mathbf{v}\,\rangle = -\bar{\mathbf{S}}_q\, \overline{\mathcal{T}}^{(SOS)}(\lambda)\, |\, \mathbf{v}\,\rangle$$

⋆ the spectrum and eigenstates of $\overline{\mathcal{T}}^{(SOS)}(\lambda)$ can be decomposed into

- **a '+' part,** with eigenvalues $t_+(\lambda)$ and eigenstates $|\,\Psi^{(SOS)}_{t_+}\rangle$,
- **a '−' part,** with eigenv. $t_-(\lambda) = -t_+(\lambda)$ and eigenst. $|\,\Psi^{(SOS)}_{t_-}\rangle = \Gamma^z |\,\Psi^{(SOS)}_{t_+}\rangle$

⋆ $\ker \mathbf{S}_q \cap \ker \bar{\mathbf{S}}^{(0)}_q = \{0\}$ and

- $\ker \mathbf{S}_q$ is generated by the type '−' eigenstates $|\,\Psi^{(SOS)}_{t_-}\rangle$
- $\ker \bar{\mathbf{S}}_q$ is generated by the type '+' eigenstates $|\,\Psi^{(SOS)}_{t_+}\rangle$

# From the antiperiodic SOS transfer matrix to the periodic 8-vertex transfer matrix in the case $N$ odd

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# From the antiperiodic SOS transfer matrix to the periodic 8-vertex transfer matrix in the case $N$ odd

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but, from the symmetry of the periodic (resp. antiper.) transfer matrices:

$$\Gamma_z \, \mathrm{T}^{(8V)}_{(0,0)}(\lambda)\, \Gamma_z = \mathrm{T}^{(8V)}_{(0,0)}(\lambda), \quad \Gamma_z \, \overline{\mathcal{T}}^{(SOS)}(\lambda)\, \Gamma_z = -\overline{\mathcal{T}}^{(SOS)}(\lambda), \quad \text{with } \Gamma_z = \otimes_{n=1}^{N} \sigma_n^z$$

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# From the antiperiodic SOS transfer matrix to the periodic 8-vertex transfer matrix in the case $N$ odd

$\rightsquigarrow$ Complete characterization of the periodic 8-vertex transfer matrix spectrum and eigenstates for $N$ odd:

The spectrum of the periodic 8-vertex transfer matrix $\mathsf{T}^{(8V)}_{(0,0)}(\lambda)$ for N odd is

$$\Sigma^{(8V)}_{(0,0)} = \bar{\Sigma}^{(SOS)}_{+}, \tag{1}$$

where $\bar{\Sigma}^{(SOS)}_{+}$ is the '+' part of the antiperiodic SOS transfer matrix spectrum $\bar{\Sigma}^{(SOS)}$.

Each of the $2^{N-1}$ $\mathsf{T}^{(8V)}_{(0,0)}(\lambda)$-eigenvalues $\mathsf{t}(\lambda) \in \Sigma^{(8V)}_{(0,0)} = \bar{\Sigma}^{(SOS)}_{+}$ is doubly degenerated, with two linearly independent $\mathsf{T}^{(8V)}_{(0,0)}(\lambda)$-eigenvectors given by

$$\mathbf{S}_q \,|\, \Psi^{(SOS)}_{\mathsf{t}} \,\rangle \qquad \text{and} \qquad \overline{\mathbf{S}}_q \,\Gamma_z \,|\, \Psi^{(SOS)}_{\mathsf{t}} \,\rangle, \tag{2}$$

where $|\, \Psi^{(SOS)}_{\mathsf{t}} \,\rangle$ denotes the $\overline{\mathcal{T}}^{(SOS)}(\lambda)$-eigenvector with eigenvalue $\mathsf{t}(\lambda)$.

# Going further: from the discrete characterization of the spectrum to a functional $T - Q$ equation

**SOV characterization of the spectrum/eigenstates of the transfer matrix:**

- ⋆ eigenvalue $\bar{\mathrm{t}}(\lambda)$ characterized by
  - its functional form ("elliptic polynomial" of a certain type)
  - the fact that it satisfies a discrete system of equations at the (shifted) inhomogeneity parameters $\xi_n - \eta h_n$, $h_n \in \{0, 1\}$, i.e. that, for each $n \in \{1, \ldots, N\}$, there exists a non-zero vector $\mathbf{Q}^{(n)} \equiv \begin{pmatrix} q_n^{(0)} \\ q_n^{(1)} \end{pmatrix}$ s.t.

$$\bar{\mathrm{t}}(\xi_n - \eta h_n)\, q_n^{(h_n)} = (-1)^{\mathrm{x+y+xy}} a(\xi_n - \eta h_n)\, q_n^{(h_n+1)} + d(\xi_n - \eta h_n)\, q_n^{(h_n-1)}, \quad h_n \in \{0, 1\}$$

- ⋆ The corresponding eigenvector $|\, \Psi_{\bar{\mathrm{t}}}^{(\mathrm{SOS})} \rangle$ is constructed in terms of $\mathbf{Q}^{(n)}$

**Question:** Does it exist, for each $\bar{\mathrm{t}}(\lambda) \in \bar{\Sigma}^{(\mathrm{SOS})}$, an entire function $Q(\lambda)$ s.t.

- ⋆ for each $n \in \{1, \ldots, N\}$, $Q(\xi_n - \eta h_n) = q_n^{(h_n)}$
- ⋆ $\bar{\mathrm{t}}(\lambda)\, Q(\lambda) = (-1)^{\mathrm{x+y+xy}} a(\lambda)\, Q(\lambda - \eta) + d(\lambda)\, Q(\lambda + \eta)$ ?

If yes, what is the functional form of $Q(\lambda)$ ( $\rightsquigarrow$ **Bethe equations** ) ?

# From the discrete characterization of the spectrum to a functional $T-Q$ equation

**Question:** Does it exist, for each $\bar{\mathrm{t}}(\lambda) \in \bar{\Sigma}^{(\mathrm{SOS})}$, an entire function $Q(\lambda)$ s.t.

$$\bar{\mathrm{t}}(\lambda)\, Q(\lambda) = (-1)^{\mathrm{x}+\mathrm{y}+\mathrm{xy}} a(\lambda)\, Q(\lambda-\eta) + d(\lambda)\, Q(\lambda+\eta) \qquad (3)$$

and $(Q(\xi_n), Q(\xi_n-\eta)) \neq (0,0)$ ?
If yes, what is the functional form of $Q(\lambda)$ ( $\rightsquigarrow$ **Bethe equations** ) ?

*Remark 1.* The algebraic construction of the Q-operator (and the knowledge of the functional form of its eigenvalues) provides in principle a solution to this problem

*Remark 2.* From analyticity/periodicity arguments, one can guess the functional form of $Q(\lambda)$.

$\rightsquigarrow$ Ansatz: $Q(\lambda) = e^{\alpha\lambda} \prod_{j=1}^{N} \theta_{\mathrm{X}}(\lambda-\lambda_j)$

with $\theta_{\mathrm{X}}(\lambda) = \begin{cases} \theta_1\left(\frac{\lambda}{2} \,\middle|\, \frac{\omega}{2}\right) & \text{if } \mathrm{x}=0, \\ \theta_1(\lambda|2\omega) & \text{if } \mathrm{y}=0, \\ e^{i\frac{\lambda}{2}}\, \theta_1\left(\frac{\lambda}{2} \,\middle|\, \omega\right) \theta_1\left(\frac{\lambda+\pi+\pi\omega}{2} \,\middle|\, \omega\right) & \text{if } \mathrm{x}=\mathrm{y}. \end{cases}$

$+$ restrictions on $\alpha$ and $\sum_i \lambda_i$.
Is it possible to prove the completeness of this solution ?

# From the discrete characterization of the spectrum to a functional $T - Q$ equation

Let $\bar{\mathsf{t}}(\lambda)$ be an eigenvalue of the antiperiodic SOS transfer matrix $\overline{\mathcal{T}}(\lambda)$.
Then, if N is even, there exists a unique function $Q(\lambda)$ of the form

$$Q(\lambda) = \prod_{j=1}^{\mathsf{N}} \theta_{\mathsf{X}}(\lambda - \lambda_j), \tag{4}$$

for some set of roots $\lambda_1, \ldots, \lambda_{\mathsf{N}} \in \mathbb{C}$, such that $\mathsf{t}(\lambda)$ and $Q(\lambda)$ satisfy the $T$-$Q$ functional equation

$$\bar{\mathsf{t}}(\lambda)\, Q(\lambda) = (-1)^{\mathsf{x}+\mathsf{y}+\mathsf{xy}} a(\lambda)\, Q(\lambda - \eta) + d(\lambda)\, Q(\lambda + \eta).$$

In (4), the notation $\theta_{\mathsf{X}}(\lambda)$ stands for the function

$$\theta_{\mathsf{X}}(\lambda) = \begin{cases} \theta_1\left(\frac{\lambda}{2} \,\middle|\, \frac{\omega}{2}\right) & \text{if } (\mathsf{x},\mathsf{y}) = (0,1), \\ \theta_1(\lambda|2\omega) & \text{if } (\mathsf{x},\mathsf{y}) = (1,0), \\ e^{i\frac{\lambda}{2}}\, \theta_1\left(\frac{\lambda}{2} \,\middle|\, \omega\right) \theta_1\left(\frac{\lambda+\pi+\pi\omega}{2} \,\middle|\, \omega\right) & \text{if } (\mathsf{x},\mathsf{y}) = (1,1). \end{cases} \tag{5}$$

Remark. In the XXZ limit ($\mathsf{y} = 1, \omega \to +i\infty$) we have shown the completeness of the solution

$$Q(\lambda) = \prod_{j=1}^{\mathsf{N}} \sin\left(\frac{\lambda - \lambda_j}{2}\right)$$

for $N$ even or odd

# From the discrete characterization of the spectrum to a functional $T-Q$ equation

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for some set of roots $\lambda_1, \ldots, \lambda_{\mathrm{N}} \in \mathbb{C}$, such that $\mathrm{t}(\lambda)$ and $Q(\lambda)$ satisfy the $T$-$Q$ functional equation

$$\bar{\mathrm{t}}(\lambda)\, Q(\lambda) = (-1)^{\mathrm{x}+\mathrm{y}+\mathrm{xy}} a(\lambda)\, Q(\lambda - \eta) + d(\lambda)\, Q(\lambda + \eta).$$

In (4), the notation $\theta_{\mathrm{X}}(\lambda)$ stands for the function

$$\theta_X(\lambda) = \begin{cases} \theta_1\left(\frac{\lambda}{2} \,\middle|\, \frac{\omega}{2}\right) & \text{if } \ (\mathrm{x},\mathrm{y}) = (0,1), \\ \theta_1(\lambda|2\omega) & \text{if } \ (\mathrm{x},\mathrm{y}) = (1,0), \\ e^{i\frac{\lambda}{2}}\, \theta_1\left(\frac{\lambda}{2} \,\middle|\, \omega\right) \theta_1\left(\frac{\lambda+\pi+\pi\omega}{2} \,\middle|\, \omega\right) & \text{if } \ (\mathrm{x},\mathrm{y}) = (1,1). \end{cases} \tag{5}$$

$\rightsquigarrow$ Complete characterization of the (SOS and hence 8V) spectrum and eigenstates in terms of the solutions of the Bethe equations:

$$(-1)^{\mathrm{x}+\mathrm{y}+\mathrm{xy}} a(\lambda_j) \prod_{\substack{k=1 \\ k\neq j}}^{\mathrm{N}} \frac{\theta_X(\lambda_j - \lambda_k - \eta)}{\theta_X(\lambda_j - \lambda_k)} + d(\lambda_j) \prod_{\substack{k=1 \\ k\neq j}}^{\mathrm{N}} \frac{\theta_X(\lambda_j - \lambda_k + \eta)}{\theta_X(\lambda_j - \lambda_k)} = 0, \quad 1 \leq j \leq \mathrm{N}$$

# Conclusion

- the **periodic 8-vertex/XYZ model** with an **even** number of sites can be solved by relating the periodic 8-vertex transfer matrix with the **periodic** SOS transfer matrix

  $\rightsquigarrow$ solution by Bethe ansatz (cf. Baxter's work...)

- the **periodic 8-vertex/XYZ model** with an **odd** number of sites, as well as the **twisted** cases, can be solved by relating the (periodic or twisted) 8-vertex transfer matrix with the **antiperiodic** SOS transfer matrix

  $\rightsquigarrow$ solution by Separation of Variables

  $\rightsquigarrow$ complete description of the transfer matrix spectrum and eigenstates in terms of solutions of discrete equations evaluated at the inhomogeneity parameters of the model

  $\rightsquigarrow$ it is possible to reformulate this description in terms of some particular classes of solutions of a functional $T$-$Q$ equation

  $\rightsquigarrow$ description in terms of Bethe-type equations enabling one to study the homogeneous/thermodynamic limit (completeness proven for $N$ even)