Constructing GGE for QFTs

Miłosz Panfil, SISSA

Fabian Essler and Giuseppe Mussardo

PRA 91 (2015) and more

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Problem

- Thermalization of a closed quantum system
- Low dimensions → extra conservation laws can coexists with nontrivial dynamics
- After a long time system thermalizes to the GGE (Rigol, Dunjko, Olshanii)

\[
\rho_{GE} = \exp(-\beta H) \quad \rho_{GGE} = \exp\left(-\sum_n \lambda_n I_n\right)
\]

How to choose conserved charges?
Integrable field theories

- Scattering is purely elastic (Zamolodchikov-Faddeev algebra)
- Momentum modes are conserved
- Construct the GGE with them

\[ \rho_{\text{GGE}} = \exp \left( - \int \frac{dk}{2\pi} \lambda(k) N(k) \right) \]

It can work! (Fioretto, Mussardo 2010)

\[ \lim_{t \to 0} \langle \Psi(t) | \mathcal{O} | \Psi(t) \rangle = \text{tr} \left( \rho_{\text{GGE}} \mathcal{O} \right) \]
Two problems and one solution

- Only works for an infinite system
- Is in the momentum space

Solution → Find better conserved charges

Require:

\[ N(k) \leftrightarrow I_n \]
Outline

1. Construction of the GGE
2. Making sense out of it
Part 1: Construction of the GGE
Ising field theory

Two interacting Majorana fermions

\[ H = \int dx \frac{i}{2} [R(x)\partial_x R(x) - L(x)\partial_x L(x)] + imR(x)L(x) \]

Bogoliubov transformation

\[ H = \int \frac{dk}{2\pi} \omega(k)N(k), \quad \omega(k) = \sqrt{m^2 + k^2}. \]

Mode occupation operator

\[ N(k) = Z^\dagger(k)Z(k), \quad \{Z^\dagger(k)Z(q)\} = 2\pi\delta(k - q) \]
Ultra-local conserved charges

In the real space

\[ I_+^n = \frac{i}{2} \int dx \left[ R(x) \partial_x^{2n+1} R(x) + L(x) \partial_x^{2n+1} L(x) \right] \]

\[ I_-^n = \frac{i}{2} \int dx \left[ R(x) \partial_x^{2n+1} R(x) - L(x) \partial_x^{2n+1} L(x) + 2mR(x) \partial_x^{2n} L(x) \right] \]
Ultra-local conserved charges

In the momentum space:

\[ I_n^{\pm} = \int \frac{dk}{2\pi} \epsilon^{\pm}_n(k) N(k), \]

\[ \epsilon^+_n(k) = \omega(k) k^{2n}, \]
\[ \epsilon^-_n(k) = k^{2n+1} \]

Incomplete basis:

\[ I_n^{\pm} = \int \frac{dk}{2\pi} \epsilon^{\pm}_n(k) [f(k) + N(k)], \]

\[ N(k) \leftrightarrow I_n \]
Lattice Ising model

\[ H_{\text{lattice}} = \frac{iJ}{2} \sum_j a_{2j} \left[ a_{2j+1} - ha_{2j-1} \right] \]

Conserved charges

\[ I^+_n = \frac{iJ}{2} \sum_{j, \sigma=\pm 1} a_{2j} \left[ a_{2j+2n\sigma+1} - ha_{2j+2n\sigma-1} \right], \]

\[ I^-_n = \frac{iJ}{2} \sum_j a_{2j} a_{2j+2n} - a_{2j-1} a_{2j+2n} \]
Continuum limit redone

a) $n$ - fixed

$a_0 \to 0$

b) $n \alpha_0$ - fixed

$a_0 \to 0$
Quasi-local charges

\[ I^+(\alpha) = \frac{i}{4} \int \, dx \left[ R(x) + L(x) \right] (\partial_x - m) \times \left[ R(x + \alpha) - L(x + \alpha) + (\alpha \to -\alpha) \right] \]

\[ I^-(\alpha) = \frac{i}{2} \int \, dx \left[ R(x)R(x + \alpha) + L(x)L(x + \alpha) \right] \]
Quasi-local charges

\[ I^{\pm}(\alpha) = \int \frac{dk}{2\pi} \epsilon^{\pm}(k, \alpha) N(k), \]
\[ \epsilon^+_n(k, \alpha) = \omega(k) \cos(k\alpha), \]
\[ \epsilon^-_n(k, \alpha) = \sin(k\alpha) \]

\[ N(k) \leftrightarrow I(\alpha) \]

Correct GGE:

\[ \rho_{\text{GGE}} = \exp \left( - \sum_{\sigma = \pm} \int d\alpha \lambda^{\sigma}(\alpha) I^{\sigma}(\alpha) \right) \]
Ultra-local charges in QFT's cannot be used to construct the GGE, are incomplete. 

Derived a complete set of quasi-local charges for the Ising 

Also for the Lieb-Liniger (PRA 91, 2015)
Part 2: making sense out of the GGE
Problem

Physical theory requiring continuum of information!
Solution

Give less information!
Mass quench

- Initial state (Rossini et al)

\[ |\Psi(0)\rangle = \exp\left(-i \int_{0}^{\infty} \frac{dk}{2\pi} K(k)Z^{\dagger}(-k)Z^{\dagger}(k)\right) |0\rangle \]

\[ K(k) = \tan\left(\frac{1}{2} \arctan(k/m) - \frac{1}{2} \arctan(k/m_0)\right) \]

- Momentum distribution

\[ N(k) = \frac{K^2(k)}{1 + K^2(k)} \]
Local correlation function

• In the real space

\[ g(x) = \langle (R(x) - iL(x)) (R(0) + iL(0)) \rangle \]

• Through mode occupation numbers

\[ g(x) = \int \frac{dk}{2\pi} \frac{N(k)}{2\omega(k)} \cos(kx) \]
Truncated GGE

\[ \rho_{\text{GGE}} = \exp \left( - \int_0^\infty d\alpha \, \lambda^\sigma(\alpha) I^+(\alpha) \right) \]

- Finite interval

\[ \rho_{\text{GGE}} = \exp \left( - \int_0^\gamma d\alpha \, \lambda^+(\alpha) I^+(\alpha) \right) \]

- Discretize

\[ \rho_{\text{GGE}} = \exp \left( - \sum_{m=1}^N \lambda^+(\alpha_m) I^+(\alpha_m) \right) \]

Motivated by M. Fagotti and F. Essler
Results

\[ g(x) \]

\[ g_{\gamma,N}(x) \text{ with } \gamma=4, N=5 \]

\[ g_{\gamma,N}(x) \text{ with } \gamma=4, N=10 \]
Results

\[ g(x) \]

with \( N = 20 \)

\[ g_N(x) \] with \( N = 40 \)

\begin{align*}
\text{with } x = & 0.002 \\
\text{with } x = & 0.004 \\
\text{with } x = & 0.006 \\
\text{with } x = & 0.008 \\
\text{with } x = & 0.010 \\
\text{with } x = & 0.012
\end{align*}
**ultra-local charges**

- $g(x)$
- Quasi-local, $N=10$
- Ultra-local, $N=10$

![Graph showing the comparison of $g(x)$, Quasi-local, and Ultra-local with $N=10$.](image-url)
Summary

• Setting up the GGE might be tricky

• In the field theory, ultra-local charges are not enough

• Quasi-local charges are a way to go

• Real space representation of the quasi-local charges for more complicated QFT's?

Thank you!