Spherical means representation of the wave propagator in a FRW background metric

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Workshop on AdS/CFT, self-adjoint extensions
and the resolution of cosmological singularities
Centre de Recherches Mathématiques, Montréal, Canada
July 6-10, 2015

Outline

Friedmann – Robertson – Walker metrics

Minkowski space-time

FRW space-time with $K = 0$

FRW space-time with $K = -1$

Conclusions

Work in collaboration with Bilal Abbasi (McGill, ISM)

Acknowledgements:
NSERC, Canada Research Chairs Program, The Fields Institute

FRW space-times

- Lorentzian line element for the classical Friedmann – Robertson – Walker space-time metrics

$$ds^2 = -dt^2 + S(t)^2 d\sigma^2 \ ,$$

where $d\sigma^2$ is the line element for each underlying spatially homogeneous time slice $\{(t, x) \ : \ t = t_0\}$.

- Cases: $\mathbb{R}^3$ with $K = 0$, and hyperbolic space $\mathbb{H}^3$ with $K = -1$
- Under the time change

$$\frac{dt}{d\tau} = S(t) \ , \qquad t(0) = 0 \ ,$$

the line element (3) is transformed to

$$ds^2 = S(\tau)^2 \left(-d\tau^2 + d\sigma^2\right) \ ,$$

conformal to the half space $\mathbb{R}_+ \times \mathbb{R}^3 = \{(t, x) : \tau > 0\}$ with the Minkowski metric

scale factor

- The scale factor

$$\frac{dt}{d\tau} = S(t)$$

- Minkowski space

$$S(\tau) = 1 \,, \qquad t = \tau$$

- Zero curvature $K = 0$

$$S(\tau) = \tau^2 \,, \qquad t = \frac{1}{3}\tau^3$$

  singular at $\qquad t = \tau = 0$

- Negative curvature $K = -1$

$$S(\tau) = \cosh(\tau) - 1 \,, \qquad t = \sinh(\tau) - \tau$$

wave propagator

- The D'Alembertian operator in Friedmann – Robertson – Walker space-times is given by

$$\Box u = -\frac{1}{S^2}\partial_\tau^2 u - \frac{2\dot{S}}{S^3}\partial_\tau u + \frac{1}{S^2}\Delta_\sigma u \, .$$

  Here Δ_σ is the Laplace – Beltrami operator for the Riemannian metric on the time slices $\{\tau = \tau_0 > 0\}$
- The wave propagator $W(\tau_0, \tau)(g, h)$ is the solution operator for the wave equation

$$\Box u = 0 \, , \qquad u(\tau_0, x) = g(x) \, , \quad \partial_\tau u(\tau_0, x) = h(x)$$

- Specifically, $W(\tau_0, \tau_1)(g, h)$ is the mapping of phase space

$$W(\tau_0, \tau_1)(g, h) := (u(\tau_1, x), \partial_\tau u(\tau_1, x)) \, , \qquad \tau_0 > 0 \, , \quad \tau_1 > 0$$

Kirchhoff's formula for Minkowski space-time

- the wave equation in Minkowski space-time $\mathbb{R}^{1+3}_{t,x}$

$$\partial_t^2 u - \Delta u = 0 \,, \qquad u(t_0, x) = g(x) \,, \quad \partial_t u(t_0, x) = h(x)$$

- Define the spherical means operator

$$M_f(r, x) := \frac{1}{4\pi r^2} \int_{S_r(x)} f(y) \, dS_r(y)$$

- The Kirchhoff formula

$$\begin{aligned} u(t,x) &= \partial_t((t-t_0)M_g(t-t_0,x)) + (t-t_0)M_h(t-t_0,x) \\ &= \frac{1}{4\pi(t-t_0)} \int_{S_{t-t_0}(x)} \nabla g(y) \cdot \frac{y-x}{|y-x|} \, dS_{t-t_0}(y) \\ &\quad + \frac{1}{4\pi(t-t_0)^2} \int_{S_{t-t_0}(x)} g(y) \, dS_{t-t_0}(y) \\ &\quad + \frac{1}{4\pi(t-t_0)} \int_{S_{t-t_0}(x)} h(y) \, dS_{t-t_0}(y) \,. \end{aligned}$$

Huygen's two principles

- This result follows directly from the Kirchhoff formula

Theorem (1)

Data $(g(x), h(x))$ *with support within* $B_R(0)$ *gives rise to solutions* $u(t, x)$ *with support within* $B_{R+|t|}(0)$ *(Huygens principle)*

For data as above, the solution $u(t, x)$ *vanishes inside the envelope of light cones with base in* $B_R(0)$ *(strong Huygens principle)*

Solutions decay in time as follows:

$$|u(t, x)| \leq \frac{C}{|t|}$$

FRW with $K = 0$

- The wave equation in FRW space-time with $K = 0$ takes the form

$$\begin{cases} \partial_\tau^2 u + \frac{4}{\tau}\partial_\tau u - \Delta u = 0 & 0 < \tau, \tau_0 \; ; \; x \in \mathbb{R}^3 \\ u(\tau_0, x) = g(x) \\ u_\tau(\tau_0, x) = h(x) \end{cases}$$

- Transformation (S. Klainerman & P. Sarnak (1982))

$$v(\tau, x) = \frac{1}{\tau}\partial_\tau(\tau^3 u) \; ,$$

which has an inverse expression for $u(\tau, x)$

$$\tau^3 u(\tau, x) = \int_0^{\tau - \tau_0} (r + \tau_0) v(r + \tau_0, x)\, dr + \tau_0^3 g(x) \; .$$

- The function $v(\tau, x)$ satisfies the wave equation

$$\partial_\tau^2 v = \Delta v \qquad \tau, \tau_0 > 0, \; x \in \mathbb{R}^3$$

Spherical means for the FRW wave propagator

- The initial data transforms to

$$\begin{cases} v(\tau_0, x) = 3\tau_0 g(x) + \tau_0^2 h(x) := \phi(x) \\ \partial_\tau v(\tau_0, x) = 3g(x) + \tau_0^2 \Delta g(x) + \tau_0 h(x) \\ \qquad := \psi(x) \end{cases}$$

so that the solution is expressed through Kirchhoff's formula

$$v(\tau, x) = \partial_\tau \left((\tau - \tau_0) M_\phi(\tau - \tau_0, x)\right) + (\tau - \tau_0) M_\psi(\tau - \tau_0, x)$$

- Expressing this in original variables in terms of $u(\tau, x)$

$$
\begin{aligned}
u(\tau,x) &= \frac{1}{\tau^3}\Bigg(\frac{1}{4\pi}\frac{\tau^3-(\tau-\tau_0)^3}{(\tau-\tau_0)^2}\int_{S_{\tau-\tau_0}} g(y)\, dS_{\tau-\tau_0}(y) \\
&\quad + \frac{1}{4\pi}\frac{\tau_0^2\tau}{(\tau-\tau_0)}\int_{S_{\tau-\tau_0}} \nabla g(y)\cdot\frac{y-x}{|y-x|}\, dS_{\tau-\tau_0}(y) \\
&\quad + \frac{3}{4\pi}\iint_{B_{\tau-\tau_0}(x)} g(y)\, dS_r(y)\, dr\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{\tau^3}\Bigg(\frac{\tau\tau_0^2}{4\pi(\tau-\tau_0)}\int_{S_{\tau-\tau_0}(x)} h(y)\, dS_{\tau-\tau_0}(y) \\
&\quad + \frac{\tau_0}{4\pi}\iint_{B_{\tau-\tau_0}(x)} h(y)\, dS_r(y)\, dr\Bigg)
\end{aligned}
$$

Huygens principle

- Theorem (2 (Abbasi & Craig 2014))

  Solutions satisfy the property of finite propagation speed; data $(g(x), h(x))$ *supported in* $B_R(0)$ *gives rise to solutions* $u(\tau, x)$ *which are supported in* $B_{R+|\tau-\tau_0|}(0)$

  Solutions do not satisfy the strong Huygens principle. In the interior of the light cone we have

$$
\begin{aligned}
u(\tau, x) &= \frac{1}{\tau^3}\left(\frac{3}{4\pi}\iint_{B_{\tau-\tau_0}(x)} g(y)\, dS_{\tau-\tau_0}(y)\, dr \right. \\
&\quad \left. + \frac{\tau_0}{4\pi}\iint_{B_{\tau-\tau_0}(x)} h(y)\, dS_{\tau-\tau_0}(y)\, dr\right)
\end{aligned}
$$

  Solutions obey the decay rate

$$
|u(\tau, x)| \leq \frac{C}{|\tau-\tau_0|^3} \sim \frac{C'}{|t|}
$$

Initial value problem for $\tau_0 = 0$

- In the limit $\tau_0 \to 0$ with τ_1 fixed, the wave propagator exists (!)

  $$\lim_{\tau_0 \to 0+} W(\tau_0, \tau_1)(g, h) = W(0, \tau_1)(g)$$

  Specifically

  $$u(\tau, x) = \frac{1}{\tau^3} \int_0^\tau 3r^2 M_g(r, x)\, dr$$

  Initial data $g(x)$ is able to be specified at the big bang $\tau = 0$

- This solution is reversible in time $\tau \mapsto -\tau$, giving a large class of solutions which are conformally regularizable to all of $\mathbb{R}^{1+3}_{\tau,x}$

FRW with $K = -1$

- Constant curvature $K = -1$, with scaling factor $S(\tau) = \cosh(\tau) - 1$.

$$\partial_\tau^2 u + 2\coth(\frac{\tau}{2})\partial_\tau u - \Delta_\sigma u = 0 \, , \quad \tau > 0 \, , \quad x \in \mathbb{H}^3$$

$$u(\tau_0, x) = g(x) \, , \quad \partial_\tau u(\tau_0, x) = h(x) \, , \qquad \tau_0 > 0$$

- Transformation (again S. Klainerman & P. Sarnak)

$$v(\tau, x) = \frac{4}{\sinh\left(\frac{\tau}{2}\right)} \partial_\tau \left(\sinh^3(\frac{\tau}{2}) u(\tau, x) \right)$$

and its inverse

$$\sinh^3(\frac{\tau}{2}) u(\tau, x) = \int_0^{\tau - \tau_0} \frac{1}{4} \sinh(\frac{r + \tau_0}{2}) v(r + \tau_0, x) \, dr + \sinh^3(\frac{\tau_0}{2}) u(\tau_0, x)$$

hyperbolic wave equation

- The function $v(\tau, x)$ satisfies the hyperbolic wave equation

$$\partial_\tau^2 v = Lv \, , \qquad L = \Delta_\sigma + 1$$

- Cauchy data is given on the hypersurface $\tau = \tau_0 > 0$

$$v(\tau_0, x) = 3\sinh(\tau_0) g(x) + 4\sinh^2(\frac{\tau_0}{2}) h(x) := \phi(x)$$

$$\partial_\tau v(\tau_0, x) = 3\cosh(\tau_0) g(x) + 4\sinh^2(\frac{\tau_0}{2}) \Delta_\sigma g + \sinh(\tau_0) h(x) := \psi(x) \, .$$

- For case $K = -1$ the geodesic spherical mean is given by an integral over the geodesic sphere $S_r(x)$ about x

$$M_f(r, x) := \frac{1}{4\pi(\sinh(r))^2} \int_{S_r(x)} f(y) \, dS_r(y) \, ,$$

where $dS_r(x)$ is the element of spherical surface area.

spherical means formulae for $K = -1$

- For (14) there is an explicit spherical means formula for the solution given in Lax & Philips, the hyperbolic analog of the Kirchhoff formula

$$v(\tau, x) = \partial_\tau \big(\sinh(\tau-\tau_0) M_\phi(\tau-\tau_0, x)\big) + \sinh(\tau-\tau_0) M_\psi(\tau-\tau_0, x)$$

- Recall that (ϕ, ψ) are functions of (g, h) as well as τ_0

- Returning to the original variables

$$
\begin{aligned}
u(\tau, x) = & \int_{S_{\tau-\tau_0}(x)} A(\tau, \tau_0)\partial_r g(y)\, dS_{\tau-\tau_0}(y) \\
& + \int_{S_{\tau-\tau_0}} B(\tau, \tau_0) g(y)\, dS_{\tau-\tau_0}(y) \\
& + \iint_{B_{\tau-\tau_0}(x)} C(r, \tau, \tau_0) g(y)\, dS_r(y)\, dr \Big) \\
& + \int_{S_{\tau-\tau_0}(x)} D(\tau, \tau_0) h(y)\, dS_{\tau-\tau_0}(y) \\
& + \iint_{B_{\tau-\tau_0}(x)} E(r, \tau, \tau_0) h(y)\, dS_r(y)\, dr \Big)
\end{aligned}
$$

Spherical means expression in original variables for $K=-1$

$$u(\tau,x)=\frac{1}{\sinh^3(\tau/2)}\left(\frac{\sinh^2(\tau_0/2)\sinh(\tau/2)}{4\pi\sinh(\tau-\tau_0)}\int_{S_{\tau-\tau_0}(x)}\partial_r g(y)\,dS_{\tau-\tau_0}(y)\right.$$

$$+\Big(\frac{1}{2}\sinh(\tau/2)\sinh(\tau_0/2)\big(3\sinh(\tau-\tau_0)\cosh(\tau_0/2)+\cosh(\tau-\tau_0)\sinh(\tau_0/$$

$$+\frac{1}{2}\sinh^3(\tau_0/2)\Big)\frac{1}{4\pi\sinh^2(\tau-\tau_0)}\int_{S_{\tau-\tau_0}}g(y)\,dS_{\tau-\tau_0}(y)$$

$$\left.+\iint_{B_{\tau-\tau_0}(x)}\frac{3}{8}\Big(\sinh(\frac{r+\tau_0}{2})+\sinh(\frac{r-\tau_0}{2})\Big)\frac{1}{4\pi\sinh(r)}g(y)\,dS_r(y)\,dr\right)$$

$$+\frac{1}{\sinh^3(\tau/2)}\left(\frac{\sinh(\tau/2))\sinh^2(\tau_0/2))}{4\pi\sinh(\tau-\tau_0)}\int_{S_{\tau-\tau_0}(x)}h(y)\,dS_{\tau-\tau_0}(y)\right.$$

$$\left.+\iint_{B_{\tau-\tau_0}(x)}\frac{1}{4}\Big(\cosh(\frac{r+\tau_0}{2})-\cosh(\frac{r-\tau_0}{2})\Big)\frac{1}{4\pi\sinh(r)}h(y)\,dS_r(y)\,dr\right)$$

Wave propagator on a FRW background metric, $K = -1$

- Theorem (2)

  Properties of the wave propagator in the case $K = -1$:

  Huygens principle holds

  But not the strong Huygens principle

  Decay rates of solutions

$$|u(\tau, x)| \leq \frac{C}{e^{2\tau}} \sim \frac{C'}{t^2}$$

Singular initial value problem

- Again taking the limit of the wave propagator for fixed τ_1, for $\tau_0 \to 0$

$$u(\tau, x) = \frac{1}{4\sinh^3(\frac{\tau}{2})} \int_0^\tau 3\sinh(\frac{r}{2})\sinh(r) M_g(r, x)\, dr$$

  Initial data $g(x)$ can be specified at the big bang $\tau = t = 0$

  Under time reversal this gives a family of solutions eminating from the singularity, which are defined for all (t, x) and are conformally regularized by the time change described at the beginning of this talk

Decay rates and theorems on global existence

- The importance of decay rates of solutions of the linearized equations in the question of stability of small solutions.
  The model problem of a scalar wave equation

$$\Box u + f(\nabla u, \nabla^2 u) = 0 \,, \quad x \in \mathbb{R}^n \tag{1}$$

  where $f(\cdot)$ is order $m-1$ in its arguments
- Angular momentum operators $\Omega_{j\ell} = x_j \partial_{x_\ell} - x_\ell \partial_{x_j}$
  Work in the invariant norm Sobolev spaces

$$Z^a := \{(u,p) := z \, : \, \Omega^\sigma \partial_x^\alpha z \in L^2(\mathbb{R}^n) \,, \, |\alpha| + |\sigma| \le a\}$$

- The standard argument for existence theory for the nonlinear wave equation (1) uses the invariant norm Sobolev estimate

$$|(u(t,\cdot), p(t,\cdot)|_{C^1} \le \frac{C}{\langle t \rangle^{(n-1)/2}} \|z\|_a$$

  with $a \ge (n+2)/2$

Global existence via an energy estimate

- Then energy estimates for (1) give

$$\begin{aligned}\|z(t,\cdot)\|_a &\leq C\exp\Big(\int_0^t |z(s',\cdot)|_{C^1}^{(m-2)}\,ds'\Big)\|z(t,0)\|_a\\ &\leq C\exp\Big(\int_0^t \frac{\|z(s',\cdot)\|_a}{\langle s'\rangle^{(n-1)(m-2)/2}}\,ds'\Big)\|z(t,0)\|_a\end{aligned}$$

This gives an *a priori* bound for $M_T := \sup_{|t|\leq T}\|z(t,\cdot)\|_a$

$$M_T\exp\Big(-M_T\int_0^T \frac{1}{\langle s'\rangle^{(n-1)(m-2)/2}}\,ds'\Big) \leq \|z(t,0)\|_a$$

This estimate is uniform in $T < +\infty$ if the integral $\int_0^T \langle s'\rangle^{-(n-1)(m-2)/2}\,ds'$ converges uniformly in T

If the integral grows logarithmically in T, it gives the lower bounds $T := T_R \geq \exp(C/R^{m-2})$

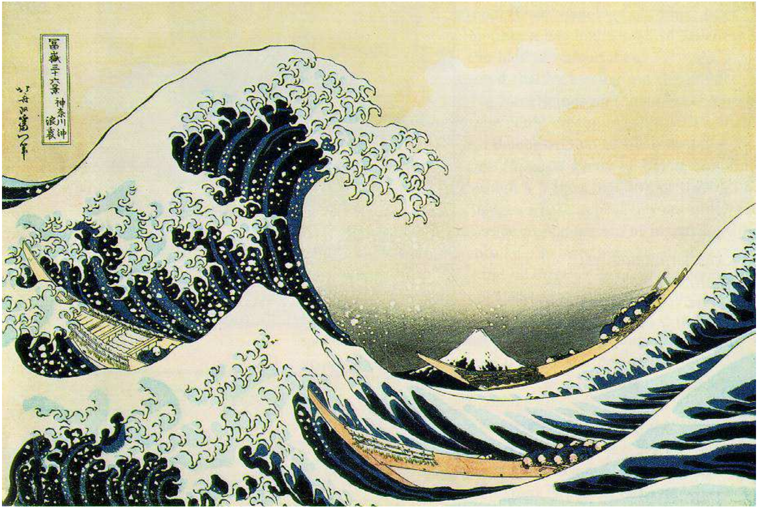
Questions and conclusions

- What are the physical manifestations of the lack of the strong Huygens principle. Can one detect a small blue shadow behind red shifted spectral lines?

  [Answer from discussions at the CRM workshop: in fact 'no'. The equations of E & M are conformally invariant, and FRW is conformal to the Minkowski metric on $\mathbb{R}_t^+ \times \mathbb{R}_x^3$. Hence the components of Maxwell's equations satisfy a conformally invariant wave equation, implying that electromagnetic radiation in an FRW background satisfies the strong Huygens principle. However other fields exhibit this behavior, such as certain components of gravitational waves.]

- Are there families of space-times that are perturbations of FRW, that exist for all $\tau \in \mathbb{R}$ and are conformally regularized at the singularity at $\tau = 0$?

- Decay rates in this expression of the wave propagator are extremely precise. It would be worthwhile to reconsider the distance-to-luminosity function that is used to compare redshift with distance.



Thank you