Microscopic approach to a class of 1D quantum critical models

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Outline

Large-distance asymptotics in massless models

- $\mathtt{H}_L$ lattice Hamiltonian in 1D with L sites and periodic bc
- Massless model ≡ continuum of excited states above $|\,G.S.\,\rangle_L$ in thermodynamic limit $L \to +\infty$
  - Zero temperature correlation functions $\Big\langle \mathrm{o}_1^{(\alpha)} \mathrm{o}_{1+x}^{(\beta)} \Big\rangle \equiv \lim_{L\to+\infty} \Big\{ {}_L\langle\, G.S.\,| \mathrm{o}_1^{(\alpha)} \mathrm{o}_{1+x}^{(\beta)} |\, G.S.\,\rangle_L \Big\}$
  - Large-x asymptotics are algebraic

$$\Big\langle \mathrm{o}_1^{(\alpha)} \mathrm{o}_{1+x}^{(\beta)} \Big\rangle \simeq \Big\langle \mathrm{o}_1^{(\alpha)} \Big\rangle \Big\langle \mathrm{o}_1^{(\beta)} \Big\rangle + \frac{\mathscr{A}_1^{(\alpha\beta)}}{x^{\delta_1^{(\alpha\beta)}}} + \frac{\mathscr{A}_2^{(\alpha\beta)}}{x^{\delta_2^{(\alpha\beta)}}} \cos\big(x\phi_2\big) + \cdots$$

- Critical exponents $\delta_k^{(\alpha\beta)}$ expected to be universal

Correspondence with a CFT

- Prediction for the structure of asymptotic expansions in massless models :
  i) ('70 **Polyakov**) Conformal invariance of correlation functions in long-distance regime.
  ii) ('86 **Cardy**,'86 **Affleck**) Central charge ⇝ finite-size corrections to ground state energy.

$$E_{G.S.} = L\varepsilon - c\frac{\pi v_F}{6L} + \mathrm{O}\Big(\frac{1}{L^2}\Big) \qquad \text{and} \qquad E_{\mathrm{ex}} - E_{G.S.} = \frac{2\pi v_F}{L}\Big(\Delta_k^+ + \Delta_k^- + n_{\mathrm{ex}}\Big) + \mathrm{O}\Big(\frac{1}{L^2}\Big)$$

- Operator $o_{1+x}^{(\beta)}$ connects $|\,G.S.\,\rangle_L$ and $|\,\Delta_k^+; n_{\mathrm{ex}}\,\rangle_L$ for $k \in I^{(\alpha)}$

$$o_{1+x}^{(\beta)} \hookrightarrow \sum_{k\in I^{(\alpha)}} C_k^{(\beta)} \cdot \Psi_{\Delta_k^{\pm}}(z,\overline{z}) \qquad z = \mathrm{e}^{\frac{2i\pi}{L}x}$$

⊛ n-point functions in original model at $x \to +\infty$ computed from CFT n-point functions.

$$\Big\langle o_1^{(\alpha)} o_{1+x}^{(\beta)} \Big\rangle \simeq \Big\langle o_1^{(\alpha)} \Big\rangle \Big\langle o_1^{(\beta)} \Big\rangle + \sum_{k\in I^{(\alpha)}\cap I^{(\beta)}} \mathcal{A}_k^{(\alpha\beta)} \frac{\mathrm{e}^{2\mathrm{i}x\ell_k p_F}}{x^{2\Delta_k^+ + 2\Delta_k^-}} \cdot \big(1 + \mathrm{o}(1)\big)$$

Various tests

- Test CFT structure of spectrum from Bethe Ansatz
  '87-'95 (**Batchelor, Destri, DeVega, Klumper, Pearce, Woynarowich, Wehner, Zittartz**)
  ✔ OK for conformal structure of the spectrum at $L \to +\infty$.
- Universality of some critical exponents for perturbation of 2D Ising
  (**'00 Penson, Spencer** , **'04 Mastropietro**)
- *Ab inicio* derivation of the large-x asymptotics in XXZ
  (**'08 Kitanine, K., Maillet, Slavnov, Terras**)

Question

- Can one provide an *ab inicio* construction of the operator correspondence ?

Answer

Question

- Can one provide an *ab inicio* construction of the operator correspondence ?

Answer : YES

$$\Big\langle \mathrm{o}^{(\alpha_1)}_{x_1+1} \cdots \mathrm{o}^{(\alpha_r)}_{x_r+1} \Big\rangle_{L\ ;\mathfrak{h}_{\mathrm{phys}}} \simeq \Big\langle \mathscr{O}^{(\alpha_1)}(\omega_1) \cdots \mathscr{O}^{(\alpha_r)}(\omega_r) \Big\rangle_{\mathfrak{h}_{\mathrm{eff}}}$$

in the limit $L >> |x_a - x_b| >> 1$

⊛ Property of **Matrix elements** and **NOT** of spectrum:

$$\mathcal{F}(\mu_h, \mu_p) \sim \frac{1}{L^2} (\mu_p - k_F)^{\Delta_k^+ - 1} (k_F - \mu_h)^{\Delta_k^- - 1} \mathscr{F}_{\mathrm{reg}}(\mu_h, \mu_p)$$

The relative excitation momentum

- Generalities
  - $\mathtt{H}_L$ 1D lattice Hamiltonian with conservation of bare-particle number (e.g. S^z)
  - G.S. built of N bare particles, $N/L \to D$, rapidities which densify on Fermi zone $[-k_F\,;k_F]$
  - Excited states in sectors with $N' = N + s$ bare-particles, s finite
  - Rapidities densify on $[-k_F\,;k_F]$ with some holes $\{\mu_{h_a^{(s)}}\}_1^n$ and particles $\{\mu_{p_a^{(s)}}\}_1^n$
  - Excitations on Fermi boundary parametrised by sets of integers $\mathcal{J}_{n;m}^{(s)} = \Big\{\{p_a^{(s)}\}_1^n\,;\,\{h_a^{(s)}\}_1^m\Big\}$

$$\mathcal{J}^{(s)}_{n_{p;-}^{(s)};n_{h;-}^{(s)}} \cup \mathcal{J}^{(s)}_{n_{p;+}^{(s)};n_{h;+}^{(s)}} \qquad \text{with} \qquad \left\{ \begin{array}{c} n_{p;+}^{(s)} + n_{p;-}^{(s)} = n_{h;+}^{(s)} + n_{h;-}^{(s)} \\ \ell_s = n_{p;+}^{(s)} - n_{h;+}^{(s)} = n_{h;-}^{(s)} - n_{p;-}^{(s)} \end{array} \right.$$

- Relative excitation momentum

$$\Delta\mathcal{P}_{\mathrm{ex}} \equiv \mathcal{P}_{\mathrm{ex}} - \mathcal{P}_{\mathrm{GS}} = \sum_{a=1}^{n} \Big\{ p(\mu_{p_a^{(s)}}) - p(\mu_{h_a^{(s)}}) \Big\} + \mathrm{O}\Big(\frac{1}{L}\Big)$$

  - Momentum of Fermi-boundary excitations

$$\Delta\mathcal{P}_{\mathrm{ex}} = 2\ell_s p_F + \frac{2\pi}{L}\Big\{ \sum_{a=1}^{n_{p;+}^{(s)}} (p_{a;+}^{(s)} - 1) + \sum_{a=1}^{n_{h;+}^{(s)}} h_{a;+}^{(s)} - \sum_{a=1}^{n_{p;-}^{(s)}} p_{a;-}^{(s)} - \sum_{a=1}^{n_{h;-}^{(s)}} (h_{a;-}^{(s)} - 1) \Big\} + \dots$$

The form factors I

- Model endowed with family of local operators $\mathrm{o}_{1+x}^{(\alpha)}$ having pure s transitions

$$\langle\, \mathrm{ex}_{\mathrm{out}}; s_{\mathrm{out}} \,|\mathrm{o}_{1+x}^{(\alpha)}|\, \mathrm{ex}_{\mathrm{in}}; s_{\mathrm{in}} \,\rangle \neq 0 \quad \text{if} \quad s_{\mathrm{in}} - s_{\mathrm{out}} = o_\alpha$$

- Form factors on Fermi boundary states in relative $\ell = \ell_{\mathrm{out}} - \ell_{\mathrm{in}}$ class

$$\mathrm{ex}_{\mathrm{out}}; s \rightsquigarrow \mathcal{J}_{m_{p;+};m_{h;+}} \cup \mathcal{J}_{m_{p;-};m_{h;-}} \quad \text{and} \quad \mathrm{ex}_{\mathrm{in}}; s + o_\alpha \rightsquigarrow \mathcal{J}_{n_{p;+};n_{h;+}} \cup \mathcal{J}_{n_{p;-};n_{h;-}}$$

$$\langle\, \mathrm{ex}_{\mathrm{out}}; s \,|\mathrm{o}_{1+x}^{(\alpha)}|\, \mathrm{ex}_{\mathrm{in}}; s + o_\alpha \,\rangle = \Big\{ \mathrm{e}^{2\mathrm{i}p_F x} \cdot \omega \Big\}^{\ell} \cdot C_\ell(\nu_\alpha^+, \nu_\alpha^-) \cdot \mathcal{F}_\ell\big(\mathrm{o}^{(\alpha)}\big) \cdot \Big(\frac{2\pi}{L}\Big)^{\Delta_\ell^{(\alpha)}}$$
$$\times \mathscr{F}\Big[\mathcal{J}_{m_{p;+};m_{h;+}}; \mathcal{J}_{n_{p;+};n_{h;+}} \mid \nu_\alpha^+, \omega\Big] \cdot \mathscr{F}\Big[\mathcal{J}_{m_{p;-};m_{h;-}}; \mathcal{J}_{n_{p;-};n_{h;-}} \mid -\nu_\alpha^-, \omega^{-1}\Big] \cdot \Big(1 + \mathrm{O}\Big(\frac{\ln L}{L}\Big)\Big).$$

- $\nu_\alpha^{\pm} = \pm\nu(\ell K + o_\alpha / K)$ excited state and operator info $\rightsquigarrow$ fix decay in volume

$$\Delta_\ell^{(\alpha)} = \frac{1}{2}\Big\{ (\nu_\alpha^+ + \ell)^2 + (\nu_\alpha^- + \ell)^2 \Big\} \qquad \text{and} \qquad \omega = \mathrm{e}^{2\mathrm{i}\pi \frac{x}{L}}$$

- Lowest energy ℓ class form factor

$$\mathcal{F}_\ell\big(\mathrm{o}^{(\alpha)}\big) = \lim_{L \to +\infty} \Big\{ \Big(\frac{L}{2\pi}\Big)^{\Delta_\ell^{(\alpha)}} \langle\, \mathrm{ex}_{\mathrm{low}}(\ell); -o_\alpha \,|\mathrm{o}_1^{(\alpha)}|\, G.S. \,\rangle \Big\}$$

- Normalisation $C_\ell(\nu, \mu) = G(1+\mu)G(1-\nu)/G(1+\mu+\ell)G(1-\nu-\ell)$

The form factors II

$$\mathcal{J}_{n_p;n_h} = \Big\{ \{p_a\}_1^{n_p} \; ; \; \{h_a\}_1^{n_h} \Big\} \qquad \text{and} \qquad \mathcal{J}_{n_k;n_t} = \Big\{ \{k_a\}_1^{n_k} \; ; \; \{t_a\}_1^{n_t} \Big\}$$

- Microscopic form factor

$$\mathscr{F}\Big(\mathcal{J}_{n_p;n_h}; \mathcal{J}_{n_k;n_t} \mid \nu, \omega\Big) = (-1)^{\varkappa} \Big(\frac{\sin[\pi\nu]}{\pi}\Big)^{n_p-n_h} \mathscr{D}\Big(\mathcal{J}_{n_p;n_h} | \nu, \omega\Big) \cdot \mathscr{D}\Big(\mathcal{J}_{n_k;n_t} | -\nu, \frac{1}{\omega}\Big) \cdot \varpi\Big(\mathcal{J}_{n_p;n_h}; \mathcal{J}_{n_k;n_t} \mid \nu\Big)$$

  - ⋆ In/out state building block (Z-measure if $n_p = n_h$)

$$\mathscr{D}\Big(J_{n_p;n_h} \mid \nu; \omega\Big) = \Big(\frac{\sin[\pi\nu]}{\pi}\Big)^{n_h} \frac{\prod\limits_{a>b}^{n_p} (p_a - p_b) \cdot \prod\limits_{a>b}^{n_h} (h_a - h_b)}{\prod\limits_{a=1}^{n_p} \prod\limits_{b=1}^{n_h} (p_a + h_b - 1)} \prod_{a=1}^{n_p} \Big\{ \omega^{p_a - 1} \frac{\Gamma(p_a + \nu)}{\Gamma(p_a)} \Big\} \prod_{a=1}^{n_h} \Big\{ \omega^{h_a} \frac{\Gamma(h_a - \nu)}{\Gamma(h_a)} \Big\}$$

  - ⋆ In/out state interaction

$$\varpi\Big(\mathcal{J}_{n_p;n_h}; \mathcal{J}_{n_k;n_t} \mid \nu\Big) = \prod_{a=1}^{n_h} \left\{ \frac{\prod\limits_{b=1}^{n_k} \big(1 - k_b - h_a + \nu\big)}{\prod\limits_{b=1}^{n_t} \big(t_b - h_a + \nu\big)} \right\} \cdot \prod_{a=1}^{n_p} \left\{ \frac{\prod\limits_{b=1}^{n_t} \big(p_a + t_b + \nu - 1\big)}{\prod\limits_{b=1}^{n_k} \big(p_a - k_b + \nu\big)} \right\}$$

The Hilbert space

- Vectors
  - Fermionic operators $\{\psi_n\}_{n\in\mathbb{Z}}$ and $*$-associates $\{\psi_n^*\}_{n\in\mathbb{Z}}$: $\{\psi_n, \psi_m^*\} = \delta_{n,m}$
  - Vacuum vector $|\,0\,\rangle$: $\psi_n|\,0\,\rangle = 0 \text{ for } n<0$ and $\psi_n^*|\,0\,\rangle = 0 \text{ for } n\geq 0$
  - s-shifted vacua $|\,s\,\rangle = \begin{cases} \psi_{s-1}\cdots\psi_0|\,0\,\rangle & s>0 \\ \psi_s^*\cdots\psi_{-1}^*|\,0\,\rangle & s<0 \end{cases}$
  - Bases labelled by sets of integers $\mathcal{J}_{n_p;n_h} = \Big\{\{p_a\}_1^{n_p}\;;\;\{h_a\}_1^{n_h}\Big\}$:

$$|\,\mathcal{J}_{n_p;n_h}\,\rangle = \psi^*_{-h_1}\cdots\psi^*_{-h_{n_h}}\cdot\psi_{p_{n_p}-1}\cdots\psi_{p_1-1}|\,0\,\rangle$$
$$|\,\mathcal{J}_{n_p;n_h};s\,\rangle = \psi^*_{s-h_1}\cdots\psi^*_{s-h_{n_h}}\cdot\psi_{p_{n_p}+s-1}\cdots\psi_{p_1+s-1}|\,s\,\rangle$$

- Hilbert space $\mathfrak{h} = \text{span}\Big\{|\,\mathcal{J}_{n_p;n_h}\,\rangle \text{ with } n_p,n_h\in\mathbb{N} \text{ and } \begin{array}{l} 1\leq p_1<\cdots<p_{n_p} \\ 1\leq h_1<\cdots<h_{n_h}\end{array} \quad p_a,h_a\in\mathbb{N}^*\Big\}.$

The vertex operators

- Operators
  - Momentum operator $\mathrm{e}^{P} | J_{n,n}; s \rangle = | J_{n,n}, s+1 \rangle$
  - Current modes $J_k \; = \; \sum_{j \in \mathbb{Z}} \psi_j \psi^*_{j+k} \quad , \; k \neq 0 \quad , \qquad [J_k, J_\ell] \; = \; k \delta_{k,-\ell}$
  - Current operators $\mathscr{J}_{\pm}(\nu, \omega) \; = \; \mp \nu \sum_{k \geq 1} \frac{1}{k} \omega^{\mp k} \cdot J_{\pm k}$
  - r-shifted vertex operators $\mathscr{V}(\nu, r \mid \omega) \; = \; \mathrm{e}^{\mathscr{J}_{-}(\nu+r, \omega)} \cdot \mathrm{e}^{\mathscr{J}_{+}(\nu+r,\omega)} \cdot \mathrm{e}^{rP}$
- Matrix elements (**'15 K, Maillet**)

$$\langle \mathcal{J}_{n_p;n_h} | \mathscr{V}(\nu, r \mid \omega) | \mathcal{J}_{n_k;n_t} \rangle \; = \; (-1)^{\frac{r(r+1)}{2}} \cdot \frac{\delta_{n_p - n_h, n_k - n_t + r}}{\omega^{\frac{r(r-1)}{2} + r(n_k - n_t)}} \cdot \frac{G(1-\nu)}{G(1-\nu-r)} \cdot \mathscr{F}\Big(\mathcal{J}_{n_p;n_h} ; \mathcal{J}_{n_k;n_t} \mid \nu, \omega \Big)$$

  - Well known result for $r = 0$ and $n_p = n_h = 0$. (Giambelli rep. Schur functions)
  - $r = 0$ and $n_p = n_h = 0$ correspond to a Z-measure on partitions.
  - General case interpreted as explicit representation for specific skew Schur functions

The correspondence of matrix elements

- Two copies of $\mathfrak{h}$: $\mathfrak{h}_{\mathrm{eff}} = \mathfrak{h}_L \otimes \mathfrak{h}_R$
- Distance dependent phase $\omega = \mathrm{e}^{2i\pi \frac{x}{L}}$
- Associate to local operator $\mathrm{o}^{(\alpha)}_{x+1}$ on $\mathfrak{h}_{\mathrm{phys}}$ the operator on $\mathfrak{h}_{\mathrm{eff}}$:

$$\mathscr{O}^{(\alpha)}(\omega) = \sum_{\ell \in \mathbb{Z}} \mathcal{F}_\ell\left(\mathrm{o}^{(\alpha)}\right) \cdot \left(\frac{2\pi}{L}\right)^{\Delta_\ell^{(\alpha)}} \cdot \mathrm{e}^{2ip_F \ell x} \cdot \mathscr{V}_L\Big(-\nu_\alpha^-, -\ell \mid \omega^{-1}\Big) \cdot \mathscr{V}_R\Big(\nu_\alpha^+, \ell - o_\alpha \mid \omega\Big) .$$

- Let $\mathcal{J}^{(s)}_{m_{p;\pm};m_{h;\pm}}$ and $\mathcal{J}^{(s+o_\alpha)}_{n_{p;\pm};n_{h;\pm}}$ parametrise low-lying excited states.

$$\langle \mathrm{ex_{out}}; s \,|\mathrm{o}^{(\alpha)}_{1+x}|\, \mathrm{ex_{in}}; s + o_\alpha \rangle = \Big\{ \langle s; \mathcal{J}_{m_{p;-};m_{h;-}} | \otimes \langle s; \mathcal{J}_{m_{p;+};m_{h;+}} | \Big\} \mathscr{O}^{(\alpha)}(\omega) \Big\{ | \mathcal{J}_{n_{p;-};n_{h;-}}; s \rangle \otimes | \mathcal{J}_{n_{p;+};n_{h;+}}; s + o_r \rangle \Big\} \cdot \Big(1 + \mathrm{O}\Big(\frac{\ln L}{L}\Big)\Big)$$

Form factor expansion and correspondence

- Heuristic saddle-point like analysis of form factor series $0_1^{(\alpha')} = \left(0_1^{(\alpha)}\right)^{\dagger}$

$$\langle\, G.S.\,|0_{1+x}^{(\alpha)} 0_1^{(\alpha')}|\,G.S.\,\rangle \;=\; \sum_{\mathrm{ex}} \mathrm{e}^{\mathrm{i}x\Delta\mathcal{P}_{\mathrm{ex}}} \langle\, G.S.\,|0_1^{(\alpha)}|\,\mathrm{ex};\, o_\alpha\,\rangle\langle\,\mathrm{ex};\, o_\alpha\,|0_1^{(\alpha')}|\,G.S.\,\rangle$$

Approximations for $x \to +\infty$ asymptotics

- Stationary points $\rightsquigarrow$ endpoints of Fermi zone $\equiv$ discrete integer spectrum;

$$\langle\, G.S.\,|0_{1+x}^{(\alpha)} 0_1^{(\alpha')}|\,G.S.\,\rangle \sim \sum_{\mathcal{J}^{(o_\alpha)}_{n_{p;\pm};n_{h;\pm}}} \langle\, G.S.\,|0_{1+x}^{(\alpha)}|\,\mathcal{J}^{(o_\alpha)}_{n_{p;\pm};n_{h;\pm}}\,\rangle\langle\,\mathcal{J}^{(o_\alpha)}_{n_{p;\pm};n_{h;\pm}}\,|0_1^{(\alpha')}|\,G.S.\,\rangle$$

$$\sim \sum_{\mathcal{J}^{(o_\alpha)}_{n_{p;\pm};n_{h;\pm}}} \langle\, 0\,|\mathscr{O}^{(\alpha)}(\omega)|\,\mathcal{J}^{(o_\alpha)}_{n_{p;+};n_{h;+}} \otimes \mathcal{J}^{(o_\alpha)}_{n_{p;+};n_{h;+}}\,\rangle\langle\,\mathcal{J}^{(o_\alpha)}_{n_{p;+};n_{h;+}} \otimes \mathcal{J}^{(o_\alpha)}_{n_{p;+};n_{h;+}}\,|\mathscr{O}^{(\alpha')}(1)|\,0\,\rangle$$

$$\sim \langle\, 0\,|\mathscr{O}^{(\alpha)}(\omega)\mathscr{O}^{(\alpha')}(1)|\,0\,\rangle$$

Comments on the realisability of the model

♦ Supporting facts

- Works for NLSM, XXZ (particle-hole spectrum) and Calogero-Sutherland ('**09-'11** **Kitanine,K.,Maillet,Slavnov,Terras** , '**12** **Shashi,Panfil,Caux,Imambekov**)
- Some features checked perturbatively for spinless fermions with pair interactions ('**11** **Shashi,Glazman,Caux,Imambekov**)
- Rigorous check of universal relations for exponents for spin systems in perturbation ('**09** **Benfatto, Mastropietro** , '**13** **Benfatto, Falco, Mastropietro**)
- Plethora of systems argued to be described by non-linear Luttinger liquid (review '**11** **Imambekov, Schmidt, Glazman**)
- Non-linear Luttinger Liquid $\subset$ physical model

Conclusion and perspectives

Review of the results

- ✔ Mapping of local operators to operators in free boson model;
- ✔ Can be extended to time dependent case to grasp non-linear Luttinger-liquid structure;

Further developments

- ⊛ Treat the case of $c \neq 1$.