Anomaly-induced Thermodynamics in Higher Dimensional AdS/CFT

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- Based on : 1311.2940, 1407.6364 and 1505.02816
  (with T. Azeyanagi, R. Loganayagam and M. J. Rodriguez)

Motivations

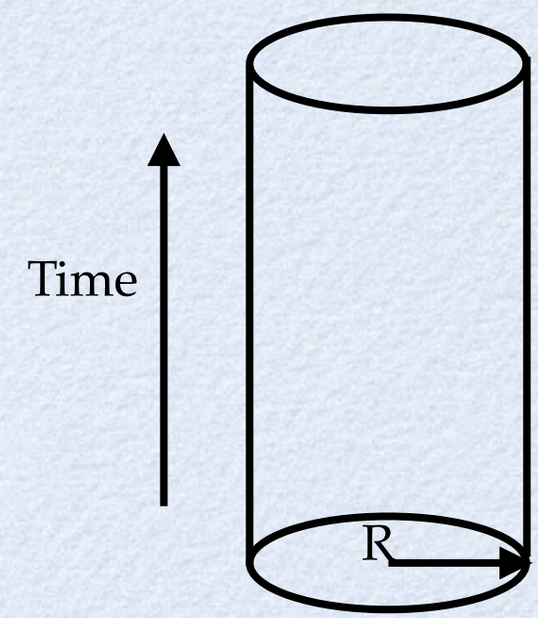
- BHs have entropy ! Entropy-matching …
- Often extremal or/and susy… How about non-extremal/non-SUSY finite temperature entropy matching?
- In 2d CFT, BTZ entropy is reproduced by the universal Cardy's formula
- Higher-dimensional generalizations of Cardy's formula?
- Higher-dimensional AdS/CFT "Cardy entropy-matching"?

OUTLINE

- Chiral Half of Cardy's Formula in CFT_{2n}
- Replacement Rule from $\mathrm{AdS}_{2n+1}/\mathrm{CFT}_{2n}$

CHIRAL HALF OF CARDY

- Consider a 2d CFT on a circle of radius R
- Let us put it at finite temperature and rotation/boost
- At high temperature: $T \gg 1/R$



$$S_{Cardy} \approx \frac{2\pi R}{1-R^2\Omega^2}\left[\frac{c_R+c_L}{24}(4\pi T)\right] + \frac{2\pi R^2\Omega}{1-R^2\Omega^2}\left[\frac{c_R-c_L}{24}(4\pi T)\right]$$

Weyl Anomaly

Gravitational Anomaly
(``Chiral Half'' / anomalous-part)

CHIRAL HALF OF CARDY

$$S_{CFT_2,anom} \approx \frac{2\pi R^2 \Omega}{1 - R^2\Omega^2}\left[\frac{c_R - c_L}{24}(4\pi T)\right]$$

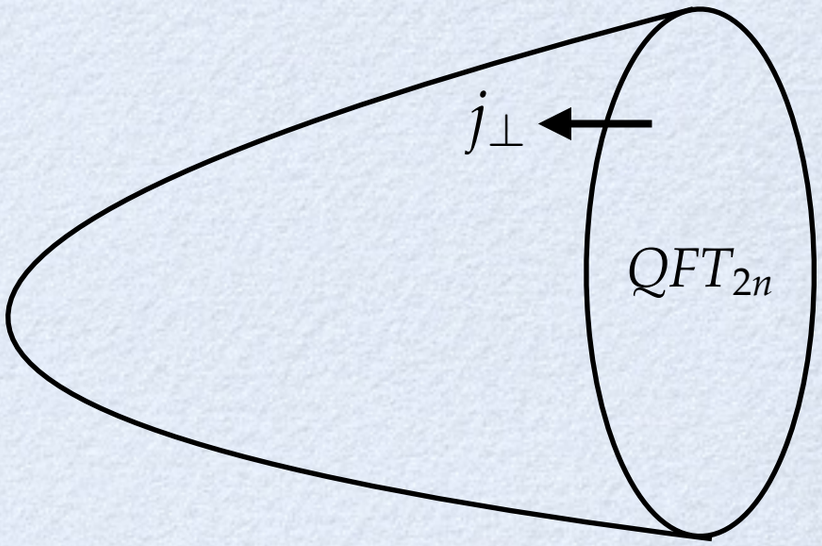
- Generalizations to higher-d CFT (on sphere)
  - Replacement rule [Surowka, Loganayagam, Jensen, Yarom,…]
- To understand the replacement rule, we need to review the following two things:
  - T=0 anomalies
  - Anomalous hydrodynamics

ANOMALY INFLOW

- Anomalies are captured by Chern-Simons terms
- Append an extra auxiliary direction.
- The (2n+1) theory is anomaly-free, but with Chern-Simons terms I_{CS}
- Non-conservation of the (2n)-theory is captured by the ``inflow'' of charges into the extra auxiliary direction:

$$\nabla_\mu J^\mu|_{QFT_{2n}} \sim j_\perp$$

$$j_\perp \sim \star \frac{\partial P_{anom}}{\partial F} \qquad P_{anom} = dI_{CS}$$



Anomalous hydro

- Hydro: effective long-wavelengths descriptions:

$$\text{Effective variables: } \{\, u^\alpha, T, \mu, \ldots \}$$
$$\text{Background fields: } \{ A_\alpha, g_{\alpha\beta}, \ldots \}$$

- Hydro derivative expansion:

$$J_\alpha = q u_\alpha + \ldots + (J_\alpha)_{anom} + \ldots$$
$$T_{\alpha\beta} = E u_\alpha u_\beta + p \left(g_{\alpha\beta} + u_\alpha u_\beta \right) + \ldots + (T_{\alpha\beta})_{anom} + \ldots$$

- Leading anomalous contribution:
  - parity-odd vorticity : $V^\alpha \equiv \varepsilon^{\alpha\beta\ldots} u_\beta \left(\nabla u \right)^{n-1}_{\ldots}$

ANOMALOUS HYDRO

$$J_\alpha = q u_\alpha + \ldots + (J_\alpha)_{anom} + \ldots$$

$$T_{\alpha\beta} = E u_\alpha u_\beta + p\left(g_{\alpha\beta} + u_\alpha u_\beta\right) + \ldots + (T_{\alpha\beta})_{anom} + \ldots$$

$$(J_\alpha)_{anom} = -\frac{\partial \mathcal{F}[T,\mu]}{\partial \mu} V_\alpha + \ldots$$

$$(T_{\alpha\beta})_{anom} = \left[\mathcal{F} - \mu\left(\frac{\partial \mathcal{F}}{\partial \mu}\right)_T - T\left(\frac{\partial \mathcal{F}}{\partial T}\right)_\mu\right]\left[u_\alpha V_\beta + V_\alpha u_\beta\right] + \ldots$$

- $\mathcal{F}$ is like an anomalous Gibbs free energy
- Recall that standard relations:

$$Q = -\frac{\partial G}{\partial \mu}, \quad E = G - \mu\left(\frac{\partial G}{\partial \mu}\right)_T - T\left(\frac{\partial G}{\partial T}\right)_\mu, \quad S = -\left(\frac{\partial G}{\partial T}\right)_\mu$$

ANOMALOUS HYDRO

$$(J_\alpha)_{anom} = -\frac{\partial \mathcal{F}[T,\mu]}{\partial \mu} V_\alpha + \ldots$$

$$(T_{\alpha\beta})_{anom} = \left[\mathcal{F} - \mu\left(\frac{\partial \mathcal{F}}{\partial \mu}\right)_T - T\left(\frac{\partial \mathcal{F}}{\partial T}\right)_\mu\right]\left[u_\alpha V_\beta + V_\alpha u_\beta\right] + \ldots$$

- Anomaly-induced entropy:

$$S_{anom} = -\left(\frac{\partial \mathcal{F}}{\partial T}\right)_\mu \int_{\text{space}} \star_{CFT} V_\mu dx^\mu = -\left(\frac{\partial \mathcal{F}}{\partial T}\right)_\mu \prod_i (2\pi R^2 \Omega_i)$$

- Question: What is $\mathcal{F}$?

CHIRAL HALF OF CARDY

- Revisit 2d CFT Cardy's formula

$$S_{CFT_2,anom} \approx 2\pi R^2 \Omega \left[\frac{c_R - c_L}{24} (4\pi T) \right]$$

$$S_{anom} = -\frac{\partial \mathcal{F}}{\partial T} (2\pi R^2 \Omega)$$

$$\mathcal{F} = -\frac{c_R - c_L}{2(2\pi)24} 2(2\pi T)^2 = c_g \ \mathrm{tr}[\mathbf{R}^2]\Big|_{trR^2 \to 2(2\pi T)^2}$$

anomaly polynomial for 2d grav. anomaly

Replacement rule or chiral half of Cardy's formula

[Surowka, Loganayagam, Jensen, Yarom,…]

$$V^\alpha \equiv \varepsilon^{\alpha\beta\cdots} u_\beta (\nabla u)^{n-1}$$

$$(J_\alpha)_{anom} = -\frac{\partial \mathcal{F}[T,\mu]}{\partial \mu} V_\alpha + \ldots$$

$$(T_{\alpha\beta})_{anom} = \left[\mathcal{F} - \mu \left(\frac{\partial \mathcal{F}}{\partial \mu}\right)_T - T\left(\frac{\partial \mathcal{F}}{\partial T}\right)_\mu\right]\left[u_\alpha V_\beta + V_\alpha u_\beta\right] + \ldots$$

$$S_{anom} = -\left(\frac{\partial \mathcal{F}}{\partial T}\right)_\mu \int_{\text{space}} \star_{CFT} V_\mu dx^\mu = -\left(\frac{\partial \mathcal{F}}{\partial T}\right)_\mu \prod_i (2\pi R^2 \Omega_i)$$

$$\mathcal{F} = P_{anom}[F,R]\big|_{F\to\mu,\,\mathrm{tr}[R^{2k}]\to 2(2\pi T)^{2k}}$$

REPLACEMENT RULE

- 2d: $P_{anom} = c_A F^2 + c_g \mathrm{tr}[R^2], \quad \mathcal{F} = c_A \mu^2 + c_g 2(2\pi T)^2$
- 4d: $P_{anom} = c_A F^3 + c_M F \mathrm{tr}[R^2], \quad \mathcal{F} = c_A \mu^3 + c_M \mu \times 2(2\pi T)^2$
- 6d: $P_{anom} \ni c_g \mathrm{tr}[R^4], \quad \mathcal{F} = c_g 2(2\pi T)^4$

$$(J_\alpha)_{anom} = -\frac{\partial \mathcal{F}[T,\mu]}{\partial \mu} V_\alpha + \ldots$$

$$(T_{\alpha\beta})_{anom} = \left[\mathcal{F} - \mu \left(\frac{\partial \mathcal{F}}{\partial \mu}\right)_T - T \left(\frac{\partial \mathcal{F}}{\partial T}\right)_\mu\right] \left[u_\alpha V_\beta + V_\alpha u_\beta\right] + \ldots$$

$$S_{anom} = -\left(\frac{\partial \mathcal{F}}{\partial T}\right)_\mu \int_{\mathrm{space}} \star_{CFT} V_\mu dx^\mu = -\left(\frac{\partial \mathcal{F}}{\partial T}\right)_\mu \prod_i (2\pi R^2 \Omega_i)$$

OUTLINE

- Chiral Half of Cardy's Formula in CFT_{2n}
- Replacement Rule from $\mathrm{AdS}_{2n+1}/\mathrm{CFT}_{2n}$
  (Q: Bulk dual of replacement rule?)

A HOLOGRAPHER'S RECIPE

1. Find your favorite AdS bulk theories

Issue 1: Need gauge-gravitational Chern-Simons terms

2. Find the relevant AdS BH solutions

Issue 2: Need AdS-Kerr-Newman+CS solutions
(with all rotations/charges turned on)

3. Calculate responses, charges, thermodynamics…

Issue 3: Holographic renormalization/charges / entropy for
Chern-Simons terms

4. Match/predict CFT results

Step 1: Favorite AdS gravity setup

- Toy model: D=2n+1 Einstein-Maxwell+negative c.c.

  +CS terms Ics

- Equations of motion:

$$R_{ab} - \frac{1}{2}\left(R - 2\Lambda\right) g_{ab} = 8\pi G_N \left[\left(T_M\right)_{ab} + \left(T_H\right)_{ab}\right]$$

$$\nabla_b F^{ab} = g_{YM}^2 \left(J_H\right)^a$$

- Maxwell contribution: $(T_M)^{ab}$
- CS/``Hall'' contributions:

$$J_H = - \star \left(\frac{\partial P_{anom}}{\partial F}\right)$$

$$(T_H)^{ab} = \nabla_c \Sigma^{(ab)c} \qquad \Sigma^{(ab)c} = -2 \star \left(\frac{\partial P_{anom}}{\partial \mathbf{R}^a{}_b}\right) \qquad P_{anom} = dI_{CS}$$

Step 2: Fluid/Gravity BH Solution (No Anomalies)

- Gravity dual of charged rotating fluid (for now, without anomalies)

$$ds^2 = -2u_\mu dx^\mu dr + r^2\left[-f(r,m,q)u_\mu u_\nu + P_{\mu\nu}\right]dx^\mu dx^\nu + \text{2nd order}$$

$$A = \Phi(r,q)u_\mu dx^\mu + \text{2nd order}$$

$$P_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$$

- Non-trivial bulk radial dependence:

$$\Phi(r,q) = \frac{q}{r^{2n-2}} \qquad f(r,m,q) = 1 - \frac{m}{r^{2n}} + \frac{1}{2}\kappa_q \frac{q^2}{r^{2(2n-1)}}$$

- Horizon values:

$$\Phi(r_H) = \mu, \quad \Phi_T(r_H) = 2\pi T \qquad \Phi_T(r) \equiv \frac{r^2}{2}\frac{df}{dr}$$

Step 2: Fluid/Gravity BH Solution (with Anomalies)

- Gravity dual of anomalous charged rotating fluid

$$ds^2 = -2u_\mu dx^\mu dr + r^2\left[-f(r,m,q)u_\mu u_\nu + P_{\mu\nu}\right]dx^\mu dx^\nu + \text{2nd order}$$
$$+g_V(r,m,q)(u_\mu V_\nu + u_\nu V_\mu)dx^\mu dx^\nu + \ldots$$

$$A = \Phi(r,q)u_\mu dx^\mu + \text{2nd order}$$
$$+a_V(r,m,q)V_\mu dx^\mu + \ldots$$

$$R_{ab} - \frac{1}{2}(R-2\Lambda)g_{ab} = 8\pi G_N\left[(T_M)_{ab} + (T_H)_{ab}\right]$$
$$\nabla_b F^{ab} = g_{YM}^2 (J_H)^a$$

- Leading contributions from the CS-terms : g_V, a_V

- Bulk replacement rule:

$$J_H \sim \partial_r\left[\frac{\partial G}{\partial \Phi}\right] \qquad T_H \sim \partial_r^2\left[\frac{\partial G}{\partial \Phi_T}\right]$$
$$G \equiv P_{anom}\left[F \to \Phi, \mathrm{tr}[R^{2k}] \to 2\Phi_T^{2k}\right]$$

STEP 3&4: CURRENTS AND REPLACEMENT RULE

Bulk replacement rule → metric corrections → boundary stress-tensor / currents

$$\left(T_{\alpha\beta}\right)_{anom} = \left(T^{\text{Brown-York}}_{\alpha\beta}\right)_{anom} = \left(G - \Phi\frac{\partial G}{\partial \Phi} - \Phi_T \frac{\partial G}{\partial \Phi_T}\right)_{r=r_H} \left(u_\alpha V_\beta + u_\beta V_\alpha\right)$$

$$\left(J_\alpha\right)_{anom} \sim g_{\mu\alpha}\left(F^{r\mu}\right)_{anom}\big|_{\text{boundary}} = -\left(\frac{\partial G}{\partial \Phi}\right)_{r=r_H} V_\alpha$$

$$G(r=r_H) = \mathcal{F}[F \to \mu,\ \text{tr}[R^{2k}] \to 2(2\pi T)^{2k}]$$

Bulk replacement rule-> boundary replacement rule !!

STEP 3&4: ENTROPY

- Tachikawa Formula (see also Solodukhin and Bonora-Cvitan-Prester-Pallua-Smolic) (note: NOT Wald's formula applied to Ics)

$$S_{CS} = \int_{\text{horizon}} \sum_{k=1}^{\infty} (8\pi k) \Gamma_N R_N^{2k-2} \frac{\partial P_{anom}}{\partial \mathrm{tr}[R^{2k}]}$$

$$P_{anom} = dI_{cs}, \quad \Gamma_N, R_N = \text{normal bundle connection/curvature}$$

- Applying this to the fluid/gravity metric we found gives

$$S_{anom} = -\left(\frac{\partial \mathcal{F}}{\partial T}\right)_\mu \int_{\text{space}} \star_{CFT} V_\mu dx^\mu = -\left(\frac{\partial \mathcal{F}}{\partial T}\right)_\mu \prod_i (2\pi R^2 \Omega_i)$$

- Computations opaque, long and tedious ...
(what bulk structures imply the replacement for entropy?)

Summary, Current and Future Work

- Higher-dimensional chiral-half of Cardy's formula (Replacement rule)
- Construct (anomalous) fluid/gravity solutions in the bulk for the Einstein-Maxwell-CS theory
- Found ``bulk replacement rule'' which implies the boundary replacement rule
- Current/Future:
  - Anomaly-induced entanglement entropy [Azeyanagi-Loganayagam-Ng, 1507.02298]
  - Time-dependence (non-stationary) and/or higher-order?
  - More realistic theories … add matter and etc (shouldn't change the conclusions)