Anomaly-induced Thermodynamics in Higher Dimensional AdS/CFT

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(with T. Azeyanagi, R. Loganayagam and M. J. Rodriguez)
Motivations

- BHs have entropy! Entropy-matching …
- Often extremal or/and susy… How about non-extremal/non-SUSY finite temperature entropy matching?
- In 2d CFT, BTZ entropy is reproduced by the universal Cardy’s formula
- Higher-dimensional generalizations of Cardy’s formula?
- Higher-dimensional AdS/CFT “Cardy entropy-matching”? 
- Chiral Half of Cardy’s Formula in $\text{CFT}_{2n}$
- Replacement Rule from $\text{AdS}_{2n+1}/\text{CFT}_{2n}$
Consider a 2d CFT on a circle of radius $R$

Let us put it at finite temperature and rotation/boost

At high temperature: $T \gg 1/R$

$$S_{\text{Cardy}} \approx \frac{2\pi R}{1 - R^2 \Omega^2} \left[ \frac{c_R + c_L}{24} (4\pi T) \right] + \frac{2\pi R^2 \Omega}{1 - R^2 \Omega^2} \left[ \frac{c_R - c_L}{24} (4\pi T) \right]$$

Weyl Anomaly

Gravitational Anomaly

(``Chiral Half’’/anomalous-part)
Generalizations to higher-d CFT (on sphere)

- Replacement rule  
  [Surowka, Loganayagam, Jensen, Yarom,…]

To understand the replacement rule, we need to review the following two things:

- T=0 anomalies
- Anomalous hydrodynamics
- Anomalies are captured by Chern-Simons terms.
- Append an extra auxiliary direction.
- The $(2n+1)$ theory is anomaly-free, but with Chern-Simons terms $I_{CS}$.
- Non-conservation of the $(2n)$-theory is captured by the "inflow" of charges into the extra auxiliary direction:

\[ \nabla_{\mu} J_{\mu}^{\mu}|_{\text{QFT}_{2n}} \sim j_{\perp} \]

\[ j_{\perp} \sim \star \frac{\partial P_{\text{anom}}}{\partial F} \quad P_{\text{anom}} = dI_{CS} \]
ANOMALOUS HYDRO

- Hydro: effective long-wavelengths descriptions:
  
  Effective variables: \( \{ u^\alpha, T, \mu, \ldots \} \)
  
  Background fields: \( \{ A_\alpha, g_{\alpha\beta}, \ldots \} \)

- Hydro derivative expansion:
  
  \[ J_\alpha = q u_\alpha + \ldots + (J^\alpha)_{\text{anom}} + \ldots \]
  \[ T_{\alpha\beta} = E u_\alpha u_\beta + p \left( g_{\alpha\beta} + u_\alpha u_\beta \right) + \ldots + (T_{\alpha\beta})_{\text{anom}} + \ldots \]

- Leading anomalous contribution:
  
  - parity-odd vorticity: \( V^\alpha \equiv \varepsilon^{\alpha\beta\ldots} u_\beta (\nabla u)^{n-1} \)
ANOMALOUS HYDRO

\[ J_\alpha = q u_\alpha + \ldots + (J_\alpha)_{\text{anom}} + \ldots \]

\[ T_{\alpha\beta} = E u_\alpha u_\beta + p (g_{\alpha\beta} + u_\alpha u_\beta) + \ldots + (T_{\alpha\beta})_{\text{anom}} + \ldots \]

\[ (J_\alpha)_{\text{anom}} = -\frac{\partial \mathcal{F}[T, \mu]}{\partial \mu} V_\alpha + \ldots \]

\[ (T_{\alpha\beta})_{\text{anom}} = \left[ \mathcal{F} - \mu \left( \frac{\partial \mathcal{F}}{\partial \mu} \right)_T - T \left( \frac{\partial \mathcal{F}}{\partial T} \right)_\mu \right] [u_\alpha V_\beta + V_\alpha u_\beta] + \ldots \]

- \( \mathcal{F} \) is like an anomalous Gibbs free energy
- Recall that standard relations:

\[ Q = -\frac{\partial G}{\partial \mu}, \quad E = G - \mu \left( \frac{\partial G}{\partial \mu} \right)_T - T \left( \frac{\partial G}{\partial T} \right)_\mu, \quad S = - \left( \frac{\partial G}{\partial T} \right)_\mu \]
\[ (J_\alpha)_{\text{anom}} = -\frac{\partial F[T, \mu]}{\partial \mu} V_\alpha + \ldots \]

\[ (T_{\alpha\beta})_{\text{anom}} = \left[ F - \mu \left( \frac{\partial F}{\partial \mu} \right)_T - T \left( \frac{\partial F}{\partial T} \right)_\mu \right] [u_\alpha V_\beta + V_\alpha u_\beta] + \ldots \]

- Anomaly-induced entropy:

\[ S_{\text{anom}} = - \left( \frac{\partial F}{\partial T} \right)_\mu \int_{\text{space}} \star_{\text{CFT}} V_\mu dx^\mu = - \left( \frac{\partial F}{\partial T} \right)_\mu \prod_i (2\pi R^2 \Omega_i) \]

- Question: What is \( F \) ?
Revisit 2d CFT Cardy’s formula

\[ S_{CFT_2,\text{anom}} \approx 2\pi R^2 \Omega \left[ \frac{c_R - c_L}{24} (4\pi T) \right] \]

\[ S_{\text{anom}} = -\frac{\partial \mathcal{F}}{\partial T} (2\pi R^2 \Omega) \]

\[ \mathcal{F} = -\frac{c_R - c_L}{2(2\pi)24} 2(2\pi T)^2 = c_g \left. \text{tr}[R^2] \right|_{trR^2 \to 2(2\pi T)^2} \]

anomaly polynomial for 2d grav. anomaly
Replacement Rule or Chiral Half of Cardy's Formula

\[ V^\alpha \equiv \varepsilon^{\alpha\beta}\ldots u_\beta (\nabla u)^{n-1} \]

\[
(J_\alpha)_{\text{anom}} = -\frac{\partial \mathcal{F}[T, \mu]}{\partial \mu} V_\alpha + \ldots
\]

\[
(T_{\alpha\beta})_{\text{anom}} = \left[ \mathcal{F} - \mu \left( \frac{\partial \mathcal{F}}{\partial \mu} \right)_T - T \left( \frac{\partial \mathcal{F}}{\partial T} \right)_\mu \right] [u_\alpha V_\beta + V_\alpha u_\beta] + \ldots
\]

\[
S_{\text{anom}} = -\left( \frac{\partial \mathcal{F}}{\partial T} \right)_\mu \int_{\text{space}} \ast_{\text{CFT}} V_\mu dx^\mu = -\left( \frac{\partial \mathcal{F}}{\partial T} \right)_\mu \prod_i (2\pi R^2 \Omega_i)
\]

\[
\mathcal{F} = P_{\text{anom}} [F, R] |_{F \rightarrow \mu, \text{tr}[R^{2k}] \rightarrow 2(2\pi T)^{2k}}
\]
2d: $P_{\text{anom}} = c_A F^2 + c_g \text{tr}[R^2], \quad \mathcal{F} = c_A \mu^2 + c_g 2(2\pi T)^2$

4d: $P_{\text{anom}} = c_A F^3 + c_M \text{tr}[R^2], \quad \mathcal{F} = c_A \mu^3 + c_M \mu \times 2(2\pi T)^2$

6d: $P_{\text{anom}} \equiv c_g \text{tr}[R^4], \quad \mathcal{F} = c_g 2(2\pi T)^4$

\[
(J_\alpha)_{\text{anom}} = -\frac{\partial \mathcal{F}[T, \mu]}{\partial \mu} V_\alpha + \ldots 
\]

\[
(T_{\alpha\beta})_{\text{anom}} = \left[ \mathcal{F} - \mu \left( \frac{\partial \mathcal{F}}{\partial \mu} \right)_T - T \left( \frac{\partial \mathcal{F}}{\partial T} \right)_\mu \right] [u_\alpha V_\beta + V_\alpha u_\beta] + \ldots 
\]

\[
S_{\text{anom}} = -\left( \frac{\partial \mathcal{F}}{\partial T} \right)_\mu \int_{\text{space}} \star_{\text{CFT}} V_\mu dx^\mu = -\left( \frac{\partial \mathcal{F}}{\partial T} \right)_\mu \prod_i (2\pi R^2 \Omega_i) 
\]
• Chiral Half of Cardy’s Formula in $\text{CFT}_{2n}$
• Replacement Rule from $\text{AdS}_{2n+1}/\text{CFT}_{2n}$
  (Q: Bulk dual of replacement rule?)
A HOLOGRAPHER’S RECIPE

1. Find your favorite AdS bulk theories
   Issue 1: Need gauge-gravitational Chern-Simons terms

2. Find the relevant AdS BH solutions
   Issue 2: Need AdS-Kerr-Newman+CS solutions
   (with all rotations / charges turned on)

3. Calculate responses, charges, thermodynamics…
   Issue 3: Holographic renormalization / charges / entropy for Chern-Simons terms

4. Match / predict CFT results
STEP 1: FAVORITE ADS GRAVITY SETUP

- Toy model: $D=2n+1$ Einstein-Maxwell+negative c.c. +CS terms $I_{cs}$

- Equations of motion:
  \[
  R_{ab} - \frac{1}{2} (R - 2\Lambda) g_{ab} = 8\pi G_N \left[ (T_M)_{ab} + (T_H)_{ab} \right]
  \]
  \[
  \nabla_b F^{ab} = g_{YM}^2 (J_H)^a
  \]

- Maxwell contribution: $(T_M)^{ab}$

- CS/``Hall’’ contributions:
  \[
  J_H = - \star \left( \frac{\partial P_{anom}}{\partial F} \right)
  \]
  \[
  (T_H)^{ab} = \nabla_c \Sigma^{(ab)c} \quad \Sigma^{(ab)c} = -2 \star \left( \frac{\partial P_{anom}}{\partial R^{a}_{\ b}} \right)
  \]
  \[
  P_{anom} = dI_{CS}
  \]
**STEP 2: FLUID/GRAVITY BH SOLUTION (NO ANOMALIES)**

- Gravity dual of charged rotating fluid (for now, without anomalies)

\[
\begin{align*}
 ds^2 &= -2u_\mu dx^\mu dr + r^2 \left[ -f(r, m, q)u_\mu u_\nu + P_{\mu\nu} \right] dx^\mu dx^\nu + 2\text{nd order} \\
 A &= \Phi(r, q)u_\mu dx^\mu + 2\text{nd order} \\
 P_{\mu\nu} &= g_{\mu\nu} + u_\mu u_\nu
\end{align*}
\]

- Non-trivial bulk radial dependence:

\[
\begin{align*}
 \Phi(r, q) &= \frac{q}{r^{2n-2}} \\
 f(r, m, q) &= 1 - \frac{m}{r^{2n}} + \frac{1}{2} \kappa q r^{(2n-1)}
\end{align*}
\]

- Horizon values:

\[
\begin{align*}
 \Phi(r_H) &= \mu, \quad \Phi_T(r_H) = 2\pi T \\
 \Phi_T(r) &\equiv \frac{r^2}{2} \frac{df}{dr}
\end{align*}
\]
STEP 2: FLUID/GRAVITY BH SOLUTION (WITH ANOMALIES)

- Gravity dual of **anomalous** charged rotating fluid

\[ ds^2 = -2u_\mu dx^\mu dr + r^2 \left[ -f(r, m, q)u_\mu u_\nu + P_{\mu\nu} \right] dx^\mu dx^\nu + 2\text{nd order} \]
\[ + g_v(r, m, q)(u_\mu V_\nu + u_\nu V_\mu) dx^\mu dx^\nu + \ldots \]

\[ A = \Phi(r, q)u_\mu dx^\mu + 2\text{nd order} \]
\[ + a_V(r, m, q)V_\mu dx^\mu + \ldots \]

- Leading contributions from the CS-terms:
  \[ g_v, a_V \]

- Bulk replacement rule:

\[ J_H \sim \partial_r \left[ \frac{\partial G}{\partial \Phi} \right] \quad T_H \sim \partial_r^2 \left[ \frac{\partial G}{\partial \Phi_T} \right] \]

\[ G \equiv P_{\text{anom}} \left[ F \rightarrow \Phi, \text{tr}[R^{2k}] \rightarrow 2\Phi_T^{2k} \right] \]
**Step 3 & 4: Currents and Replacement Rule**

Bulk replacement rule $\rightarrow$ metric corrections $\rightarrow$ boundary stress-tensor / currents

\[
(T_{\alpha\beta})_{anom} = (T_{\alpha\beta}^{\text{Brown-York}})_{anom} = \left( G - \Phi \left( \frac{\partial G}{\partial \Phi} - \Phi_T \frac{\partial G}{\partial \Phi_T} \right) \right)_{r=r_H} (u_\alpha V_\beta + u_\beta V_\alpha)
\]

\[
(J_\alpha)_{anom} \sim g_{\mu\alpha} (F^r_{\mu})_{anom} \big|_{\text{boundary}} = - \left( \frac{\partial G}{\partial \Phi} \right)_{r=r_H} V_\alpha
\]

\[
G(r = r_H) = \mathcal{F}[F \rightarrow \mu, \ \text{tr}[R^{2k}] \rightarrow 2(2\pi T)^{2k}]
\]

Bulk replacement rule $\rightarrow$ boundary replacement rule !!
**STEP 3 & 4: ENTROPY**

- **Tachikawa Formula** (see also Solodukhin and Bonora-Cvitan-Prestes-Pallua-Smolic)  
  (note: NOT Wald’s formula applied to Ics)

  \[
  S_{CS} = \int_{\text{horizon}} \sum_{k=1}^{\infty} (8\pi k) \Gamma_N R_N^{2k-2} \frac{\partial P_{\text{anom}}}{\partial \text{tr}[R^{2k}]}
  \]

  \[
P_{\text{anom}} = dI_{cs}, \quad \Gamma_N, R_N = \text{normal bundle connection/curvature}
  \]

- Applying this to the fluid / gravity metric we found gives

  \[
  S_{\text{anom}} = - \left( \frac{\partial F}{\partial T} \right)_\mu \int_{\text{space}} \star_{\text{CFT}} V_\mu dx^\mu = - \left( \frac{\partial F}{\partial T} \right)_\mu \prod_i (2\pi R^2 \Omega_i)
  \]

- Computations opaque, long and tedious …  
  (what bulk structures imply the replacement for entropy?)
SUMMARY, CURRENT AND FUTURE WORK

- Higher-dimensional chiral-half of Cardy’s formula (Replacement rule)

- Construct (anomalous) fluid/gravity solutions in the bulk for the Einstein-Maxwell-CS theory

- Found “bulk replacement rule” which implies the boundary replacement rule

- Current/Future:
  - Anomaly-induced entanglement entropy [Azeyanagi-Loganayagam-Ng, 1507.02298]
  - Time-dependence (non-stationary) and/or higher-order?
  - More realistic theories … add matter and etc (shouldn’t change the conclusions)