Dirty Holographic Superconductors

Leopoldo Pando Zayas
ICTP, Trieste/University of Michigan

Applications of AdS/CFT to QCD and Condensed Matter Physics
October 20, CRM, Montreal, Canada

Based on 1308.1920 (PRD), 1407.7526 (JHEP), 1507.02280
Work in Progress, D. Areán, LPZ, I. Salazar, A. Scardicchio

Outline

- Motivation: Anderson localization, Many Body Localization, Dirty Superconductors.
- A numerical approach to disorder in AdS/CFT.
- Disordered Holographic s- and p-wave Superconductor.
- Enhancement of superconductivity for mild disorder and a universality in the responses.
- Conductivity: Superconductor-Metal Transition.
- Outlook

Motivation: Big Picture

- What is the AdS/CFT correspondence an answer to?
- QCD?
- Phase transitions and infinite correlation length: $\xi \sim |p - p_c|^{-\nu}$ (CFT).
- K. Wilson already answered this question with RG. Critical exponents are universal, independent of the microscopic details (WF). Permission to approach the fixed points any way you can, including AdS/CFT.
- Critical exponents for strongly coupled systems.
- Critical exponents for disorder-driven strongly coupled transitions.
- Dynamical critical exponents in time-dependent systems (quenches, thermalization ...).

Motivation: Transport and Disorder

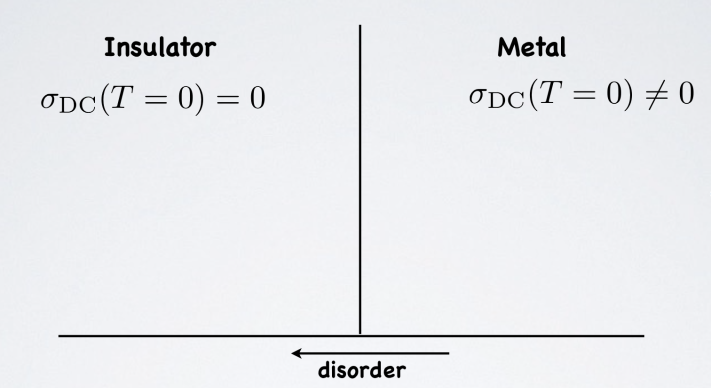
- Most systems are not clean!
- Problem (AdS/CM): Transport and translational symmetry/absence of momentum dissipation.
- Anderson Localization: The conductivity can be completely suppressed by quantum effects.
- Many Body Localization for isolated quantum systems: Anderson localization persists in the presence of finite interactions.

Anderson Localization

- Anderson's orginal model (1958): a single electron moves on a regular lattice, e.g. a hypercubic lattice, where each lattice point carries a random on-site potential V_i

$$H = \sum_{i,j,\sigma} t_{ij} c^{\dagger}_{i\sigma} c_{j\sigma} + \sum V_i n_{i\sigma} \tag{1}$$

- $c^{\dagger}_{i\sigma}, c_{i\sigma}$ create/annihilate a particle of spin σ on site i.
- $n_{i\sigma} = c^{\dagger}_{i\sigma} c_{i\sigma}$ is the number operator.
- The random energies V_i are characterized by their distribution $P(V_i)$ (Uniform, Gaussian, etc.).


Insulator
$\sigma_{\rm DC}(T=0)=0$
Metal
$\sigma_{\rm DC}(T=0)\neq 0$
disorder

Motivation: Many Body Localization

- The rapidly emerging field of many-body localization is concerned with the fate of the Anderson insulator under electron-electron interaction and the characterization of the possible resulting many-body localized phase.
- Basko-Aleiner-Altshuler ('06): Presented compelling evidence in favor of a many body localized phase, based on an analysis of the perturbation theory in electron-electron interaction to all orders

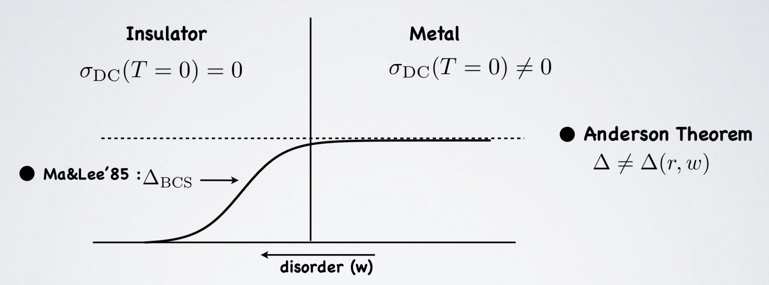
$$H = \sum_{\alpha} \xi_\alpha c^\dagger_\alpha c_\alpha + \frac{1}{2} \sum_{\alpha\beta\gamma\delta} V_{\alpha\beta\gamma\delta} c^\dagger_\alpha c^\dagger_\beta c_\gamma c_\delta \tag{2}$$

- $c^\dagger_\alpha |\Psi_k\rangle$ Single particle state above a certain eigenstate of the interacting system.

Anderson's Theorem

- Anderson's Theorem (Journal of Physics and Chemistry, 58): Superconductivity is insensitive to perturbations that do not destroy time-reversal invariance (pair breaking). This provided the central intuition.
- Critiques to Anderson's argument were raised, for example, by considering the effects of strong localization: `A. Kapitulnik, G. Kotliar, Phys. Rev. Lett. 54, 473, (1985). G. Kotliar, A. Kapitulnik, Phys. Rev. B 33, 3146 (1986), M. Ma, P.A. Lee, Phys. Rev. B 32, 5658, (1985).`
- More generally, the question of the role of interactions, in particular, the Coulomb interaction in dirty superconductors cannot be considered settled.

Anderson's theorem



What is AdS/CFT doing about momentum dissipation?

- Massive gravity:
  - Vegh: Holography without translational symmetry
  - Tong, Andrade,Davidson, ...
- Disorder (After our work):
  - Hartnoll and Santos: Disordered horizons: Holography of randomly disordered fixed points
  - Lucas, Sachdev and Schalm: Scale-invariant hyperscaling-violating holographic theories and the resistivity of strange metals with random-field disorder
  - O'Keeffe and Peet: Perturbatively charged holographic disorder

Disordered holographic s-wave superconductor

- The gravity model (Hartnoll-Herzog-Horowitz)

$$S = \int d^4x \sqrt{-g} \left(-\frac{1}{4} F_{ab} F^{ab} - (D_\mu \Psi)(D^\mu \Psi)^\dagger - m^2 \Psi^\dagger \Psi \right) . \quad (3)$$

- The background Schwarzschild-AdS metric:

$$\begin{aligned} ds^2 &= \frac{1}{z^2}\left(-f(z)dt^2 + \frac{dz^2}{f(z)} + dx^2 + dy^2 \right), \\ f(z) &= 1 - z^3 \,, \end{aligned} \quad (4)$$

- The fields

$$\Psi(x,z) = \psi(x,z)\,, \quad \psi(x,z) \in \mathbb{R}\,, \quad (5)$$

$$A = \phi(x,z)\, dt\,. \quad (6)$$

Boundary Conditions: Spontaneously broken symmetry

- UV asymptotics ($z = 0$):

$$\phi(x,z) = \mu(x) + \rho(x)\, z + \phi^{(2)}(x)\, z^2 + o(z^3)\,, \tag{7}$$
$$\psi(x,z) = \psi^{(1)}(x)\, z + \psi^{(2)}(x)\, z^2 + o(z^3)\,, \tag{8}$$

- $\mu(x)$ and $\rho(x)$ are space-dependent chemical potential and charge density respectively.
- The functions $\psi^{(1)}(x)$ and $\psi^{(2)}(x)$ are identified, under the duality, with the source (vanishes) and VEV of an operator of dimension 2.
- IR regularity implies that A_t vanishes at the horizon ($z_h = 1$).

$$\phi(x,z) = (1-z)\,\phi_h^{(1)}(x) + (1-z)^2\,\phi_h^{(2)}(x) + \dots\,,$$
$$\psi(x,z) = \psi_h^{(0)}(x) + (1-z)\,\psi_h^{(1)}(x) + (1-z)^2\,\psi_h^{(2)}(x) + \dots\,,$$

Introducing disorder in the holographic s-wave superconductor

- What? Promote the chemical potential in the holographic superconductor to a random space-dependent function.
- Why? The chemical potential defines the local energy of a charged carrier placed at a given position x coupling with the particle number $n(x)$ locally (proportional to charge density $\rho(x)$). This choice of disorder replicates a local disorder in the on-site energy just as in the Anderson's model: $(V_i, n_i) \mapsto (\mu(x), \rho(x))$.
- Moreover, once disorder is introduced in such an interacting system, all observables will become disordered and, therefore, the physics is not expected to depend on the way disorder is implemented.

Disorder in details

- The noisy chemical potential:

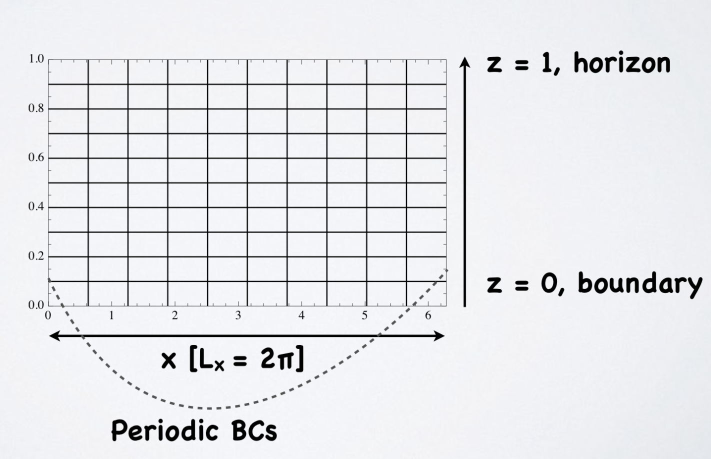
$$\mu(x) \;=\; \mu_0 + \epsilon \sum_{k=k_0}^{k_*} \frac{1}{k^\alpha} \cos(k\,x + \delta_k)\,,$$

- For $\alpha \neq 0$ the correlation length is proportional to $1/k_0$ which is the system size.
- $\delta_k \in [0, 2\pi]$ are random phases. Ensemble averages means averaging over these i.i.d. phases.
- We discretize the space, and impose periodic boundary conditions in the x direction. An IR scale k_0 and a UV scale $k_* = \frac{2\pi}{a}$.
- Our definition of $w \sim \epsilon/\mu_0$ corresponds, in the standard solid state notation, to $1/k_F l$, where k_F is the Fermi momentum and l is the mean free path.

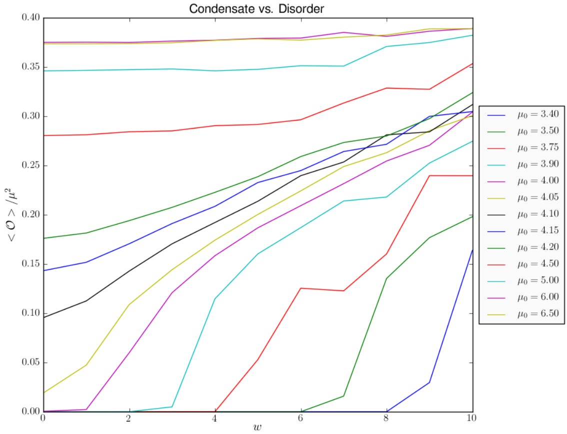
Numerics

- Most of the simulations were done independently in Mathematica and in Python. The latter ones ran in the University of Michigan Flux cluster.
- Our typical result contains a grid of 100×100 points but we have gone up to 200×200 to control issues of convergence and optimization.
- We used a relaxation algorithm to search for the solution and use an $\mathcal{L}_2$ measure for convergence which in most cases reached 10^{-16}. As the source of randomness we used $\mu(x)$ (sum of cosines) and also for uniform and Gaussian distributions.

Relaxation method

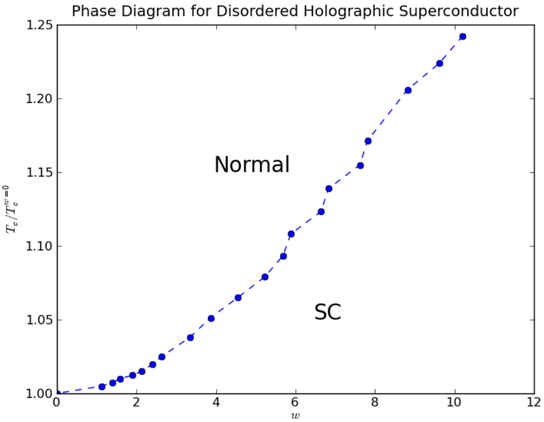


Condensate versus noise strength



- The value of the condensate grows with increasing disorder strength, w

Disordered phase diagram



- Phase diagram, dependence of the critical temperature on the strength of the noise.

Disorder renormalization?

- For highly discontinuous functions of the boundary value of the chemical potential, we find very smooth dependence of the condensate on the coordinate x. A typical form of $\mu(x)$ and its corresponding $\mathcal{O}(x)$:

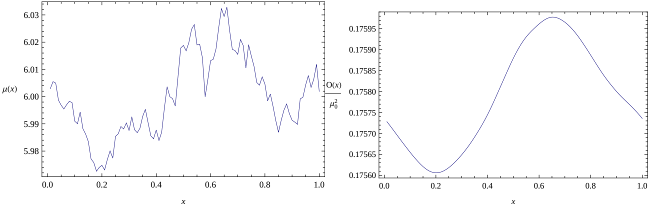


Figure: Initial chemical potential profile $\mu(x) = 6.0 + 0.1 \sum_{n=1}^{100} \frac{1}{2\pi n} \cos(2\pi n x + \delta_n)$ (left panel) and the corresponding condensate profile (right panel).

Disorder renormalization II

- A noisy chemical potential will translate into an even noisier charge density. A typical form of $\mu(x)$ and its corresponding $\rho(x)$:

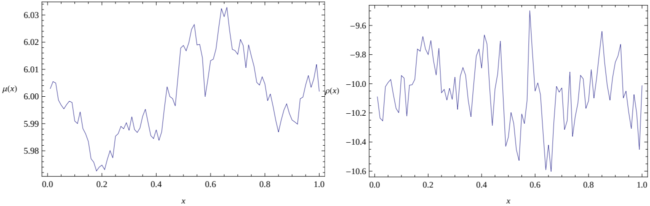
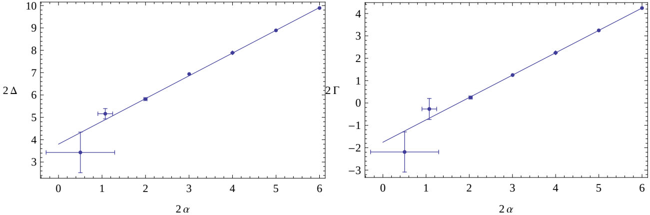


Figure: Initial chemical potential profile $\mu(x) = 6.0 + 0.1 \sum_{n=1}^{100} \frac{1}{2\pi n} \cos(2\pi n x + \delta_n)$ (left panel) and the corresponding charge density profile (right panel).

Universality?

- Power spectra: For a given random signal with power spectrum of the form $k^{-2\alpha}$ we study the power spectrum of the condensate $k^{-2\Delta(\alpha)}$ and of the charge density $k^{-2\Gamma(\alpha)}$



- Renormalization of the disorder: Condensate $\Delta = 3.8 + 1.0\,\alpha$ (left panel) and charge density $\Gamma = -1.75 + 1.0\,\alpha$ (right panel).

Holographic p-wave sc

- What about p-wave? Anderson's theorem doesn't even apply!
- Action (Gubser):

$$S = \int d^4x \sqrt{-g} \left(-\frac{1}{4} F^c_{\mu\nu} F^{\mu\nu}_c + R - \Lambda \right). \tag{9}$$

- Field content:
  - $A^3_t(z) \sim \mu$ (Chemical potential), breaks $SU(2) \to U(1)$
  - $A^1_x(z) \sim < J^1_x >$ (p-wave condensate) breaks $U(1)$ and rotational invariance.

Set up

$$A = \phi(x,z)\,dt\,T_3 + w_x(x,z)\,T_1\,dx + w_y(x,z)\,T_1\,dy + \theta(x,z)\,T_2\,dt\,, \quad (10)$$

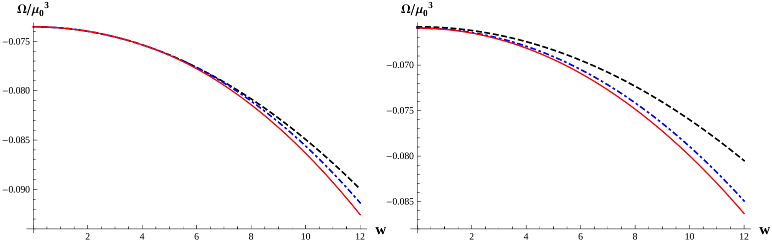
$$\phi(x,z) = \mu(x) - \rho(x)\,z + o(z^2)$$

- Disorder as before:

$$\begin{aligned} \mu(x) &= \mu_0 + \epsilon \sum_{k=k_0}^{k_*} \sqrt{S_k}\,\cos(k\,x + \delta_k) \\ &= \mu_0 + \epsilon \sum_{k=k_0}^{k_*} \frac{1}{k^\alpha}\,\cos(k\,x + \delta_k)\,, \end{aligned}$$

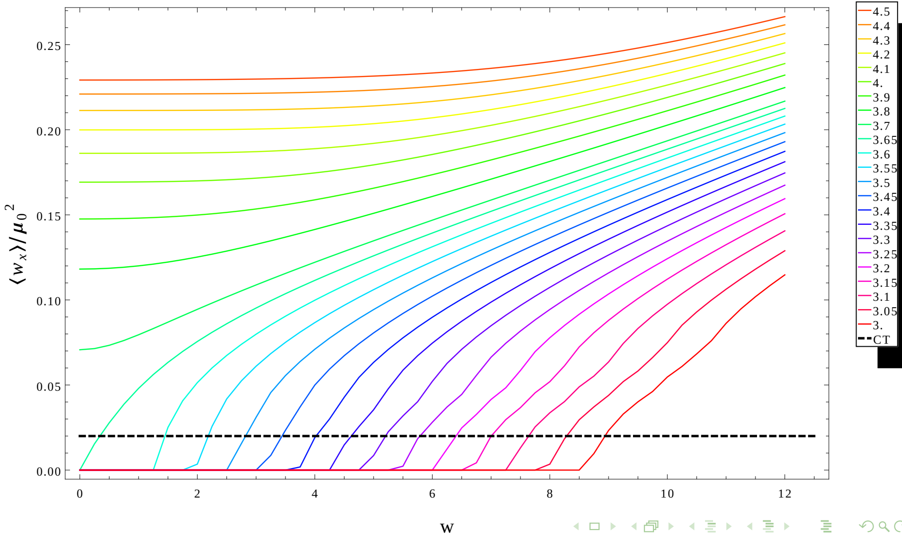
Free energy and competing solutions

$$
\begin{aligned}
\Omega &= -\frac{T S_{\mathrm{on-shell}}}{L_y\, L} = \\
&= l - \frac{1}{4L}\int_0^L dx\,\mu\,\rho + \frac{1}{4L}\int_0^L dx\int_0^1 dz\,\frac{1}{f}\left[(\theta^2+\phi^2)\,(w_x^2+w_y^2) + w_x(\phi\,\partial_x\theta - \theta\,\partial_x\phi)\right]\ ,
\end{aligned}
$$



- The left panel corresponds to $\mu_0 = 3.4 < \mu_c$, and the right one to $\mu_0 = 3.8 > \mu_c$ (both with $\alpha = 1.50$). Black dashed line – the normal phase solution, the blue dot-dashed line – Y solution, red solid line – X solution.

Condensate versus disorder



Disordered p-wave phase diagram

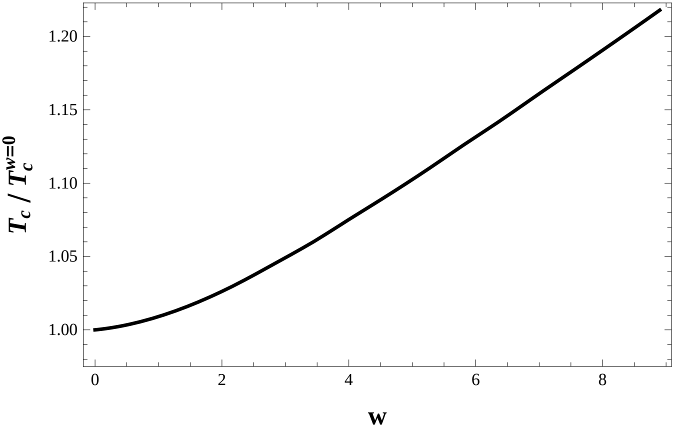


Figure: Enhancement of T_c with the noise strength w ($T_c^{w=0}$ stands for the critical temperature in the absence of disorder).

The order parameter and superconducting islands

- Superconducting islands [Ma-Lee, PRB '85], [Spivak-Zhou, PRL '95], [Galitski-Larkin, PRL '01] and a mechanism for enhancement on superconductivity: There are spatial regions where local upper critical field exceeds the system-wide average value.
- Superconducting islands weakly coupled by Josephson effect.

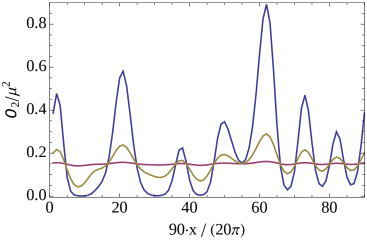


Figure: The condensate as a function of the coordinate x for various realization of disorder $w = 0.1, 0.9, 3$ (with the same set δ_k).The other parameters $\mu = 5$ and $L = 20\pi$. This plot shows the appearance of islands, that is, of spatial fluctuations in the condensate.

Conductivities holographically

- Perturbation: A_μ;

$$A_\mu = A_\mu^{(0)}(x,z) + a_\mu(x,z)\, e^{-i\omega\, t}\,. \tag{11}$$

- Near boundary values

$$a_i(x,z) = a_i^{(0)}(x) + a_i^{(1)}(x) z + O(z^2)\,, \tag{12}$$

- Ingoing boundary conditions at the horizon

$$a_i(x,r) = (1-z)^{\frac{i\omega}{3}} \left(a_i^{h,0}(x) + a_i^{h,1}(x)(1-z) + O((1-z)^2) \right). \tag{13}$$

- Holographic definition

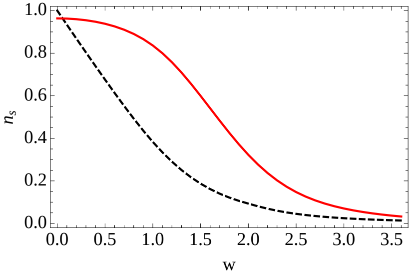
$$\sigma_{ii}(x) = \frac{\langle J_i \rangle}{E_i} = -\frac{i a_i^{(1)}(x)}{\omega a_i^{(0)}(x)}, \tag{14}$$

Conductivities holographically: Superfluid Density

- Conductivities at low frequencies and Kramers-Kronig

$$\sigma \approx n_s \left(\pi \delta(\omega) + \frac{i}{\omega} \right). \tag{15}$$

- The superfluid density n_s



- The dependence of the superfluid density and the minimum value of the condensate as a function of disorder strength.
- For low w, n_s does not change much in agreement with Anderson's theorem even in a strongly coupled system.
- Basko-Aleiner-Altschuler: For weak disorder the transport properties persist in the metal-insulator transition with interactions.
- For large w the superfluid density and the minimum of the condensate decay exponentially.

Conductivities holographically

- The Plot $\Re(\sigma)$
- New resonances
- spectral shift

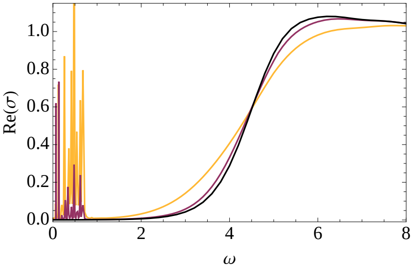


Figure: Real part of the AC conductivity for $T/T_c^{w=0} = 0.45$. The black line corresponds to the homogeneous case, whereas the purple and yellow lines denote the disordered $w = 1$ and $w = 2.4$ cases respectively.

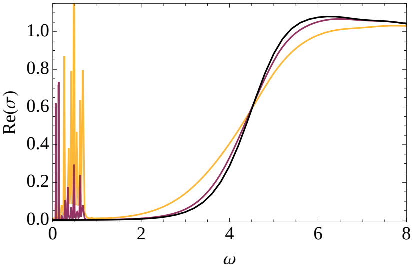
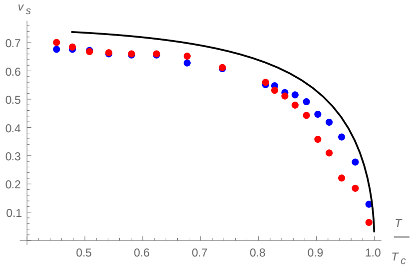


Figure: Real part of the AC conductivity for $T/T_c^{w=0} = 0.45$. The black line corresponds to the homogeneous case, whereas the purple and yellow lines denote the disordered $w = 1$ and $w = 2.4$ cases respectively.

- The real part of the conductivity. There are also some resonances that can be understood as related to the hydrodynamics holographic modes Amado '09 and to the superfluid velocity in Amado '13.
- The shift in the spectral weight clearly depends on the strength of disorder and we view it as evidence of the Higgs mode associated with the spontaneous breaking of the $U(1)$ symmetry.



- The resonances present in the conductivity plots for small frequencies have origin in the presence of hydrodynamic quasi-normal modes. The resonances correspond to gapless modes in the dual field theory $\omega = v\,k$.
- Second sound velocity – $v_s(T, w = 0, 0.1, 1)$ follows the standard temperature dependence discussed in [Herzog-Kovtun-Son '08, Amado et al. '09] modulo its dependence on the disorder.
- Disorder is suppressed for small T. For higher temperatures we see that v_s decreases with increasing disorder strength except very near the critical temperature.

Phase Transitions and Mean Field Theory

- Landau theory and critical exponents:

$$F = a + b\Psi^2 + c\Psi^4 + \dots \tag{16}$$

- At the critical temperature, T_c, the minimum goes from $\Psi = 0$ to $\Psi =$ when b changes sign.

$$\Psi = \pm\sqrt{\frac{-b_0(T - T_c)}{a}} \tag{17}$$

- Critical exponent $\beta = 1/2$, generally $\Psi \sim |T - T_c|^\beta$.
- Wilson: β from RG, that is, scaling.
- What about disordered phase transitions?

Field theory approach to disordered fixed points

- Relativistic $O(N)$ model ($\alpha = 1 \dots N$)

$$\int d^d x \int d\tau \left[(\partial_\tau \phi_\alpha)^2 + c^2 (\nabla_x \phi_\alpha)^2 + (r_0 + r(x))\phi_\alpha^2 + \frac{u}{4!}(\phi_\alpha^2)^2 \right]. \quad (18)$$

- Integrate $r(x)$ with a Gaussian probability distribution $P[r(x)] \sim \exp\left(-\int d^d x r^2(x)/(2\delta^2)\right)$

$$\begin{aligned} &\int d^d x \int d\tau \sum_{a=1}^{n} \left[(\partial \phi_{\alpha a})^2 + r_0 \phi_{\alpha a}^2 + \frac{u}{4!}(\phi_{\alpha a}^2)^2 \right], \\ &-\frac{\delta^2}{2} \int d^d x \int d\tau d\tau' \sum_{a,b} \phi_{\alpha a}^2(x,\tau)\phi_{\beta b}^2(x,\tau') \end{aligned} \quad (19)$$

- Compute and $n \to 0$ ($\ln Z = \lim_{n\to 0} \frac{Z^n - 1}{n}$).

The disordered fixed point: Status

- RG equations for the couplings (r_0, u, δ^2).

$$
\begin{aligned}
\frac{dr_0}{dl} &= 2r_0 + c_1 u - c_2 \delta^2, \\
\frac{du}{dl} &= (3-d)u - c_3 u^2 + c_4 u \delta^2, \\
\frac{d\delta^2}{dl} &= (4-d)\delta^2 + c_5 \delta^4 - c_6 u \delta^2.
\end{aligned}
$$

- These equations do not allow for a fixed point.
- Dorogovstev ('80) and Boyanovsky-Cardy ('83) found a solution by allowing an expansion in ϵ_τ $(d^{\epsilon_\tau}\tau)$. Double epsilon expansion.

Quantum Phase Transition is Smeared by Disorder

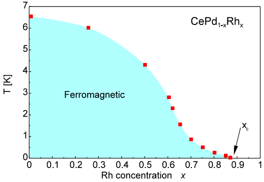


Fig. 4 (Color online) Experimental phase diagram of CePd$_{1-x}$Rh$_x$ as function of Rh concentration x and temperature T (redrawn using data from Ref. [103]). Note the pronounced "tail" of the ferromagnetic phase towards large x.

- "Theory of Smeared Quantum Phase Transitions" J. A. Hoyos and T.Vojta, PRL 100, 240601 (2008)
- Experiment: Ferromagnetic quantum phase transitions in $Ni_{1-x}Pd_x$ and $CePd_{1-x}Rh_x$
- The holographic prediction is coming !!??

The Holographic Disorder-Driven Transition is Smeared

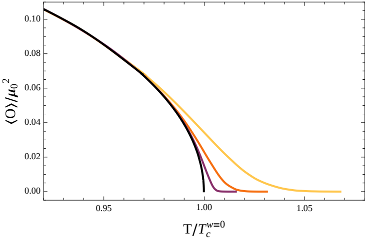


Figure: The disorder-driven transition is smeared. Plot of the spatial average of the condensate close to the critical temperature for $w = 0$, 0.1, 0,2 and 0.4 (black, purple, orange, and yellow lines respectively

The Holographic Disorder-Driven Transition is Smeared: BKT

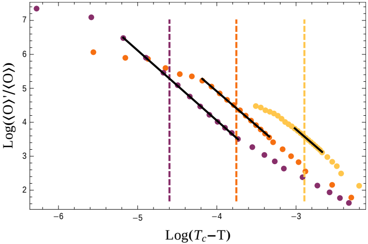


Figure: We plot $\log\left(\frac{1}{\mathcal{O}}\frac{d\mathcal{O}}{dT}\right)$ *versus* $\log(T_c - T)$ for $w = 0.1, 0.2, 0.4$ (purple, orange, and yellow respectively).

- The order parameter behaves as $\langle\mathcal{O}\rangle \sim \exp\left(-A|T - T_c|^{-\nu}\right)$.
- T_c is different for each value of w, and is always higher than $T_c^{w=0}$.
- $\nu = 1.03 \pm 0.02$ independent of the disorder w.

Dirty holographic thin films

- Holographic superconductor model

$$S = \int d^4x \sqrt{-g} \left(-\frac{1}{4} F_{ab} F^{ab} - (D_\mu \Psi)(D^\mu \Psi)^\dagger - m^2 \Psi^\dagger \Psi \right) .$$

- Background metric: Schwarzschild-AdS

$$ds^2 = -\frac{1}{z^2} \left(-f(z) dt^2 + \frac{dz^2}{f(z)} + dx^2 + dy^2 \right) , \quad \text{with} \quad f(z) = 1 - z^3 ,$$

- Ansatz

$$\Psi(x, y, z) = \psi(x, y, z) , \quad A = \phi(x, y, z)\, dt .$$

- Equations of motion The resulting equations of motion read:

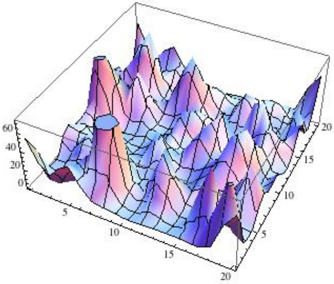
$$\partial_z^2 \phi + f^{-1} \partial_x^2 \phi + f^{-1} \partial_y^2 \phi - \frac{2\psi^2}{z^2 f} \phi = 0 ,$$

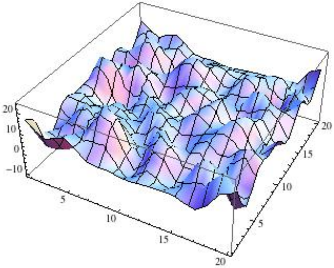
$$\partial_z^2 \psi + f^{-1} \partial_x^2 \psi + f^{-1} \partial_y^2 \psi + \left(\frac{f'}{f} - \frac{2}{z} \right) \partial_z \psi + f^{-2} \left(\phi^2 - \frac{m^2 f}{z^2} \right) \psi = 0 .$$

- Boundary conditions:

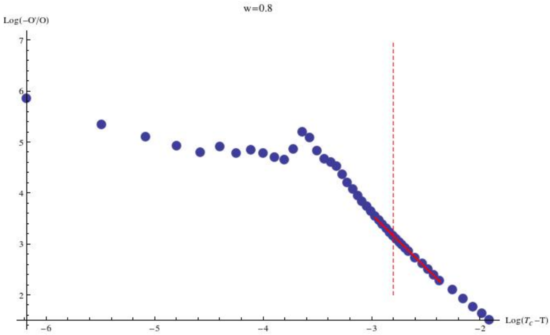
$$\phi(x,y,z) = \mu(x,y) - \rho(x,y)\,z + \phi^{(2)}\,z^2 + o(z^3)\,,$$
$$\psi(x,y,z) = \psi^{(1)}(x,y)\,z + \psi^{(2)}(x,y)\,z^2 + o(z^3)\,,$$

- Chemical potential and order parameter:





The Holographic Disorder-Driven Transition in Thin Films



- The order parameter behaves as $\langle\mathcal{O}\rangle \sim \exp\left(-A|T-T_c|^{-\nu}\right)$.
- T_c is different for each value of w, and is always higher than $T_c^{w=0}$.
- $\nu \approx 1.00$ independent of the disorder w.

Outlook

- Implementation of disorder in AdS/CFT for a s- and p-wave holographic superconductor.
- Better understanding of the x-dependent behavior; power spectra of responses.
- The holographic disorder-driven superconductor-metal transition is BKT type.
- Disordered thin film superconductors [Areán, Salazar].
- Disordered holographic graphene [Araújo, Areán, Erdmenger, PZ, Salazar].
- The fully back-reacted setup is needed to make definitive RG claims about the quantum ($T = 0$) fixed point.