Non-equilibrium steady states in many-body quantum systems

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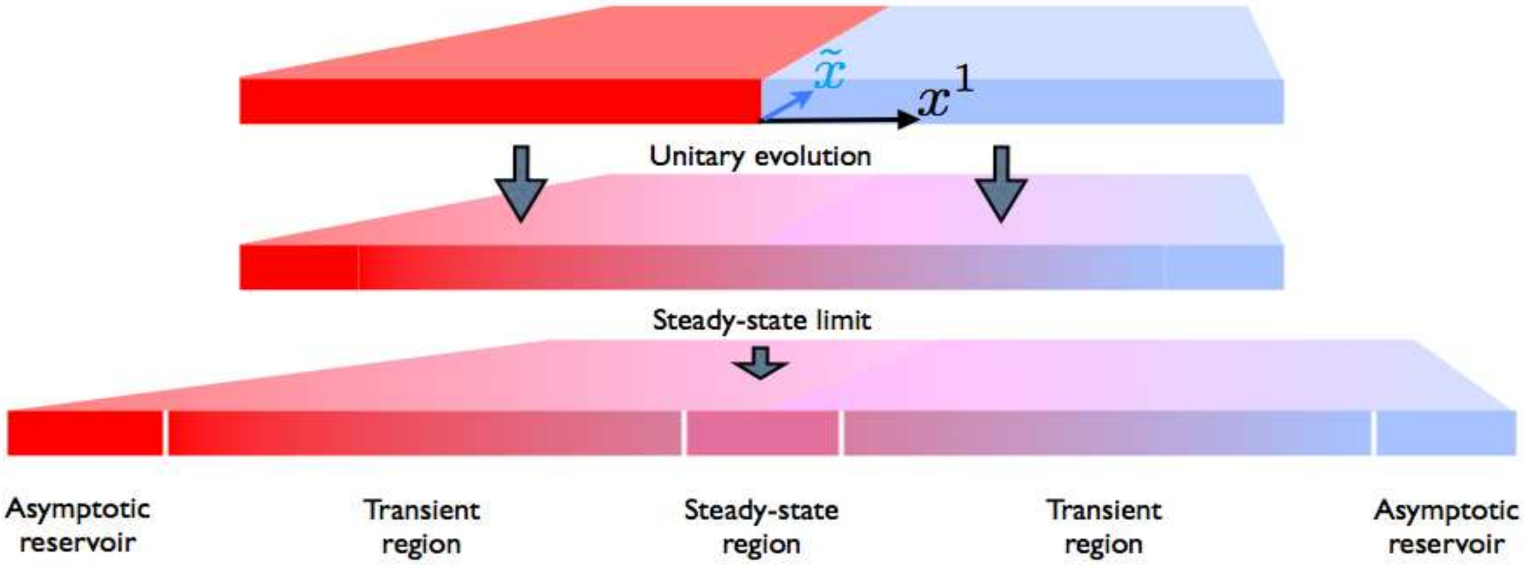
Partitioning approach

[Caroli et. al. 1971; Rubin et. al. 1971; Spohn et. al. 1977]

Consider some extended, local many-body quantum system separated into two halves, independently thermalized. Then suddenly connect them (local quench) and wait for a long time (unitary evolution).

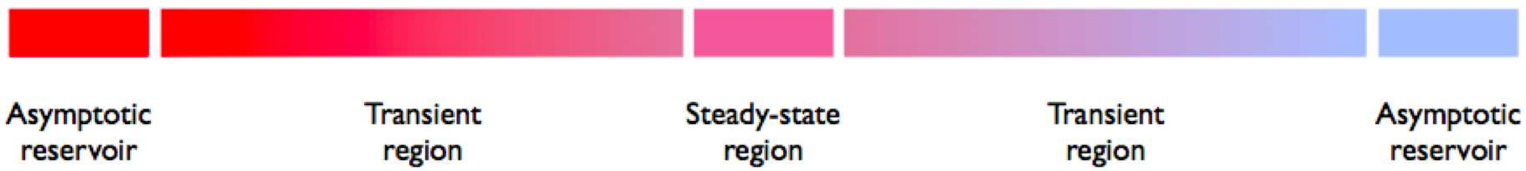


Generically, expect steady state to be trivial: thermalization, no flows.

In what situation can there be a nontrivial current?

Asymptotic baths very far; steady state translation invariant ⇒ No gradients ⇒ no diffusive transport (cf Fourier's law).

Current emerges in steady-state region iff there is ballistic transport



Ballistic steady state

- By stationarity and Eigenstate Thermalization Hypothesis [Deutsch 1991, Srednicki 1994, Rigol, Dunjko, Olshanii 2008], steady state described by **(semi-)local conserved charges**.
- By cluster property, steady states is **exponential of local conserved charges** (cf GGE).

Need a parity-odd conserved charge P:

$$e^{-\beta H+\nu P+\ldots},\quad \langle\mathcal{O}\rangle_{\rm stat} = \frac{{\rm Tr}\left(e^{-\beta H+\nu P+\ldots}\,\mathcal{O}\right)}{{\rm Tr}\left(e^{-\beta H+\nu P+\ldots}\right)}$$

Steady-state limit: only in central region, for local observables,

$$\langle\mathcal{O}\rangle_{\rm stat} = \lim_{vL\gg t\to\infty}\langle e^{{\rm i}Ht}\,\mathcal{O}\,e^{-{\rm i}Ht}\rangle_0,\quad \rho_0 = e^{-\beta_{\rm l}H_{\rm l}-\beta_{\rm r}H_{\rm r}},\quad H = H_{\rm l}+\delta H_{\rm l\,r}+H_{\rm r}$$

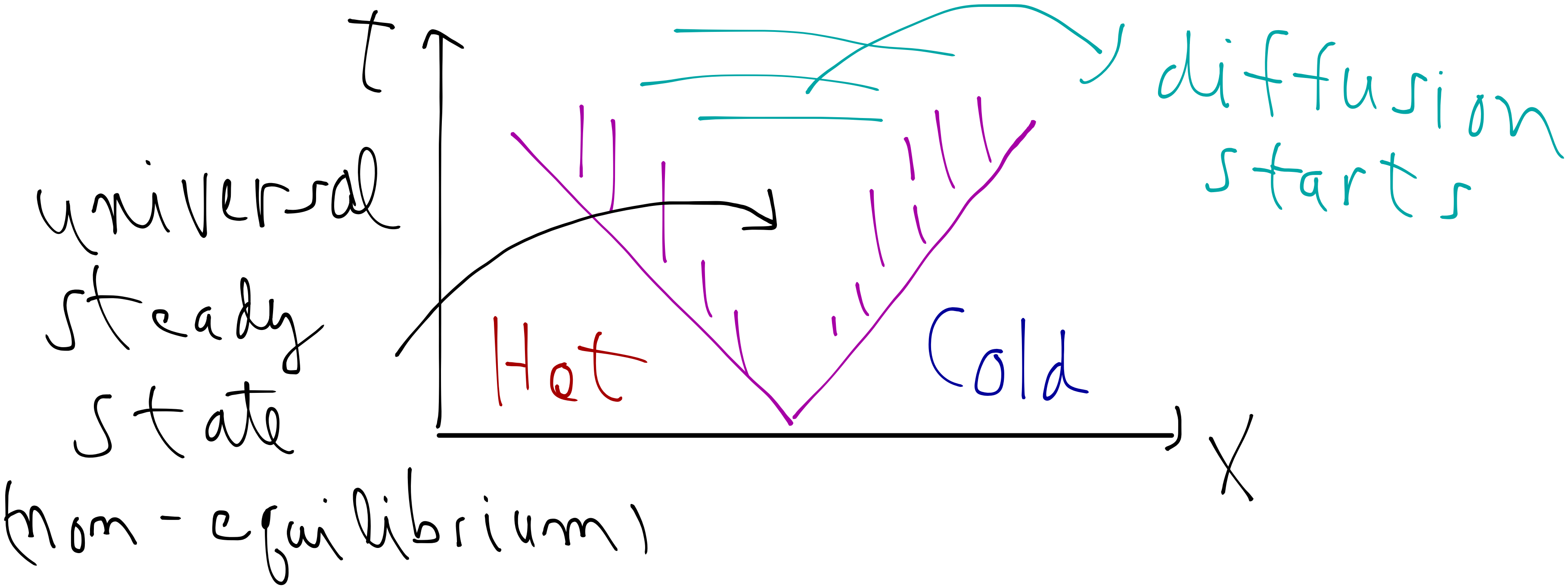
Near quantum criticality

Near zero-temperature quantum criticality: continuous translation invariance emerges

Momentum P

Universal steady state near criticality, with “diffusion time” $t_{\rm diff}(T_{\rm l}, T_{\rm r})$ set by temperatures,

$$\langle \mathcal{O}\rangle_{\rm stat} = \lim_{t_{\rm diff}(T_{\rm l},T_{\rm r}),vL\gg t\to\infty} \langle e^{{\rm i}Ht}\,\mathcal{O}\,e^{-{\rm i}Ht}\rangle_0.$$



If the total current is a conserved quantity

Let $\underline{\mathtt{j}}$ be a **current observable** for transport of quantity $\mathtt{q}$, i.e. $\partial_t \mathtt{q} + \nabla \cdot \underline{\mathtt{j}} = 0$.

Let $\mathtt{j} := \underline{\mathtt{j}}^1$ be longitudinal component, and assume that there is some $\underline{\mathtt{k}}$ such that

$$\partial_t \mathtt{j} + \nabla \cdot \underline{\mathtt{k}} = 0.$$

$\int \mathrm{d}^d x \, \mathtt{j}$ is **conserved** $\Rightarrow$ nonzero Drude peak, linear-response conductivity

Example: Lorentz invariant energy transport ($z = 1$ near-critical systems),

$$\partial_\mu \mathtt{T}^{\mu\nu} = 0 \text{ and } \mathtt{T}^{\mu\nu} = \mathtt{T}^{\nu\mu}$$

Set $\mathtt{q} = \mathtt{h} := \mathtt{T}^{00}$, $\underline{\mathtt{j}} = \underline{\mathtt{p}} := \mathtt{T}^{0i}$, $\underline{\mathtt{k}} = \mathtt{T}^{1i}$, and we have $P = \int \mathrm{d}^d x \, \mathtt{j}$.

Linear response: sound velocity

Take small variations about local Gibbs equilibrium

$$\langle \mathtt{q}(x,t)\rangle_0 \approx \langle \mathtt{q}\rangle + \delta\mathtt{q}(x,t), \quad \langle \mathtt{j}(x,t)\rangle_0 \approx \delta\mathtt{j}(x,t), \quad \langle \mathtt{k}(x,t)\rangle_0 \approx \langle \mathtt{k}\rangle + \delta\mathtt{k}(x,t)$$

Assume local thermalization: Equation of state $\langle \mathtt{k}\rangle = F(\langle \mathtt{q}\rangle)$ valid at every point:

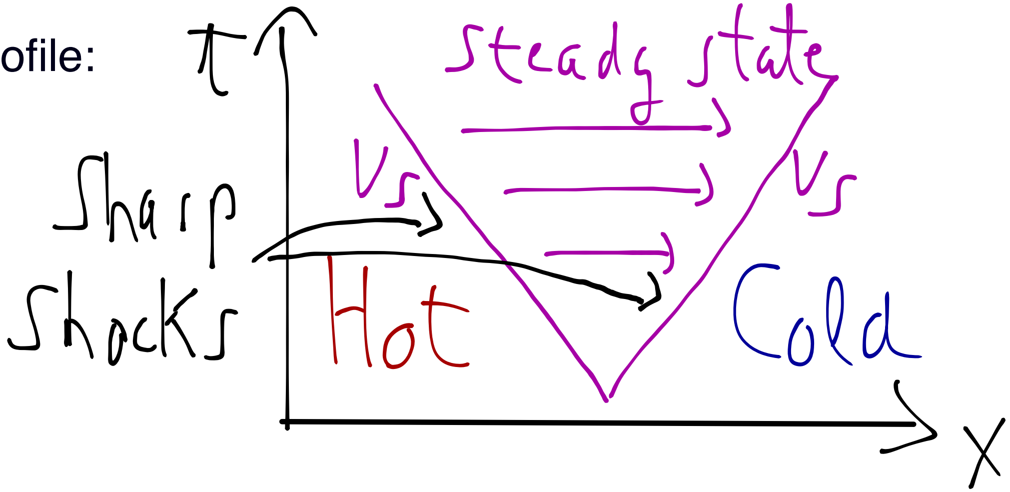
$$\delta\mathtt{k}(x,t) = F'(\langle \mathtt{q}\rangle)\,\delta\mathtt{q}(x,t)$$

Conservation equations imply wave equation with sound velocity $v_{\mathrm{s}} = \sqrt{F'(\langle q\rangle)}$:

$$\delta\mathtt{q}(x,t) = f(x - v_{\mathrm{s}}t) + g(x + v_{\mathrm{s}}t), \quad \delta\mathtt{j}(x,t) = v_{\mathrm{s}}(f(x - v_{\mathrm{s}}t) - g(x + v_{\mathrm{s}}t))$$

Solving with initial zero-current step profile:

$$\boxed{\delta\mathtt{j}_{\mathrm{stat}} = \frac{\delta\mathtt{k}_{\mathrm{l}} - \delta\mathtt{k}_{\mathrm{r}}}{2v_{\mathrm{s}}}.}$$



An inequality that quantifies non-equilibrium ballistic transport

[BD 2014]

If “pressure” $\mathtt{k}$ is monotonic on large scales in transient regions, then

$$\mathtt{j}_{\rm stat} \geq \frac{\mathtt{k}_{\rm l} - \mathtt{k}_{\rm r}}{2v}$$

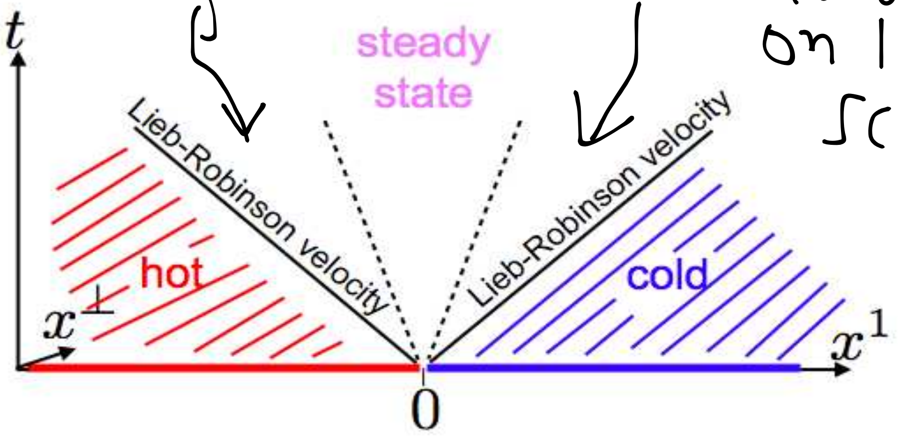
where v is **Lieb-Robinson velocity** and $\mathtt{k}_{\rm l,r}$ are thermal averages in left and right reservoir.

Can define “transient velocities”:

$$v_{\rm l,r} := \pm \frac{\mathtt{k}_{\rm l,r} - \mathtt{k}_{\rm stat}}{\mathtt{j}_{\rm stat}}, \quad v_{\rm l,r} \leq v.$$

From the linear response calculation:

$$\lim_{\text{equilibrium}} v_{\rm l,r} = v_{\rm s} = \text{ sound velocity.}$$



pressure monotonic on large scales

Lieb-Robinson bound and light-cone effect in many-body systems

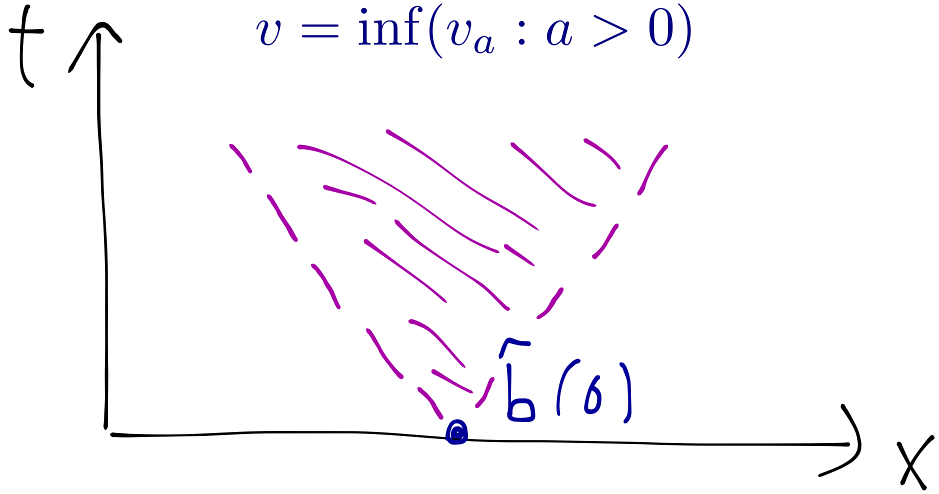
[Lieb, Robinson 1972]

$$\forall\, a>0,\ \exists\, v_a\ :\ \forall\, \mathtt{b},\tilde{\mathtt{b}}; t>0\ :\ \exists A>0 \mid \|[\mathtt{b}(t),\tilde{\mathtt{b}}(0)]\| \le A e^{-a(D(\mathtt{b},\tilde{\mathtt{b}})-v_a t)}$$

$||\cdot||$: operator norm. $D(\mathtt{b},\tilde{\mathtt{b}})$: distance between supports of $\mathtt{b}$ and $\tilde{\mathtt{b}}$.

Lieb-Robinson velocity:

$$v=\inf(v_a : a>0)$$



L-R bound: Cold atoms experimental verification [Cheneau, Barmettler, Poletti, Endres, Schauss, Fukuhara, Gross, Bloch, Kollath, Kuhr 2012].

Shocks

Suppose two-shock picture of linear response remains “mostly true”: $o(t)$ **transient regions**.

Take integral form of conservation equations through shocks

$$\partial_t \mathtt{h} + \partial_x \mathtt{j} = 0, \quad \partial_t \mathtt{p} + \partial_x \mathtt{k} = 0$$

Four connection equations (Rankine-Hugoniot):

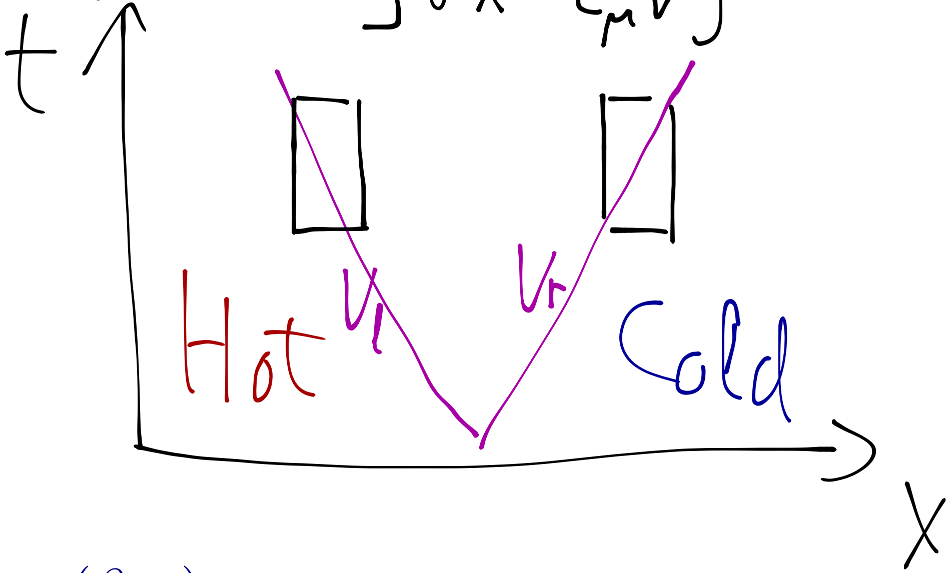
$$\int dx^\mu \varepsilon_{\mu\nu} j^\nu = 0$$

$$v_\mathrm{l}(\mathtt{h}_\mathrm{l} - \mathtt{h}_\mathrm{stat}) = \mathtt{j}_\mathrm{stat}$$

$$v_\mathrm{l}\mathtt{p}_\mathrm{stat} = \mathtt{k}_\mathrm{l} - \mathtt{k}_\mathrm{stat}$$

$$v_\mathrm{r}(\mathtt{h}_\mathrm{stat} - \mathtt{h}_\mathrm{r}) = \mathtt{j}_\mathrm{stat}$$

$$v_\mathrm{r}\mathtt{p}_\mathrm{stat} = \mathtt{k}_\mathrm{stat} - \mathtt{k}_\mathrm{r}$$



If we know $\mathtt{h}(\beta,\nu)$, $\mathtt{j}(\beta,\nu)$, $\mathtt{p}(\beta,\nu)$ and $\mathtt{k}(\beta,\nu)$:

Four equations, four unknowns β_stat, ν_stat, v_l, v_r $\Rightarrow$ unique solution (?)

Relativistic thermodynamics

If $\mathtt{j} = \mathtt{p} + \partial_x(\cdots)$ then:

- Stress-energy tensor in state $e^{-\beta H + \nu P}$ (with $\beta = \beta_{\text{rest}} \cosh\theta$, $\nu = \beta_{\text{rest}} \sinh\theta$, $u = \begin{pmatrix} \cosh\theta \\ \sinh\theta \end{pmatrix}$):

$$\mathtt{T}^{\mu\nu} = \mathtt{k}_{\text{rest}} \eta^{\mu\nu} + (\mathtt{h}_{\text{rest}} + \mathtt{k}_{\text{rest}}) u^\mu u^\nu$$

where $\mathtt{k}_{\text{rest}} = \mathtt{k}(T_{\text{rest}})$, $\mathtt{h}_{\text{rest}} = \mathtt{h}(T_{\text{rest}})$ (thermal averages)

- Temperature dependence in thermal state $e^{-\beta H}$:

$$T \frac{\mathrm{d}}{\mathrm{d}T} \mathtt{k}(T) = \mathtt{h}(T) + \mathtt{k}(T) \quad \text{(thermal averages)}$$

$\Rightarrow$ Thermal equation of state $\mathtt{k}(T) = F(\mathtt{h}(T))$ fixes everything.

$$\log T = \int^{\mathtt{k}(T)} \frac{\mathrm{d}\ell}{\ell + F^{-1}(\ell)} = \int^{\mathtt{h}(T)} \frac{\mathrm{d}\ell \, F'(\ell)}{\ell + F(\ell)}.$$

Example: conformal relativistic fluid in d dimensions, $\mathtt{k}(T) = d\,\mathtt{h}(T)$.

Refinement: pure hydrodynamics

- Assume local generalized thermalization: $\beta = \beta(x,t)$ and $\nu = \nu(x,t)$.
- Hydrodynamic equations are
  $\partial_t \mathtt{h}(\beta,\nu) + \partial_x \mathtt{j}(\beta,\nu) = 0,\ \partial_t \mathtt{p}(\beta,\nu) + \partial_x \mathtt{k}(\beta,\nu) = 0$
- Solve using step-profile initial condition
- Shocks are weak self-similar solutions

Further refinement: viscous hydrodynamics, entropy considerations

- Viscosity terms (higher-derivatives)...
- 2^{nd} law of thermodynamics (entropy production)...
- Rarefaction waves (other self-similar solutions)...

Example 1: 1+1-dimensional conformal field theory

[...; Sotiriadis, Cardy 2008; Bernard, BD 2012; ...]

Here $\mathtt{j} = \mathtt{p}$ and $\mathtt{k} = \mathtt{h}$. **Right- and left-moving combinations**:

$$\mathtt{h}_+ = \frac{\mathtt{h}+\mathtt{p}}{2} = \mathtt{h}_+(x-t), \quad \mathtt{h}_- = \frac{\mathtt{h}-\mathtt{p}}{2} = \mathtt{h}_-(x+t).$$

Same as linear-response calculation!

$$\mathtt{j}_{\rm stat} = \lim_{t\to\infty} \langle \mathtt{h}_+(-t) - \mathtt{h}_-(t) \rangle_0 = \langle \mathtt{h}_+ \rangle_{\rm l} - \langle \mathtt{h}_- \rangle_{\rm r} = \frac{\mathtt{k}_{\rm l} - \mathtt{k}_{\rm r}}{2}.$$

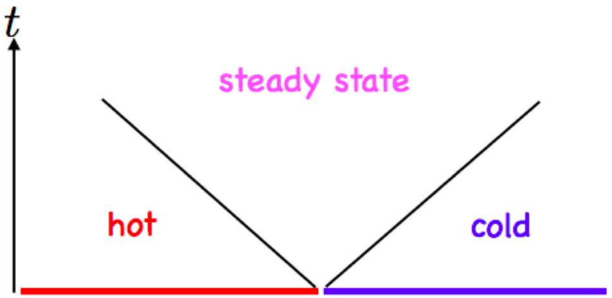
Using CFT results, $\mathtt{j}_{\rm stat} = \frac{\pi c k_B^2}{12\hbar}\left(T_{\rm l}^2 - T_{\rm r}^2\right)$. Verified numerically [Karrasch, Ilan, Moore 2012] and experimentally [Jezouin, Parmentier, Anthore, Gennser, Cavanna, Jin, Pierre 2013].

Remarks:

Inequalities **saturated**, $v_{\rm l} = v_{\rm r} = v_{\rm s} = v$.

Sharp shock waves (up to non-universal scales)

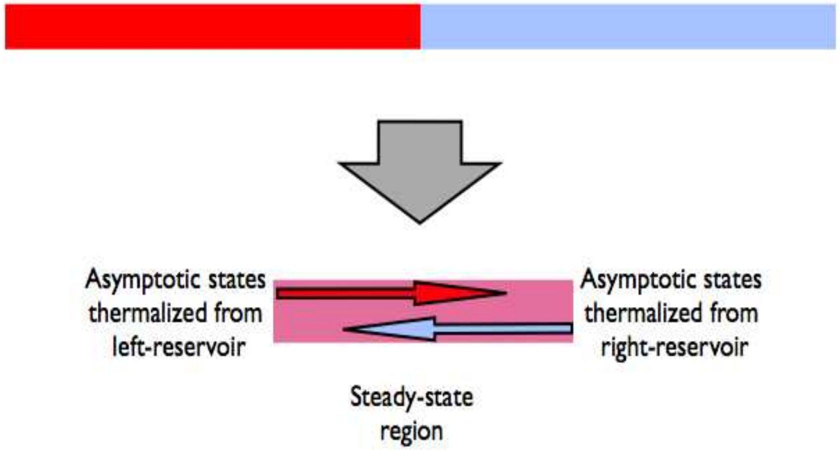
Steady state reached “immediately” (idem)

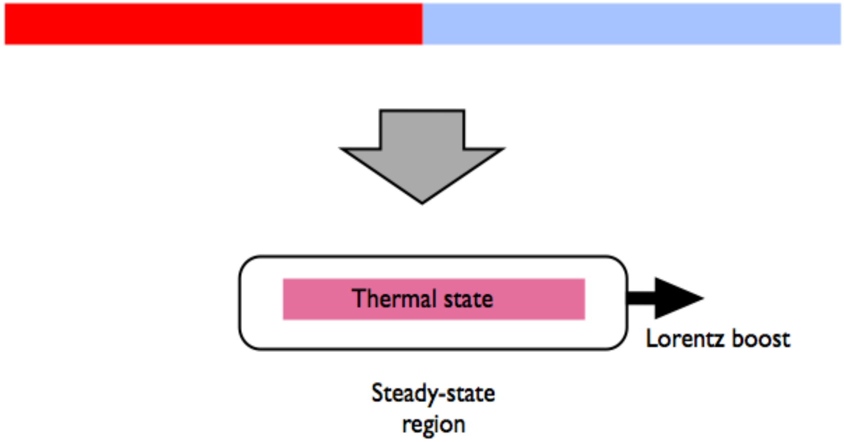


Density matrix for steady state [Bernard, BD 2012; Bhaseen, BD, Lucas, Schalm 2015]:

$$e^{-\beta_\mathrm{l} H_+ - \beta_\mathrm{r} H_-} = \exp - \left[\frac{\beta_\mathrm{l} + \beta_\mathrm{r}}{2} H + \frac{\beta_\mathrm{l} - \beta_\mathrm{r}}{2} P \right]$$

$H_\pm =$ total energy of right- / left- moving modes;
boost of a thermal state with $\beta_\mathrm{rest} = \sqrt{\beta_\mathrm{l}\beta_\mathrm{r}}$, $\tanh\theta = \frac{\beta_\mathrm{r} - \beta_\mathrm{l}}{\beta_\mathrm{l} + \beta_\mathrm{r}}$.





Example 2: $T\bar{T}$-perturbation of CFT

[Bernard, BD in preparation]

$$H = \int dx\,(T(x) + \bar{T}(x)) + g \int dx\,T(x)\bar{T}(x).$$

Irrelevant perturbation: low-energy correction to universal behaviour.

Currents at $O(g)$:

$$\mathtt{h}(x) = T(x) + \bar{T}(x) + gT(x)\bar{T}(x), \quad \mathtt{p}(x) = T(x) - \bar{T}(x)$$

$$\mathtt{j}(x) = \mathtt{p}(x) + \partial_x(\cdots), \quad \mathtt{k}(x) = \mathtt{h}(x) + 2gT(x)\bar{T}(x) + \partial_x(\cdots)$$

$\Rightarrow$ Thermodynamics is relativistic: $\langle \mathtt{j} \rangle = \langle \mathtt{p} \rangle$, eqn of state $\langle \mathtt{k} \rangle = \langle \mathtt{h} \rangle + \frac{g}{2} \langle \mathtt{h} \rangle^2$

Can determine exact thermal averages, e.g.

$$\mathtt{h}(T) = \frac{c\pi}{6} T^2 \left(1 - \frac{gc\pi}{8} T^2\right).$$

Speed of sound is $v_{\mathrm{s}}(T) = 1 + \frac{gc\pi}{12} T^2$, and we find

- Shocks with velocities $v_{\mathrm{l}} = v_{\mathrm{s}}(T_{\mathrm{l}})$ and $v_{\mathrm{r}} = v_{\mathrm{s}}(T_{\mathrm{r}})$
- Current $\mathtt{j}_{\mathrm{stat}} = \frac{c\pi}{12}\left(T_{\mathrm{l}}^2/v_{\mathrm{l}} - T_{\mathrm{r}}^2/v_{\mathrm{r}}\right)$: still left-right separation in agreement with numerics [Karrasch, Ilan, Moore 2012]
- Steady state density matrix with $T_{\mathrm{rest}} = \sqrt{T_l T_r}\left(1 - \frac{gc\pi}{48}(T_l - T_r)^2\right)$ and $\tanh\theta = \frac{T_l - T_r}{T_l + T_r}\left(1 - \frac{gc\pi}{12} T_l T_r\right)$; we still have $\beta = \frac{\beta_l + \beta_r}{2}$
- Shocks of sublinear extent $O(t^{1/3})$ (conjecture)
- Generic approach $O(1/\sqrt{t})$ (conjecture)

Example 3: free higher-dimensional QFT (massive Klein-Gordon model)

Steady state can be described by **independently thermalizing right- and left-movers** (modes with positive and negative longitudinal momenta) **with left and right temperatures**

[Spohn, Lebowitz 1977; ...; Collura, Martelloni 2014; BD, Lucas, Schalm, Bhaseen 2014]

$$e^{-\beta_{\rm l} H_+ - \beta_{\rm r} H_-},\quad H_\pm = \int_{p^1 \gtrless 0} {\rm d}^d p\, \sqrt{p^2+m^2}\, A^\dagger(p) A(p)$$

Equivalently [BD, Lucas, Schalm, Bhaseen 2014]:

$$\exp - \left[\frac{\beta_{\rm l}+\beta_{\rm r}}{2}\, H + \frac{\beta_{\rm l}-\beta_{\rm r}}{2}\,(P_1 + Q)\right]$$

with **semi (or non?)-local parity-odd conserved charge**

$$Q = \int d^dx d^dy\ :\phi(x)\pi(y):\, Q(x-y),\quad Q(x-y) \overset{\text{at d=1}}{\sim} -\frac{m}{\pi(x^1-y^1)}$$

[BD, Lucas, Schalm, Bhaseen 2014]

Current and pressure (here at $m = 0$):

$$\mathtt{j}_{\rm stat} = d\,\Gamma(d/2)\zeta(d+1)/(2\pi^{\frac{d}{2}+1})\,(T_{\rm l}^{d+1} - T_{\rm r}^{d+1}),$$

$$\mathtt{k}_{\rm l,r} = \Gamma((d+1)/2)\zeta(d+1)/(\pi^{\frac{d+1}{2}})\,T_{\rm l,r}^{d+1}$$

Remarks:

Inequality ok: $2\mathtt{j}_{\rm stat} > \mathtt{k}_{\rm l} - \mathtt{k}_{\rm r}$.

Equilibrium limit **does not give the sound velocity**, $\lim_{\text{equilibrium}} v_{\rm l,r} \neq 1/\sqrt{d}$. Signal of generalized Gibbs thermalization (GGE fluid...).

No shock waves, rather large transition regions.

Generic approach to steady state is either $O(1/\sqrt{t})$ or $O(1/t)$ depending on initial boundary conditions at $x^1 = 0$.

Example 4: non-integrable higher-dimensional CFT

[Bhaseen, BD, Lucas, Schalm 2015; in preparation]

Relativistic system, thermal eqn of state $\mathtt{k}(T) = d\,\mathtt{h}(T)$. Pure conformal hydrodynamics:

$$\langle \mathtt{T}^{\mu\nu}(x,t)\rangle \approx a_d T_{\text{rest}}^{d+1}(x,t)(\eta^{\mu\nu} + (d+1)u^{\mu}(x,t)u^{\nu}(x,t)), \quad \partial_\mu \langle \mathtt{T}^{\mu\nu}\rangle = 0$$

(a_d: model-dependent normalization) with initial conditions:

$$T_{\text{rest}}(x,0) = \left\{ \begin{array}{ll} T_{\text{l}} & (x^1 < 0) \\ T_{\text{r}} & (x^1 > 0) \end{array} \right\}, \quad \theta(x,0) = 0.$$

Assuming two shocks [Bhaseen, BD, Lucas, Schalm 2015; Chang, A. Karch and A. Yarom 2014]:

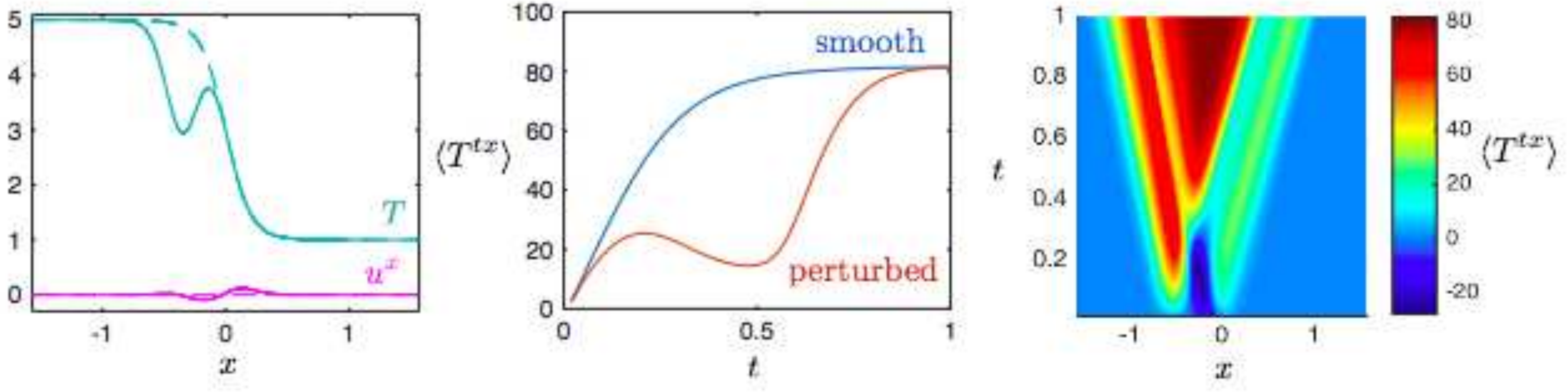
$$v_{\text{l}} = \frac{1}{d}\sqrt{\frac{\tau_{\text{l}} + d\tau_{\text{r}}}{\tau_{\text{l}} + d^{-1}\tau_{\text{r}}}}, \quad v_{\text{r}} = \sqrt{\frac{\tau_{\text{l}} + d^{-1}\tau_{\text{r}}}{\tau_{\text{l}} + d\tau_{\text{r}}}}, \quad \tau_{\text{l,r}} = T_{\text{l,r}}^{\frac{d+1}{2}}$$

$$T_{\text{rest}} = \sqrt{T_{\text{l}}T_{\text{r}}}, \quad \tanh\theta = \frac{\tau_{\text{l}} - \tau_{\text{r}}}{\sqrt{(\tau_{\text{l}} + d\tau_{\text{r}})(\tau_{\text{l}} + d^{-1}\tau_{\text{r}})}}$$

$$\mathtt{j}_{\text{stat}} = \frac{d a_d}{d+1}(\tau_{\text{l}} - \tau_{\text{r}})\sqrt{(\tau_{\text{l}} + d\tau_{\text{r}})(\tau_{\text{l}} + d^{-1}\tau_{\text{r}})}.$$

Remarks:

- Emerges naturally under a wide range of initial smoothed-out conditions.



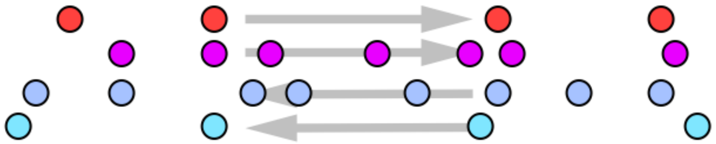
- Verified by AdS/CFT numerics [I. Amado, A. Yarom 2015]
- Inequalities are satisfied: $v_{\mathrm{l,r}} < 1$.
- Equilibrium limit gives the sound velocity, $\lim_{\text{equilibrium}} v_{\mathrm{l,r}} = 1/\sqrt{d}$.

Conclusions

- Hydrodynamics gives general framework in non-integrable models, including perturbed CFT. Integrable models: generalized Gibbs thermalization, generalized hydro?

- Fluctuations and fluctuation relations: **Poisson process interpretation** for energy transport. For instance in 1+1-dimensional CFT: [Bernard, BD 2012; Bernard, BD 2014, BD, Hoogeveen, Bernard 2014]: $F(z) := \sum_{n=1}^{\infty} c_n \frac{z^n}{n!}$

$$F(z) = \int dq\,\omega(q)\,(e^{zq}-1)\,,\quad \omega(q) = \frac{c\pi}{12}\,e^{-\beta_{\mathrm{l,r}}|q|}\ \ (q>0,\ q<0).$$



- Charge transfer (in one dimension [Gutman, Gefen, Mirlin 2010; Bernard, BD 2014]); presence of impurities (in 1-d CFT [Bernard, BD, Viti 2014]); other dynamical exponents; curved connection hypersurface; integrable massive QFT (conjecture [Castro-Alvaredo, Chen, BD, Hoogeveen 2014]); integrable spin chains (conjecture [De Luca, Viti, Mazza, Rossini 2014]); entanglement evolution (in 1-d CFT [Hoogeveen, BD 2015]);...