Eigenvalue-based determinants for quantum integrable models derived from the Generalized Gaudin Algebra

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- Time-evolution of an observable

Form factors

$$\langle \Psi(t)| \mathcal{O} |\Psi(t)\rangle = \sum_{m,n} \langle \Psi(0) \mid \psi_m\rangle \langle \psi_m| \mathcal{O} |\psi_n\rangle \langle \psi_n \mid \Psi(0)\rangle \, e^{i(E_m - E_n)t}$$

Overlaps

- Decomposing the initial state on the eigenbasis of the Hamiltonian requires the eigenstates and eigenvalues + **Overlaps** + **Form factors** (matrix elements).

We want to use integrability and the algebraic Bethe ansatz to **numerically** compute these quantities as **efficiently** as possible.

- XXX Generalized Gaudin algebra:

$$\left[S^{\pm}(u), S^{\pm}(v)\right] = 0$$

Spectral parameters

$u, v \in \mathbb{C}$

$$\left[S^{+}(u), S^{-}(v)\right] = \pm 2 \frac{g}{u-v} \left(S^{z}(u) - S^{z}(v)\right)$$

$$\left[S^{z}(u), S^{\pm}(v)\right] = \pm \frac{g}{u-v} \left(S^{\pm}(u) - S^{\pm}(v)\right)$$

- Spin-only realization

Spin-only realization of the algebra can written in terms of the local generators for N local SU(2) spins:

$$\mathrm{S}^{+}(u) = \sum_{i=1}^{N} \frac{S_i^{+}}{u - \epsilon_i}$$

$$\mathrm{S}^{-}(u) = \sum_{i=1}^{N} \frac{S_i^{-}}{u - \epsilon_i}$$

$$\mathrm{S}^{z}(u) = \frac{1}{g} - \sum_{i=1}^{N} \frac{S_i^{z}}{u - \epsilon_i}$$

- Central-spin models
- BCS model (in terms of Anderson pseudo-spins)

- Conserved quantities

"Transfer matrix": $$\mathrm{S}^2(u) \equiv S^z(u)S^z(u) + \frac{1}{2}\left(S^+(u)S^-(u) + S^-(u)S^+(u)\right)$$

which can be shown to commute for arbitrary u,v:

$$\left[\mathrm{S}^2(u), \mathrm{S}^2(v)\right] = 0 \qquad \forall\ u, v \in \mathbb{C}$$

The conserved charges R_k (with $[R_k, R_{k'}] = 0$) can be found as the operator-valued residues of this transfer matrix:

$$\mathrm{S}^2(u) = \sum_{k=1}^{N} \frac{\hat{R}_k}{u - \epsilon_k} + \hat{P}(u) \qquad\qquad R_k = \oint_C \mathrm{S}^2(u)du$$

Since it is just a u-parametrized linear combination of the R_i operators, finding eigenstates common to every $S^2(u)$ is equivalent to finding eigenstates common to the set of N conserved charges R_i which, in our case, read:

$$R_i = S_i^z + \sum_{j \neq i} \frac{g}{\epsilon_i - \epsilon_j} \vec{S}_i \cdot \vec{S}_j ,$$

- Quantum Inverse Scattering Method

Build a generic state by creating M excitations $|\{\lambda_1 ... \lambda_m\}\rangle \equiv \prod_{i=1}^{M} S^+(\lambda_i)\,|0\rangle$

above a pseudo-**vacuum such that**:

$$S^-(u)\,|0\rangle = 0 \quad \forall \quad u \in \mathbb{C}$$
$$\mathrm{S}^2(u)\,|0\rangle = \ell(u)\,|0\rangle$$
$$\mathrm{S}^z(u)\,|0\rangle = F(u)\,|0\rangle\,.$$

The fully down-polarized state is a proper vacuum in our case:

$$S^-(u)\,|\downarrow\downarrow \ldots \downarrow\rangle = 0$$
$$S^2(u)\,|\downarrow\downarrow \ldots \downarrow\rangle = \ell(u)\,|\downarrow\downarrow \ldots \downarrow\rangle$$
$$S^z(u)\,|\downarrow\downarrow \ldots \downarrow\rangle = F(u)\,|\downarrow\downarrow \ldots \downarrow\rangle$$

Acting on our ansatz with $S^2(u)$ gives:

$$\mathrm{S}^2(u)\,|\lambda_1 \ldots \lambda_M\rangle = \left(\prod_{i=1}^{M} \mathrm{S}^+(\lambda_i)\right) \mathrm{S}^2(u)\,|0\rangle + \left[\mathrm{S}^2(u), \left(\prod_{i=1}^{M} \mathrm{S}^+(\lambda_i)\right)\right] |0\rangle$$
$$= E(\{\lambda\}, u)\,|\lambda_1 ... \lambda_M\rangle + \sum_{k=1}^{M} F_k(\{\lambda\}, u)\,|\lambda_1 ..., \lambda_k \to u, ... \lambda_M\rangle\,,$$

- Quantum Inverse Scattering Method

The generic Bethe state $|\{\lambda_1 ... \lambda_m\}\rangle \equiv \prod_{i=1}^{M} S^+(\lambda_i)\,|0\rangle$

therefore becomes an eigenstate common to every $S^2(u)$ (and therefore R_k):

unwanted terms

$$\mathrm{S}^2(u)\,|\lambda_1 \ldots \lambda_M\rangle = E(\{\lambda\},u)\,|\lambda_1 ... \lambda_M\rangle + \sum_{k=1}^{M} F_k(\{\lambda\},u)\,|\lambda_1 ..., \lambda_k \to u, ... \lambda_M\rangle\,,$$

eigenvalue

whenever $\{\lambda_1 ... \lambda_M\}$ is a solution of the Bethe equations (when every $F_k = 0$) explicitly given by

$$-\frac{1}{2}\sum_{k=1}^{N}\frac{1}{\epsilon_k - \lambda_j} + \frac{1}{g} = \sum_{k\neq j}^{M}\frac{1}{\lambda_j - \lambda_k}$$

for our current realization.

- Quantum Inverse Scattering Method

The resulting Bethe equations remain tricky to deal with because of the potential canceling divergences which can easily become a numerical nightmare.

$$-\frac{1}{2}\sum_{k=1}^{N}\frac{1}{\epsilon_k-\lambda_j}+\frac{1}{g}=\sum_{k\neq j}^{M}\frac{1}{\lambda_j-\lambda_k}$$

- 'THE' transformation

For systems described by M Bethe equations $F(\lambda_i)=\sum_{j=1(\neq i)}^{M}\frac{g}{\lambda_i-\lambda_j}$, with

$$F(u)=-\sum_{i=1}^{N}\frac{A_i}{\epsilon_i-u}+\frac{B}{2g}u+\frac{C}{2g}$$

... it is possible to rewrite the problem using a set of N **quadratic** Bethe equations (which for our N spins-1/2 realization reads):

$$\Lambda^2(\epsilon_j) + \frac{B}{g}M - \frac{B\epsilon_i + C}{g}\Lambda(\epsilon_j) - \sum_{i\neq j}^{N} \frac{\Lambda(\epsilon_j) - \Lambda(\epsilon_i)}{\epsilon_j - \epsilon_i} = 0,$$

using the **variables** $\Lambda(\epsilon_i) = \sum_{j=1}^{N} \frac{1}{\epsilon_i - \lambda_j}$. These actually turn out to be the state-dependent part of the **eigenvalues r_i of the conserved charges R_i**.

$$r_i^{\lambda} = -\sum_{j=1}^{M} \frac{1}{\epsilon_i - \lambda_j} + \frac{1}{2}\sum_{j=1(\neq i)}^{N} \frac{1}{\epsilon_i - \epsilon_j} + \frac{1}{g}$$

- O. Babelon and D. Talalaev, *On the Bethe ansatz for the Jaynes–Cummings–Gaudin model*, J. Stat. Mech, P06013 (2007)
- A. Faribault, O. El Araby, C. Sträter, and V. Gritsev, *Gaudin models solver based on the correspondence between Bethe ansatz and ordinary differential equations*, Phys Rev. B **83**, 235124 (2011)

Since quadratic equations are much easier to solve than the original ones, this rewriting has been extremely helpful for numerical work.

However, computing scalar products and form factors, required the actual Bethe roots:

$$\langle \Psi(t)| \mathcal{O} |\Psi(t)\rangle = \sum_{m,n} \langle \Psi(0) \mid \psi_m\rangle \langle \psi_m| \mathcal{O} |\psi_n\rangle \langle \psi_n \mid \Psi(0)\rangle e^{i(E_m - E_n)t}$$

$$\langle \Psi(0) \mid \psi_m\rangle , \langle \psi_m| \mathcal{O} |\psi_n\rangle , \langle \psi_n \mid \Psi(0)\rangle = \det G$$

Slavnov's determinant of an M x M matrix whose elements depend on the λ_j

N. A. Slavnov, *Calculation of scalar products of wave functions and form factors in the framework of the algebraic Bethe ansatz*, Teor. Mat. Fiz. **79**, 502 (1989)

The inversion of $\Lambda(\epsilon_i) = \sum_{j=1}^{N} \frac{1}{\epsilon_i - \lambda_j}$, remains a complicated numerical operation due to stability issues. Having similar determinants in terms of our new variables would save computation time.

- Partition functions

Domain wall boundary partition functions, i.e. scalar products between a generic Bethe state (arbitrary set {λ_1 ... λ_M}) with a canonical-basis state containing M up-pointing spins, the other ones pointing down,:

$$\left\langle \uparrow_{\{i_1 \ldots i_M\}} | \lambda_1 \ldots \lambda_M \right\rangle = \left\langle \uparrow_{\{i_1 \ldots i_M\}} \right| \prod_{i=1}^{M} \mathrm{S}^+(\lambda_i) \left| \downarrow \ldots \downarrow \right\rangle = \mathrm{Det} J_{M \times M}$$

$$J_{ab} = \begin{cases} \displaystyle\sum_{c=1(\neq a)}^{M} \frac{1}{\epsilon_{i_a} - \epsilon_{i_c}} - \Lambda^{\lambda}(\epsilon_{i_a}) & a = b \\ \frac{1}{\epsilon_{i_a} - \epsilon_{i_b}} & a \neq b \end{cases}$$

was the first **eigenvalue-based determinant** we were able to find ... first by trial and error but then rigorously through a **recursive proof** using properties of rational functions.

A. Faribault and D. Schuricht, *On the determinant representations of Gaudin models' scalar products and form factors*, J. Phys. A: Math. Theor. **45**, 485202 (2012).

- Partition functions

Basically, we showed that the **permanent** of a NxN **Cauchy matrix**:

$$\langle\uparrow\uparrow\uparrow| S^+(\nu_1)S^+(\nu_2)S^+(\nu_3) |\downarrow\downarrow\downarrow\rangle = \qquad \mathrm{S}^+(u) = \sum_{i=1}^{N} \frac{S_i^+}{u-\epsilon_i}$$

$$\frac{1}{\nu_1-\epsilon_1}\frac{1}{\nu_2-\epsilon_2}\frac{1}{\nu_3-\epsilon_3} + \frac{1}{\nu_1-\epsilon_2}\frac{1}{\nu_2-\epsilon_1}\frac{1}{\nu_3-\epsilon_3} + \quad \text{Permutations } \ldots$$

can be written as a NxN **determinant** (recursively check that both sides have the same poles, residues and limit at infinity):

$$\mathrm{Perm}_{N\times N}\left(\frac{1}{\nu_i-\epsilon_j}\right) = \mathrm{Det}_{N\times N} J$$

whose entries only depend explicitly on:

$$\Lambda_i = \sum_{k=1}^{N} \frac{1}{\epsilon_i - \nu_k}$$

- Dual representation

To get to scalar products and form factors in terms of $\Lambda(\epsilon_i)$, we used the obvious: spins are symmetric ! One can flip the quantization axis ($z \to -z$) and nothing changes: $S^+ \Leftrightarrow S^-$ $\quad$ $S^z \to -S^z$

One can build a dual ABA starting from a fully up-polarized vacuum, which is an eigenstate of $S^2(u)$ of $-S^z(u)$ and is annihilated by every $S^+(u)$:

Particle-like state of the form:

$$|\{\lambda_1 \dots \lambda_M\}\rangle \equiv \prod_{j=1}^{M} \mathrm{S}^+(\lambda_j)\,|\downarrow \dots \downarrow\rangle$$

proper vacuum:

$$S^-(u)\,|\downarrow\downarrow \dots \downarrow\rangle = 0$$
$$S^2(u)\,|\downarrow\downarrow \dots \downarrow\rangle = \ell(u)\,|\downarrow\downarrow \dots \downarrow\rangle$$
$$S^z(u)\,|\downarrow\downarrow \dots \downarrow\rangle = F(u)\,|\downarrow\downarrow \dots \downarrow\rangle$$

Hole-like state of the form:

$$|\mu_1 \dots \mu_{N-M}\rangle = \prod_{i=1}^{N-M} \mathrm{S}^-(\mu_i)\,|\uparrow \dots \uparrow\rangle$$

proper vacuum:

$$S^+(u)\,|\uparrow\uparrow \dots \uparrow\rangle = 0$$
$$S^2(u)\,|\uparrow\uparrow \dots \uparrow\rangle = \tilde{\ell}(u)\,|\uparrow\uparrow \dots \uparrow\rangle$$
$$S^z(u)\,|\uparrow\uparrow \dots \uparrow\rangle = \tilde{F}(u)\,|\uparrow\uparrow \dots \uparrow\rangle$$

Spin-only models

M particles

States of the form:

$$|\{\lambda_1 ... \lambda_m\}\rangle \equiv \prod_{j=1}^{M} \mathrm{S}^{+}(\lambda_j)\,|\downarrow \ldots \downarrow\rangle$$

become eigenstates when

$$-\frac{1}{2}\sum_{k=1}^{N}\frac{1}{\epsilon_k - \lambda_j} + \frac{1}{g} = \sum_{k\neq j}^{M}\frac{1}{\lambda_j - \lambda_k}$$

with eigenvalues

$$r_i^{\lambda} = -\sum_{j=1}^{M}\frac{1}{\epsilon_i - \lambda_j} + \frac{1}{2}\sum_{j=1(\neq i)}^{N}\frac{1}{\epsilon_i - \epsilon_j} + \frac{1}{g}$$

which themselves obey the BEs:

$$\left[\Lambda^{\lambda}(\epsilon_i)\right]^2 = \sum_{j\neq i}^{N}\frac{\Lambda^{\lambda}(\epsilon_i) - \Lambda^{\lambda}(\epsilon_j)}{\epsilon_i - \epsilon_j} + \frac{2}{g}\Lambda^{\lambda}(\epsilon_i)$$

N-M holes

States of the form:

$$|\mu_1 \ldots \mu_{N-M}\rangle = \prod_{i=1}^{N-M} \mathrm{S}^{-}(\mu_i)\,|\uparrow \ldots \uparrow\rangle$$

become eigenstates when

$$-\frac{1}{2}\sum_{k=1}^{N}\frac{1}{\epsilon_k - \mu_j} - \frac{1}{g} = \sum_{k\neq j}^{N-M}\frac{1}{\mu_j - \mu_k}$$

with eigenvalues

$$r_i^{\mu} = -\sum_{j=1}^{N-M}\frac{1}{\epsilon_i - \mu_j} + \frac{1}{2}\sum_{j=1(\neq i)}^{N}\frac{1}{\epsilon_i - \epsilon_j} - \frac{1}{g}$$

which themselves obey the BEs:

$$\left[\Lambda^{\mu}(\epsilon_i)\right]^2 = \sum_{j\neq i}^{N}\frac{\Lambda^{\mu}(\epsilon_i) - \Lambda^{\mu}(\epsilon_j)}{\epsilon_i - \epsilon_j} - \frac{2}{g}\Lambda^{\mu}(\epsilon_i)$$

Go from the normal to the dual representation of a given eigenstate using the transformation $\Lambda^{\mu}(\epsilon_i) = \Lambda^{\lambda}(\epsilon_i) - \frac{2}{g}$ found by imposing equality of eigenvalues.

- Scalar products (as partition functions):

Mixing representations allows us to write scalar products not as:

$$\langle \lambda'_1 \dots \lambda'_M \mid \lambda_1 \dots \lambda_M \rangle = \langle \downarrow \dots \downarrow | \prod_{i=1}^{M} \mathrm{S}^-(\lambda'_i) \prod_{j=1}^{M} \mathrm{S}^+(\lambda_j) \, | \downarrow \dots \downarrow \rangle$$

but as:

$$\left\langle \mu'_1 \dots \mu'_{N-M} \mid \lambda_1 \dots \lambda_M \right\rangle = \langle \uparrow \dots \uparrow | \prod_{i=1}^{N-M} \mathrm{S}^+(\mu'_i) \prod_{j=1}^{M} \mathrm{S}^+(\lambda_j) \, | \downarrow \dots \downarrow \rangle$$

which is just a partition function whose determinant expression we found before. The resulting NxM determinant is then **valid for arbitrary sets** $\{\lambda_1 \dots \lambda_M\}$ and $\{\mu_1 \dots \mu_{M-N}\}$, evidently including those which define eigenstates.

$$\langle \{\mu_1 \dots \mu_{N-M}\} | \{\lambda_1 \dots \lambda_M\} \rangle = \mathrm{Det}_{N \times N} J$$

$$J_{ab} = \begin{cases} \sum_{c \neq a} \frac{1}{\epsilon_a - \epsilon_c} - \Lambda^{\lambda}(\epsilon_a) - \Lambda^{\mu}(\epsilon_a) & a = b \\ \frac{1}{\epsilon_a - \epsilon_b} & a \neq b \end{cases}$$

- Form factors:

Since $\lim_{u\to\epsilon_k}(u-\epsilon_k)\mathrm{S}^+(u) = S_k^+$ the form factors for local spin raising/lowering operators are trivially given by partition functions as well:

$$\langle\mu_1\dots\mu_N|\,\mathrm{S}^+(u)\,|\lambda_1\dots\lambda_{M-1}\rangle = \langle M;\uparrow\dots\uparrow|\left(\prod_{i=1}^{N}\mathrm{S}^+(\mu_i)\right)\mathrm{S}^+(u)\left(\prod_{j=1}^{M-1}\mathrm{S}^+(\lambda_j)\right)|0;\downarrow\dots\downarrow\rangle$$

On the other hand, the local S^z form factors can be shown to be given by a derivative of the partition function determinant which can be explicitly written as a sum of N similar determinants.

- XXZ models:

Similar results have been obtained recently for **XXZ-Gaudin models**, by reusing mapping the permanent which gives the partition function to that of a slightly more complex Cauchy matrix.

P. W. Claeys, S. De Baerdemacker, M. Van Raemdonck and D. Van Neck, *An eigenvalue-based method and determinant representations for general integrable XXZ Richardson-Gaudin models*, Phys. Rev. B **91**, 155102 (2015)

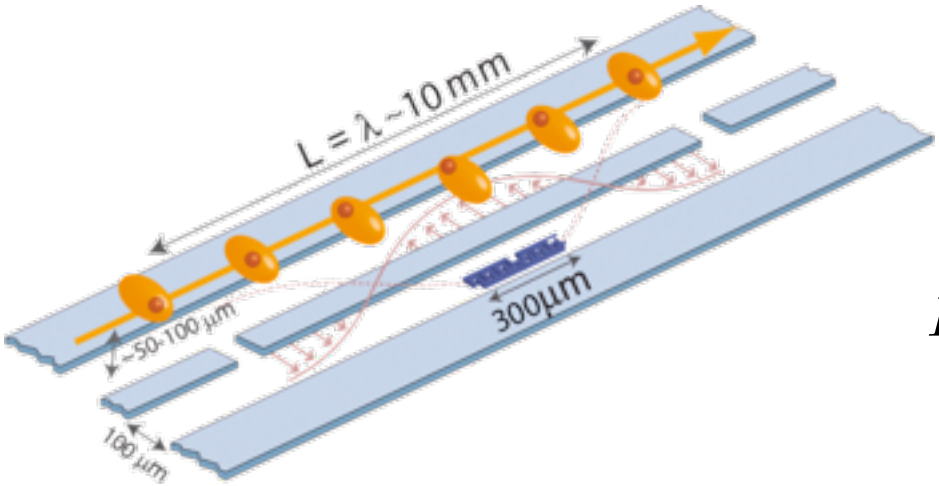
- Spin-boson models

Do the same for the following spin-boson realization of the XXX Generalized Gaudin algebra (N spins + 1 bosonic mode):

$$\mathrm{S}^{+}(u) = b^{\dagger} + \sum_{j=1}^{N} \frac{V}{u - \quad_{j}} S_j^{+} \qquad\qquad {}^{-}(\ \ ) = \quad + \sum_{j=1}^{N} \frac{}{\quad - \epsilon_j} \quad {}_{j}^{-}$$

$$\mathrm{S}^{z}(u) = \frac{\omega - u}{2V} - \sum_{j=1}^{N} \frac{V}{u - \epsilon_j} S_j^{z}$$

which leads to integrable Generalized Tavis-Cummings / Dicke models useful for cavity QED systems



$$H = \ \omega b^{\dagger} b + \sum_{k=1}^{N} \epsilon_k S_k^{z} + V \sum_{k=1}^{N} \left(S_k^{+} b + b^{\dagger} S_k^{-} \right).$$

- The normal QISM

Particle-like eigenstate: $$|\lambda_1 \dots \lambda_M\rangle = \prod_{i=1}^{M} \mathrm{S}^{+}(\lambda_i)\,|0;\downarrow,\dots\downarrow\rangle$$

Since the vacuum $|0;\downarrow,\dots\downarrow\rangle$ is an actual eigenstate of both $S^2(u)$, $S^z(u)$ and annihilated by every $^{-}(\ ) = \ +\sum_{j=1}^{N} \frac{}{-\epsilon_j}\ ^{-}_{j}$, the usual QISM works and leads to Bethe equations:

$$\frac{\omega-\lambda_j}{V}+\sum_{i=1}^{N}\frac{V}{\lambda_j-\epsilon_i}+\sum_{i\neq j}\frac{2V}{\lambda_i-\lambda_j}=0$$

The bosonic mode however kills the highest/lowest weight symmetry we used to define the dual ABA since the number of bosonic excitations is not bounded.

- Numerical checks for small systems

1. Find eigenstates the normal way: find solutions {λ_1 ... λ_M} to the Bethe equations and build the eigenstate:

$$|\lambda_1 \dots \lambda_M\rangle = \prod_{i=1}^{M} \mathrm{S}^+(\lambda_i)\,|0;\downarrow,\dots\downarrow\rangle$$

2. For a given particle eigenstate {λ_1 ... λ_M} one can check numerically whether there exists a set {μ_1 ... μ_N} such that:

$$|\{\mu_1 \dots \mu_N\rangle = \prod_{i=1}^{N} \mathrm{S}^-(\mu_i)\,|M;\uparrow\dots\uparrow\rangle = C\,|\lambda_1 \dots \lambda_M\rangle$$

... and it systematically worked ... so it might be possible to build a second ABA.

We chose this particular dual vacuum and use N hole-excitations since the state then spans the same subspace of the Hilbert space as the original particle-like state.

- The “abnormal” ABA

Since the hole-like ansatz seems valid, we can start looking for a second ABA in to find hole-like Bethe Equations that the set {μ_1 ... μ_N} would have to obey.

However our “hole pseudo-vacuum” is **not annihilated by $S^+(u)$**, since we can always keep adding bosons:

$$\begin{aligned} \mathrm{S}^+(u)\,|M;\uparrow\ldots\uparrow\rangle &= \left[b^\dagger + \sum_{i=1}^{N} \frac{V}{u-\epsilon_i} S_i^+\right] |M;\uparrow\ldots\uparrow\rangle \\ &= \sqrt{M+1}\,|M+1;\uparrow\ldots\uparrow\rangle \neq 0 \end{aligned}$$

... and is **not an eigenstate of $S^2(u)$**:

$$\mathrm{S}^2(u)\,|M;\uparrow\ldots\uparrow\rangle = \ell(u)\,|M;\uparrow\ldots\uparrow\rangle + \underline{\sqrt{M+1}\,\mathrm{S}^-(u)\,|M+1;\uparrow\ldots\uparrow\rangle}\,.$$

At least it is an eigenstate of $S^z(u)$, but that is not sufficient for the ABA to work ...

QISM gives:

$$\mathrm{S}^2(u)\left[\prod_{i=1}^{N}\mathrm{S}^-(\mu_i)\right]|M;\uparrow\uparrow\ldots\uparrow\rangle=\left[\prod_{i=1}^{N}\mathrm{S}^-(\mu_i)\right]\mathrm{S}^2(u)\,|M;\uparrow\uparrow\ldots\uparrow\rangle+$$
$$[\mathrm{S}^2(u),\prod_{i=1}^{N}\mathrm{S}^-(\mu_i)]\,|M;\uparrow\uparrow\ldots\uparrow\rangle$$

but here the first term is not trivially proportional to: $\left[\prod_{i=1}^{N}\mathrm{S}^-(\mu_i)\right]|M;\uparrow\uparrow\ldots\uparrow\rangle$

We should find (with a proper vacuum):

Eigenvalue — Unwanted terms

$$S^2(u)\,|\{\mu_1\ldots\mu_N\}\rangle \;=\; \tilde{E}(u,\{\mu\})\,|\{\mu_1\ldots\mu_N\}\rangle+\sum_{k=1}^{N}F_k(u,\{\mu\})\,|\{\mu_1\ldots\mu_k\to u,\ldots\mu_N\}\rangle$$

QISM gives:

$$\mathrm{S}^2(u)\left[\prod_{i=1}^{N}\mathrm{S}^-(\mu_i)\right]|M;\uparrow\uparrow\ldots\uparrow\rangle = \left[\prod_{i=1}^{N}\mathrm{S}^-(\mu_i)\right]\mathrm{S}^2(u)\,|M;\uparrow\uparrow\ldots\uparrow\rangle + [\mathrm{S}^2(u),\prod_{i=1}^{N}\mathrm{S}^-(\mu_i)]\,|M;\uparrow\uparrow\ldots\uparrow\rangle$$

but here the first term is not trivially proportional to: $\left[\prod_{i=1}^{N}\mathrm{S}^-(\mu_i)\right]|M;\uparrow\uparrow\ldots\uparrow\rangle$

We do find (with our **faulty** vacuum):

Eigenvalue ? Unwanted terms ?

$$S^2(u)\,|\{\mu_1\ldots\mu_N\}\rangle = \tilde{E}(u,\{\mu\})\,|\{\mu_1\ldots\mu_N\}\rangle + \sum_{k=1}^{N}F_k(u,\{\mu\})\,|\{\mu_1\ldots\mu_k\to u,\ldots\mu_N\}\rangle + \sqrt{M+1}\left(\prod_{i=1}^{N}S^-(\mu_i)\right)S^-(u)\,|M+1,\uparrow\ldots\uparrow\rangle$$

Annoying term !

Numerically one can easily show that the last two terms **cannot** be cancelled for arbitrary spectral parameter u:

$$S^2(u)\,|\{\mu_1 \dots \mu_N\}\rangle = \tilde{E}(u,\{\mu\})\,|\{\mu_1 \dots \mu_N\}\rangle + \sum_{k=1}^{N} F_k(u,\{\mu\})\,|\{\mu_1 \dots \mu_k \to u, \dots \mu_N\}\rangle$$
$$+\sqrt{M+1}\left(\prod_{i=1}^{N} S^-(\mu_i)\right) S^-(u)\,|M+1,\uparrow \dots \uparrow\rangle$$

The unwanted terms should therefore actually be:

$$S^2(u)\,|\{\mu_1 \dots \mu_N\}\rangle = E(u,\{\mu\})\,|\{\mu_1 \dots \mu_N\}\rangle + \Delta E(u,\{\mu\})\,|\{\mu_1 \dots \mu_N\}\rangle$$
$$+\sum_{k=1}^{N} F_k(u,\{\mu\})\,|\{\mu_1 \dots \mu_k \to u, \dots \mu_N\}\rangle$$
$$+\sqrt{M+1}\left(\prod_{i=1}^{N} S^-(\mu_i)\right) S^-(u)\,|M+1,\uparrow \dots \uparrow\rangle = 0$$

The problem is that we do not know, for a given value of u, how much ΔE(u) we need to grab from the first term to make it possible to cancel the unwanted terms. Things just do not neatly fall into place as the would for a proper vacuum.

However, if we work with the conserved charges themselves R_i then things just fall into place much better and we DO find (as can be confirmed by small-size numerics):

Eigenvalue r_k

$$R_k|\mu\rangle = \lim_{u\to\epsilon_k}(u-\epsilon_k)\mathrm{S}^2(u)|\mu\rangle$$

$$= V^2\left[\frac{1}{2}\sum_{j\neq k}\frac{1}{\epsilon_k-\epsilon_j}-\frac{\omega-\epsilon_k}{2V^2}-\sum_{j=1}^{N}\frac{1}{\epsilon_k-\mu_j}\right]|\mu\rangle$$

$$+V^2\left[\sum_{j=1}^{N}\frac{B_j}{\epsilon_k-\mu_j}\left(\prod_{i\neq j}^{N}\mathrm{S}^-(\mu_i)\right)\right]S_k^-|M;\uparrow\ldots\uparrow\rangle$$

$$+V\sqrt{M+1}\left(\prod_{i=1}^{N}\mathrm{S}^-(\mu_i)\right)S_k^-|M+1;\uparrow\ldots\uparrow\rangle$$

} Unwanted terms

Where we defined: $B_j=\left(-\frac{\omega-\mu_j}{V}+\sum_{i=1}^{N}\frac{V}{\mu_j-\epsilon_i}+\sum_{i\neq j}\frac{2V}{(\mu_i-\mu_j)}\right).$

The unwanted terms found from the application of R_k are:

$$V^2\left[\sum_{j=1}^{N}\frac{B_j}{\epsilon_k-\mu_j}\left(\prod_{i\neq j}^{N}\mathrm{S}^-(\mu_i)\right)\right]S_k^-|M;\uparrow\ldots\uparrow\rangle$$

$$+V\sqrt{M+1}\left(\prod_{i=1}^{N}\mathrm{S}^-(\mu_i)\right)S_k^-|M+1;\uparrow\ldots\uparrow\rangle$$

We need to find Bethe equations for B_j such that the unwanted terms, coming from every R_k, cancel out.

Each $S^-(\mu_j)$ can either lower the bosonic number or flip down a spin

$$^-(\ \ )=\ +\sum_{j=1}^{N}\frac{}{-\epsilon_j}\ \ ^-_j$$

if N < M+1, there is a **term which comes from the bosonic action of every $S^-(\mu_j)$**, and it is easy to write the coefficient in front as:

$$V^2\left[\sum_{j=1}^{N}\frac{B_j}{\epsilon_k-\mu_j}\frac{\sqrt{M!}}{\sqrt{(M+1-N)!}}\right]+V\sqrt{M+1}\frac{\sqrt{(M+1)!}}{\sqrt{(M+1-N)!}}$$

which cancels when: $$\sum_{j=1}^{N}\frac{B_j}{\epsilon_k-\mu_j}+\frac{M+1}{V}=0.$$

One needs these terms to cancel after the application of every conserved operator R_k which leads to a system of linear equations for the B_j:

$$\begin{pmatrix} 1/(\epsilon_1-\mu_1) & 1/(\epsilon_1-\mu_2) & \dots & 1/(\epsilon_1-\mu_N) \\ \vdots & \vdots & \dots & \vdots \\ 1/(\epsilon_N-\mu_1) & 1/(\epsilon_N-\mu_2) & \dots & 1/(\epsilon_N-\mu_N) \end{pmatrix} \begin{pmatrix} B_1 \\ \vdots \\ B_N \end{pmatrix} = \begin{pmatrix} -\frac{M+1}{V} \\ \vdots \\ -\frac{M+1}{V} \end{pmatrix}$$

Cauchy Matrix, whose inverse we know

$$B_j \equiv \boxed{\frac{-\omega+\mu_j}{V} = \sum_{i=1}^{N} \frac{V}{\mu_j-\epsilon_i} + \sum_{i\neq j}^{N} \frac{2V}{\mu_i-\mu_j} = -\frac{(M+1)}{V} \frac{\prod_{k=1}^{N}(\epsilon_k-\mu_j)}{\prod_{a\neq j}^{N}(\mu_a-\mu_j)}}$$

Dual Bethe Equations ?

To compare to the particle-like BE: $$\frac{\omega-\lambda_j}{V} + \sum_{i=1}^{N} \frac{V}{\lambda_j-\epsilon_i} + \sum_{i\neq j} \frac{2V}{\lambda_i-\lambda_j} = 0$$

$$B_j \equiv \boxed{\frac{-\omega+\mu_j}{V} = \sum_{i=1}^{N} \frac{V}{\mu_j - \epsilon_i} + \sum_{i\neq j}^{N} \frac{2V}{\mu_i - \mu_j} = -\frac{(M+1)}{V} \frac{\prod_{k=1}^{N}(\epsilon_k - \mu_j)}{\prod_{a\neq j}^{N}(\mu_a - \mu_j)}}$$

Therefore sets {μ_1 ... μ_N} which are solution to these equations will cancel one small subset of coefficients which MAY appear in the unwanted terms. We still need to prove that it cancels every coefficient which actually appears in the vector:

$$V^2 \left[\sum_{j=1}^{N} \frac{B_j}{\epsilon_k - \mu_j} \left(\prod_{i\neq j}^{N} \mathrm{S}^-(\mu_i)\right)\right] S_k^- |M;\uparrow \ldots \uparrow\rangle$$

$$+ V\sqrt{M+1} \left(\prod_{i=1}^{N} \mathrm{S}^-(\mu_i)\right) S_k^- |M+1;\uparrow \ldots \uparrow\rangle$$

Summing up the ways to go from $|M;\uparrow\ldots\uparrow\rangle$ or $|M+1;\uparrow\ldots\uparrow\rangle$ to a given canonical basis state: $|m;\uparrow\ldots\downarrow_{i_1}\ldots\uparrow\ldots\downarrow_k\ldots\uparrow\ldots\downarrow_{i_r}\ldots\uparrow\rangle$

a_k excludes μ_j

$$C^r_{\{i_1\ldots i_r,k\}} = \frac{\sqrt{M!}}{\sqrt{m!}} V^{r+2} \left(\sum_{j=1}^{N} \frac{B_j}{\epsilon_k - \mu_j} \left[\sum_{A_{\hat{j}} \in S^r_{\hat{j}}} \frac{1}{\prod_{k=1}^{r}(a_k - \epsilon_{i_k})} \right] + \frac{(M+1)}{V} \left[\sum_{A \in S^r} \frac{1}{\prod_{k=1}^{r}(\alpha_k - \epsilon_{i_k})} \right] \right)$$

α_k includes μ_j

gives the coefficients we need to cancel.

We showed that, provided the B_j (and therefore the set $\{\mu_1 \ldots \mu_N\}$) are a solution to our equations, every one of these coefficients turns out to be (a complicated representation of) 0.

H. Tschirhart and A. Faribault, *Algebraic Bethe ansätze and eigenvalue-based determinants for Dicke–Jaynes–Cummings–Gaudin quantum integrable models*, J. Phys. A: Math. Theor. **47**, 405204 (2014).

Algebraic Bethe Ansätze:

Particle-like

States of the form:

$$|\lambda_1 \dots \lambda_M\rangle = \prod_{i=1}^{M} \mathrm{S}^{+}(\lambda_i)\,|0;\downarrow,\dots\downarrow\rangle$$

become eigenstates when

$$\frac{\omega-\lambda_j}{V} + \sum_{i=1}^{N}\frac{V}{\lambda_j-\epsilon_i} + \sum_{i\neq j}\frac{2V}{\lambda_i-\lambda_j} = 0$$

with eigenvalues

$$r_i = \frac{V^2}{2}\left(\sum_{j\neq i}^{N}\frac{1}{\epsilon_i-\epsilon_j}\right) - \frac{\epsilon_i-\omega}{2} - V^2\sum_{j=1}^{M}\frac{1}{\epsilon_i-\lambda_j}$$

which themselves obey the BEs:

$$[\Lambda(\epsilon_i)]^2 = \sum_{j\neq i}^{N}\frac{\Lambda(\epsilon_i)-\Lambda(\epsilon_j)}{\epsilon_i-\epsilon_j} - \frac{\epsilon_i-\omega}{V^2}\Lambda(\epsilon_i) + \frac{M}{V^2}$$

Hole-like (starting from a “faulty” vacuum)

States of the form:

$$|\{\mu_1 \dots \mu_N\rangle = \prod_{i=1}^{N} \mathrm{S}^{-}(\mu_i)\,|M;\uparrow\dots\uparrow\rangle$$

become eigenstates when

$$\frac{-\omega+\mu_j}{V} = \sum_{i=1}^{N}\frac{V}{\mu_j-\epsilon_i} + \sum_{i\neq j}^{N}\frac{2V}{\mu_i-\mu_j} = -\frac{(M+1)}{V}\frac{\prod_{k=1}^{N}(\epsilon_k-\mu_j)}{\prod_{a\neq j}^{N}(\mu_a-\mu_j)}$$

with eigenvalues

$$r_k = V^2\left[\frac{1}{2}\sum_{j\neq k}\frac{1}{\epsilon_k-\epsilon_j} - \frac{\omega-\epsilon_k}{2V^2} - \sum_{j=1}^{N}\frac{1}{\epsilon_k-\mu_j}\right]$$

which themselves obey the BEs:

$$[\Lambda^{\mu}(\epsilon_i)]^2 = \sum_{j\neq i}\frac{\Lambda^{\mu}(\epsilon_i)-\Lambda^{\mu}(\epsilon_j)}{\epsilon_i-\epsilon_j} + \frac{\epsilon_i-\omega}{V^2}\Lambda^{\mu}(\epsilon_i) + \frac{M-N+1}{V^2}$$

The transformation:

$$\Lambda^{\lambda}(\epsilon_k) = \Lambda^{\mu}(\epsilon_k) + \frac{\omega-\epsilon_k}{V^2}$$

allows one to go from one representation of a given eigenstate to the other.

Again one can now easily compute scalar products by mixing the two representations of eigenstates turning them into a partition function:

$$\langle \mu_1 \dots \mu_N \mid \lambda_1 \dots \lambda_M \rangle = \langle M; \uparrow \dots \uparrow | \prod_{i=1}^{N} \mathrm{S}^+(\mu_i) \prod_{j=1}^{M} \mathrm{S}^+(\lambda_j) \, |0; \downarrow \dots \downarrow \rangle$$

This time, the same iterative proof allows us to demonstrate that summing the Cauchy permanents built out of every subsets of cardinality N (taken from the set of N+M rapidities) is given by the same form of determinant:

$$\mathrm{S}^+(u) = b^\dagger + \sum_{j=1}^{N} \frac{V}{u - \ _j} S_j^+$$

$$\langle M; \uparrow \dots \uparrow | \prod_{i=1}^{N} \mathrm{S}^+(\mu_i) \prod_{j=1}^{M} \mathrm{S}^+(\lambda_j) \, |0; \downarrow \dots \downarrow \rangle =$$

$$\sum_{S \in \binom{\{\nu_1 \dots \nu_{N+M}\}}{N}} \mathrm{Perm}_{N \times N} \left(\frac{1}{\nu_{i_a} - \epsilon_j} \right) = \mathrm{Det}_{N \times N} J$$

So that:

$$\langle \mu_1 \dots \mu_N \, | \lambda_1 ... \lambda_M \rangle = \sqrt{M!} V^{M+N} \mathrm{Det} J$$

$$J_{ab} = \begin{cases} \sum_{c \neq a} \frac{1}{\epsilon_a - \epsilon_c} - \Lambda^{\lambda}(\epsilon_a) - \Lambda^{\mu}(\epsilon_a) & a = b \\ \frac{1}{\epsilon_a - \epsilon_b} & a \neq b \end{cases}$$

Identically, we find that the form-factors of local spin or bosonic operators can be written as eigenvalue-dependent determinants.

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Conclusions

- The quantum inverse scattering method can (at least sometimes) be carried out on an “faulty” pseudo-vacuum.

- Here it allowed us to build **two alternative Algebraic Bethe Ansätze** leading to two distinct representations of the eigenstates of a **spin-boson Gaudin model**.

- Mixing the normal and dual representations allowed us to find **determinant** expressions for scalar products and **form factors** which **only depend on the eigenvalues** of the conserved charges. By evacuating the need for explicit Bethe roots, this can hugely help in the numerical treatment of such systems.

Next question:

Can one do the same for spins-only but with $S > 1/2$?

The dual representation is trivial, but the recursive proof for the partition function is not as nicely behaved as I expected ...