Conformal Anomalies, Renormalized area and bulk re-construction.

Spyros Alexakis

CRM AdS-CFT conference.

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- Renormalized area of minimal surfaces of minimal surfaces in asymtpoticaly AdS manifolds, and the expectation values on Wilson loops at conformal infinity.
- Reconstruction of bulk metrics from area data. *Equivalently:* Knowledge of the expectation values of Wilson loops determines the AdS metric uniquely, and *one can reconstruct the bulk metric*.

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$\int \mathcal{L}[\mathbf{g}]\phi$ is the *conformal anomaly* of the action $\mathcal{A}$. Note that the anomaly is invariant under the symmetry it breaks:

$$\int \mathcal{L}[e^{\psi}\mathbf{g}, \dots] dV_{e^{\psi}\mathbf{g}} = \int \mathcal{L}[\mathbf{g}, \dots] dV_{\mathbf{g}}. \quad (1)$$

The structure of conformal anomalies.

Conjecture (Deser-Schwimmer, '93.)

Assume that $\int_{M^n} P(\mathbf{g})dV_{\mathbf{g}}$ *satisfies:*

$$\int_{M^n} P(e^{2\psi}\mathbf{g})dV_{e^{2\psi}\mathbf{g}} = \int_{M^n} P(\mathbf{g})dV_{\mathbf{g}}. \qquad (2)$$

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- *$divT$ is a divergence of a vector field, and*
- Gauss-Bonnet$^n(\mathbf{g})$ *integrates to the Euler number:* $\int_{M^n} \text{Gauss-Bonnet}^n(\mathbf{g}) = \mathrm{C}_n \cdot \chi(M^n)$.

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Note: A variant of the above is also conjectured, where $\mathcal{L}[\mathbf{g}]$ is assumed to arise from an action. (This assumption is stronger). The conclusion is then also strengthened, to assume that the term $divT(\mathbf{g})$ also has extra structure. (It arises via the conformal variation of some other action). This other version is still open.

Applications: Volume re-normalization for asymptotically AdS space-times.

Work in Riemannian signature (for asymptotically hyperbolic metrics), in vacuum, $(M^n, \mathbf{g})$, with $\partial_\infty M$ inheriting a *conformal class* of metrics $[h]$.

Given a metric $h_0 \in [h]$ recall the expansion of $\mathbf{g}$ in the Fefferman-Graham coordinate x associated to h_0:

$$\mathbf{g} = x^{-2}(dx^2 + \sum_{i,j=1}^{n-1} [(h_0)_{ij}(y)dy^i dy^j + x^2 h^{(2)}_{ij}(y)dy^i dy^j + x^4 + \dots)]$$

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$$\mathrm{Vol}_{\mathbf{g}}[\{x \geq \epsilon\}] \sim \epsilon^{-n+1} + C\epsilon^{-n+3} + \dots + V + o(1).$$

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While for n odd:

$$\mathrm{Vol}_{\mathbf{g}}[\{x \geq \epsilon\}] \sim \epsilon^{-n+1} + C\epsilon^{-n+3} + \dots + V + logx \cdot L + o(1).$$

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For n odd it is NOT; The conformal anomaly is encapsulated by L. The key to both V and L is a quantity called the Q-curvature, defined for all *even* dimensions:

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\begin{aligned}
&n = 2 : Q^2 = R(\mathbf{g}). \\
&n = 4 : Q^4(\mathbf{g}) = \Delta R(\mathbf{g}) + c|W|^2 + c'[\mathrm{Gauss} - \mathrm{Bonnet}^4][\mathbf{g}], \\
&\dots, \\
&n = 2k : Q^{2k} = \Delta^{k-1} R(\mathbf{g}) + \dots.
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and $\int_{M^n} Q^n(\mathbf{g}) dV_{\mathbf{g}}$ *is* conformally invariant.

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Using the decomposition proven in the Theorem above, we derive:

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Chang-Qing-Yang have used the above to derive that for $(M^n, \mathbf{g})$ vacuum and asymptotically hyperbolic, when n even:

$$\mathcal{RV}[\mathbf{g}] = \int_M W^n(\mathbf{g}) dV_{\mathbf{g}} + \mathrm{C}_n \chi[M^n]. \tag{5}$$

When n is odd, the conformal anomaly L is encoded by the total Q^{n-1}-curvature *on the boundary*:

$$L = \int_{\partial M^n} Q^{n-1}(h) dV_h.$$

Minimal surfaces and area re-normalization.

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Theorem (A.-Mazzeo '08)

Let A be the extrinsic curvature of $Y \subset M^3$; then:

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In particular $\mathcal{A}[Y]$ depends explicitly on the topology, and the extrinsic geometry of the surface Y, as well as the Weyl curvature of the bulk M^3.

Reconstruction of the bulk metric from area data.

Question: Assume that for all 1-dimensional loops $\gamma^1 \subset \partial_\infty M^n$ we know $\mathcal{A}[Y]$, where Y is any minimal surface with $\partial_\infty Y = \gamma$. Can we then reconstruct the bulk metric $\mathbf{g}$ from this information?

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(Tentative) answer: Yes (forthcoming with Balehowsky, Nachman), at least for *some* manifolds. Consider any foliation of ∂M^n by an $(n-2)$-family of loops, $\gamma_t, t \in [0,1]^{n-2}$. Assume that the γ_t's bound a family of minimal surfaces Y_t, which foliate the entire bulk M^n. Assume that the areas of all Y_t and all their perturbations are known. Then $\mathbf{g}$ can be reconstructed from this data.