Thermal QCD at Finite Gauge Coupling from String/M Theory
Involving Six-/Seven-Folds of $SU(3)$-/G$_2$-Structure

Based on arXiv:1306.4339, 1406.6076 (with M. Dhuria); arXiv:1507.02692 (with K. Sil)

[Application of AdS/CFT to QCD and Condensed Matter Physics, CRM, Montreal, Canada; Oct 19-23, 2015]

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(Deformed/Resolved) Conifolds and fluxed type IIB Resolved Warped Deformed Conifold (RWDC) background with $N$ $D3$, wrapped $M$ $D5$ and $N_f$ $D7$ branes in small coupling + large t’Hooft couplings K. Dasgupta et al [2010], and finite coupling + large t’Hooft coupling/MQGP M. Dhuria, AM [2013] limits, relevant to gravity dual of large-N thermal QCD.

Local type IIA mirror a la S(trominger) Y(au) Z(aslow) prescription of triple T-dualities and its uplift to black $M3$-branes which are $M5$-branes wrapping a two-cycle, $SU(3)$-structure torsion classes of delocalized type IIA mirror and $G_2$-structure torsion classes of its M-theory uplift M. Dhuria, AM [2014]; AM, K.Sil [2015] and supersymmetry, Thermodynamical stability of stringy background and its M-theory uplift M.Dhuria, AM [2013, 2014].

QCD Deconfinement Temperature $T_c$ compatible with lattice results for right number of light quark flavors and right light quark mass scale AM, K.Sil [2015]

Transport Coefficients: $\sigma, \chi$ and Einstein’s relation; R-charge diffusion constant from zero/pole of correlators; $\eta$ and $\frac{\eta_s}{s} = \frac{1}{4\pi}$ from vector and tensor mode fluctuations M.Dhuria, AM [2013, 2014]; speed of sound AM, K.Sil [2015].
The Physics results are influenced by the inherent non-Kählerity (apart from non-conformality) of the type IIB background which is captured by $i^*g, i^*B, ...$ Cool to see the reflection of this before and after delocalized SYZ mirror symmetry by explicitly working out the $G$-structure of the type IIB/IIA mirror and its M theory uplift.
Background of Background: Conifolds

- Analogous to a two-dimensional cone embedded in $\mathbb{R}^3: x^2 + y^2 - u^2 = 0$, a real six-dimensional conifold can be represented as a quadric: $\sum_{i=1}^{4} z_i^2 = 0$ in $\mathbb{C}^4$, which is smooth everywhere except at the apex/node: $\vec{Z} = 0$ which is a double-point.

- Writing $z_i = x_i + iy_i$, the base of the cone is $S^2(y_i)$ fibered over $S^3(x_i)$ and as all fibrations are trivial, hence $S^2 \times S^3$: Sasaki-Einstein spaces $T^{p,q}$, $p$ and $q$ being relatively prime integers, which means $ds^2 = dr^2 + r^2 g_{T^{p,q}}$, $g_{T^{p,q}} = g_{ab}dx^a dx^b = \lambda^2 (d\psi + p \cos \theta_1 d\phi_1 + q \cos \theta_2 d\phi_2)^2 + \Lambda^{-1}_1 (d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2) + \Lambda^{-1}_2 (d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2)$. 
An $n$-conifold is Ricci-flat if the base is Einsteinian: $R_{ab}(g) = (n - 2)g_{ab}$. For $n = 6$, one obtains:

$$4 = \frac{\lambda^2}{2} \{(p\Lambda_1)^2 + (q\Lambda_2)^2\} = \Lambda_1 - \frac{1}{2}(\lambda p\Lambda_1)^2 = \Lambda_2 - \frac{1}{2}(\lambda q\Lambda_2)^2.$$ For $p = 1, q = 0$ one obtains: $\lambda = \frac{1}{2\sqrt{2}}, \Lambda_1 = 8, \Lambda_2 = 4$, i.e.,

$$g_{T^1,0} = \frac{1}{8}(d\psi + \cos\theta_1 d\phi_1)^2 + \frac{1}{8}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \frac{1}{8}(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2);$$ similarly

$$g_{T^1,1} = \frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 + \frac{1}{6}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \frac{1}{6}(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2)$$

P. Candelas, X. de la Ossa [2000].
The conifold base can be viewed as a coset space $\frac{SU(2) \times SU(2)}{U(1)}$. To see this, convenient to introduce $W = \frac{1}{\sqrt{2}} \begin{pmatrix} Z_3 + \i Z_4 & Z_1 - \i Z_2 \\ Z_1 + \i Z_2 & -Z_3 + \i Z_4 \end{pmatrix}$. By a rescaling $Z = \frac{W}{r}$, the defining equation for the conifold and the base $\det Z = 0, \text{tr} Z^\dagger Z = 1$. Given a particular solution $Z_0$, say $Z_0 = \frac{1}{2}(\sigma_1 + \i \sigma_2)$, the general solution can be written as $Z = LZ_0 R^\dagger$, where $L = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}, \quad R = \begin{pmatrix} k & -\bar{l} \\ l & \bar{k} \end{pmatrix}$, where $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cos \frac{\theta_1}{2} e^{\i/2(\psi_1 + \phi_1)} \\ \sin \frac{\theta_1}{2} e^{\i/2(\psi_1 - \phi_1)} \end{pmatrix}$ and $\begin{pmatrix} k \\ l \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} (1 \rightarrow 2)$. $L, R \in SU(2)$. 
When \((L, R) = (\Theta, \Theta^\dagger)\) with \(\Theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\), \(Z_0 = \Theta Z_0 \Theta\). This means that we can identify \((L, R)\) and \((L\Theta, R\Theta^\dagger)\), i.e. the base is the coset space \(\frac{SU(2) \times SU(2)}{U(1)} = S^3 \times S^3\). \(\text{tr} (dZ^\dagger dZ)\) and \(|\text{tr} (Z^\dagger dZ)|^2\) are invariants P. Candelas, X. de la Ossa [2000].

The metric is Kähler iff

\[
g_{\bar{i}j} = \partial_i \bar{\partial}_j K(r^2) \Rightarrow ds^2 = K'(r^2) \text{tr}(dW^\dagger dW) + K''(r^2)|\text{tr}(W^\dagger dW)|^2.\]

Defining \(\gamma \equiv r^2 K'\), \(R_{\bar{i}j} = 0\) implies \(\gamma = r^{\frac{4}{3}}\). Redefining \(\rho = \sqrt{\frac{3}{2} r^\frac{2}{3}}\), \(ds^2 = d\rho^2 + \rho^2 g_{T^{1,1}}\) implying \(T^{1,0}\) does not correspond to a Kähler Ricci-flat metric.
Deformed Conifolds

- The singular conifold can be smoothed by deforming as: $\sum_{i=1}^{4} z_i^2 = \epsilon^2$ or $\det W_\epsilon = -\frac{\epsilon^2}{2}$. For $r \neq \epsilon$ one gets back $T^{1,1}$ and for $r = \epsilon$ (‘origin of coordinates’), $W_\epsilon \sim \sigma_3 \rightarrow LW_\epsilon R^\dagger = W_\epsilon$, $R = \sigma_3 L \sigma_3$ and $\tilde{y} = 0$, $S^3(\tilde{x})$.

- A solution $W_\epsilon = rZ_\epsilon$ where

  $Z_\epsilon = LZ_\epsilon^{(0)} R^\dagger$, $Z_\epsilon^{(0)} = \begin{pmatrix} 0 & \alpha \\ \frac{\epsilon^2}{2r^2 \alpha} & 0 \end{pmatrix}$, $\alpha = \frac{1}{2} \left( \sqrt{1 + \frac{\epsilon^2}{r^2}} + \sqrt{1 - \frac{\epsilon^2}{r^2}} \right)$
The deformed conifold metric is given by R. Miniasin, D. Tsimpis [2000]

\[ ds_{\text{def}}^2 = |\text{tr}(W_e^\dagger dW_e)|^2 K''(r^2) + \text{tr}(dW_e^\dagger dW_e) K'(r^2) \]

\[ = \left[ \left( r^2 \hat{\gamma}' - \hat{\gamma} \right) \left( 1 - \frac{\epsilon^4}{r^4} \right) + \hat{\gamma} \right] \]

\[ \times \left( \frac{dr^2}{r^2(1 - \epsilon^4/r^4)} + \frac{1}{4} (d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)^2 \right) \]

\[ + \frac{\hat{\gamma}}{4} \left[ (\sin \theta_1^2 d\phi_1^2 + d\theta_1^2) + (\sin \theta_2^2 d\phi_2^2 + d\theta_2^2) \right] \]

\[ + \frac{\hat{\gamma} \epsilon^2}{2r^2} [\cos \psi (d\theta_1 d\theta_2 - \sin \theta_1 \sin \theta_2 d\phi_1 d\phi_2) \]

\[ + \sin \psi (\sin \theta_1 d\phi_1 d\theta_2 + \sin \theta_2 d\phi_2 d\theta_1) \], \]

\[ \psi \equiv \psi_1 + \psi_2, \hat{\gamma} \equiv r^2 K \text{ determined by the solution of the Ricci-flatness condition:} \]

\[ r^2 (r^4 - \epsilon^4)(\hat{\gamma}^3)' + 3\epsilon^4 \hat{\gamma}^3 = 2r^8 ; \text{ as } r \to \infty, \hat{\gamma} \to r^\frac{4}{3} \text{ P. Candelas and X. de la Ossa [1990].} \]
The $e_i$'s are one-forms on $S^2$

$$e_1 \equiv d\theta_1 , \quad e_2 \equiv -\sin \theta_1 d\phi_1 ,$$

and the $\epsilon_i$'s a set of one-forms on $S^3$

$$\epsilon_1 \equiv \sin \psi \sin \theta_2 d\phi_2 + \cos \psi d\theta_2 ,$$
$$\epsilon_2 \equiv \cos \psi \sin \theta_2 d\phi_2 - \sin \psi d\theta_2 ,$$
$$\epsilon_3 \equiv d\psi + \cos \theta_2 d\phi_2 .$$
Define: R. Miniasin, D. Tsimpis [2000]

\[
\begin{pmatrix}
g_1 \\
g_2 \\
g_3 \\
g_4 \\
g_5 \\
\end{pmatrix} = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} \begin{pmatrix}
e_2 \\
e_1 \\
\epsilon_2 \\
\epsilon_1 \\
\epsilon_3 + \cos \theta_1 d\phi_1 \\
\end{pmatrix}.
\]
For $r \sim \epsilon$, also defining: $\delta \equiv r - \epsilon$, $\nu \equiv \sqrt{\frac{2\delta}{\epsilon}}$, the deformed conifold metric:

$$ ds_6^2 \sim R_{\epsilon}^2 \left[ dv^2 + d\Omega_3^2 + \frac{v^2}{2} ds_2^2 \right], $$

$$ d\Omega_3^2 \equiv \frac{1}{2} g_3^2 + g_4^2 + g_5^2, \quad ds_2^2 \equiv g_1^2 + g_2^2; $$

$S^3$ has non-vanishing radius: $R_{\epsilon} = \frac{1}{\sqrt{2}} \left( \frac{2\epsilon^4}{3} \right)^{\frac{1}{6}}.$
Resolved Conifolds

- Replace $\sum_{i=1}^{4} z_i^2 = 0$ by $\begin{pmatrix} x & u \\ v & y \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0$, $(\lambda_1, \lambda_2) \neq (0, 0)$. So, for $(u, v, x, y) \neq 0$ (away from the tip), again a conifold. But at $(u, v, x, y) = 0$ solved by any pair $(\lambda_1, \lambda_2)$. Due to the overall scaling freedom $(\lambda_1, \lambda_2) \sim (\lambda \lambda_1, \lambda \lambda_2)$, $(\lambda_1, \lambda_2)$ actually describe a $\mathbb{C}P^1 \sim S^2$ at the tip of the cone.

- As $c_1(TM_{RG}) = c_1(T\mathbb{C}P_1 \oplus N_{\mathbb{C}P_1}) = 0$, $c_1(T\mathbb{C}P_1) = 2$, therefore $c_1(N_{\mathbb{C}P_1}) = -2$. As rank($N_{\mathbb{C}P_1}$)=2, and a holomorphic line bundle over $\mathbb{C}P_1$ can be written as a sum of line bundles, $N_{\mathbb{C}P_1} = \mathcal{O}(-p) \oplus \mathcal{O}(-q) : p + q = 2$. For isolated $\mathbb{C}P_1$, $p = q = 1$.

- $H_+ : \lambda_1 \neq 0$, $\lambda \equiv \frac{\lambda_2}{\lambda_1}$, $W = \begin{pmatrix} -u \lambda & u \\ -y \lambda & y \end{pmatrix}$; $(u, y, \lambda)$ : coordinates, $\Omega_+ = du \wedge dy \wedge d\lambda$. $H_- : \lambda_2 \neq 0$, $\mu \equiv \frac{\lambda_1}{\lambda_2}$, $W = \begin{pmatrix} x & -x \mu \\ v & -v \mu \end{pmatrix}$; $(v, x, \mu)$ : coordinates, $\Omega_- = dv \wedge dx \wedge d\mu$. 
For $\mathcal{O}(s)$-bundle over $\mathbb{CP}^1$: if on $H_+$ coordinates are $(z_+, \lambda)$ and if on $H_-$ coordinates are $(z_-, \mu)$, then on $H_+ \cap H_- : (z_-, \mu) = (\lambda^{-s} z_+, \frac{1}{\lambda})$.

On $H_+ \cap H_- : (v, x, \mu) = (-y \lambda, -u \lambda, \frac{1}{\lambda})$. Thus:

$$
\begin{array}{cccc}
\mathbb{C}^2 & \longrightarrow & \mathcal{O}(-1) & \oplus \mathcal{O}(-1) \\
& & \downarrow & \\
& & \mathbb{CP}^1
\end{array}
$$

The conifold, instead of being thought of as a real cone over $S^2 \times S^3$, can be thought of as a complex cone over $\mathbb{CP}^1 \times \mathbb{CP}^1$. The quadric in $\mathbb{CP}^3[2]$, rewritten as $xy = uv$ is $\mathbb{CP}^1 \times \mathbb{CP}^1$ via Segré-embedding:

$\mathbb{CP}^1(A_1, A_2) \times \mathbb{CP}^1(B_1, B_2) \hookrightarrow \mathbb{CP}^3(x = A_1 B_1, y = A_2 B_2, u = A_1 B_2, v = A_2 B_1)$.

The node can be replaced by either $\mathbb{CP}^1$ - one could use $(\hat{\lambda}_1 \hat{\lambda}_2) \begin{pmatrix} x & u \\ v & y \end{pmatrix} = 0$. 


The most general Kähler potential \( K = F(r^2) + 4a^2 \ln(1 + |\lambda|^2) \) implying the resolved conifold metric L.P. Zayas, A. Tseytlin; G. Papadopoulos, A. Tseytlin [2000] is given by:

\[
\begin{align*}
\text{ds}^2_{\text{res}} &= F'(r^2) \text{tr}(dW^\dagger dW) + F''(r^2) \left| \text{tr}(W^\dagger dW) \right|^2 + 4a^2 \frac{|d\lambda|^2}{(1 + |\lambda|^2)^2} \\
&= \tilde{\gamma}' \, dr^2 + \frac{\tilde{\gamma}'}{4} \, r^2 \left( d\psi + \cos \theta_1 \, d\phi_1 + \cos \theta_2 \, d\phi_2 \right)^2 \\
&\quad + \frac{\tilde{\gamma}}{4} \left( d\theta_1^2 + \sin^2 \theta_1 \, d\phi_1^2 \right) + \frac{\tilde{\gamma} + 4a^2}{4} \left( d\theta_2^2 + \sin^2 \theta_2 \, d\phi_2^2 \right); \\
\end{align*}
\]

\( \tilde{\gamma} \xrightarrow{r \to \infty} r^{\frac{4}{3}} \). Defining \( \rho^2 = 3/2 \tilde{\gamma} \):

\[
\begin{align*}
\text{ds}^2_{\text{res}} &= \frac{\kappa(\rho)}{9} \, \rho^2 \left( d\psi + \cos \theta_1 \, d\phi_1 + \cos \theta_2 \, d\phi_2 \right)^2 \\
&\quad + \frac{\rho^2}{6} \left( d\theta_1^2 + \sin^2 \theta_1 \, d\phi_1^2 \right) + \frac{\rho^2 + 6a^2}{6} \left( d\theta_2^2 + \sin^2 \theta_2 \, d\phi_2^2 \right) + \kappa(\rho)^{-1} \, d\rho^2,
\end{align*}
\]

with \( \kappa(\rho) = (\rho^2 + 9a^2)/(\rho^2 + 6a^2) \).
Complex Structures on (Resolved/Deformed) Conifold

Left-invariant one-forms on the conifold base:

\[
\begin{align*}
\sigma_1 &= \cos \psi_1 \, d\theta_1 + \sin \psi_1 \sin \theta_1 \, d\phi_1, \\
\sigma_2 &= -\sin \psi_1 \, d\theta_1 + \cos \psi_1 \sin \theta_1 \, d\phi_1, \\
\sigma_3 &= d\psi_1 + \cos \theta_1 \, d\phi_1, \\
\Sigma_1 &= \cos \psi_2 \, d\theta_2 + \sin \psi_2 \sin \theta_2 \, d\phi_2, \\
\Sigma_2 &= -\sin \psi_2 \, d\theta_2 + \cos \psi_2 \sin \theta_2 \, d\phi_2, \\
\Sigma_3 &= d\psi_2 + \cos \theta_2 \, d\phi_2.
\end{align*}
\]

They satisfy a Maurer–Cartan equation \(d\xi_i = -i/2 \epsilon_{ij}^{\,\,k} \xi_j \wedge \xi_k\). The one-forms give rise to vielbeins on the six-dimensional conifold:

\[
\begin{align*}
e_1 &= \frac{r}{\sqrt{6}} \sigma_1, \\
e_2 &= \frac{r}{\sqrt{6}} \sigma_2, \\
e_3 &= \frac{r}{\sqrt{6}} \Sigma_1; \\
e_4 &= \frac{r}{\sqrt{6}} \Sigma_2, \\
e_5 &= \frac{r}{3} (\sigma_3 + \Sigma_3), \\
e_6 &= dr,
\end{align*}
\]

and the conifold metric is diagonal in these vielbeins: \(ds^2 = \sum_{i=1}^{6} e_i^2\).
An (almost) complex structure on this real six-dimensional manifold is defined by choosing complex vielbeins

\[ E_1 = e_1 + \iota e_2, \quad E_2 = e_3 + \iota e_4, \quad E_3 = e_5 + \iota e_6. \]

In terms of these complex vielbeins, the fundamental two-form \( J \) and holomorphic three form \( \Omega \) are defined as

\[
\begin{align*}
J^{(1,1)} & = \frac{\iota}{2} (E_1 \wedge \bar{E}_1 + E_2 \wedge \bar{E}_2 + E_3 \wedge \bar{E}_3) \\
\Omega^{(3,0)} & = E_1 \wedge E_2 \wedge E_3 \\
dJ & = d\Omega = 0.
\end{align*}
\]
The complex structure induced by these vielbeins is identical with the one coming from the holomorphic homogenous coordinates $z_i$:

$$z_1 = \frac{1}{2} (x - iy), \quad z_2 = \frac{1}{2i} (x + iy), \quad z_3 = \frac{1}{2} (u + iv), \quad z_4 = -\frac{1}{2i} (u - iv).$$

used to define the singular conifold. Up to a numerical factor, $\Omega = \frac{dz_1 \wedge dz_2 \wedge dz_3}{z_4}$, which agrees with above coordinate expression if the holomorphic coordinates are parameterized as

$$x = r^{3/2} e^{\psi/2} e^{\phi_1 - \phi_2} \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2},$$
$$y = r^{3/2} e^{\psi/2} e^{\phi_1 + \phi_2} \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2},$$
$$u = r^{3/2} e^{\psi/2} e^{\phi_1 - \phi_2} \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2},$$
$$v = r^{3/2} e^{\psi/2} e^{\phi_1 + \phi_2} \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2}.$$
The singular conifold complex structure can be used for the resolved conifold after scaling the vielbeins according to the metric R. Gwyn, A. Knauf [2008]:

\[
\begin{align*}
\mathbf{e}_1 &= \frac{\rho}{\sqrt{6}} \sigma_1, \\
\mathbf{e}_2 &= \frac{\rho}{\sqrt{6}} \sigma_2, \\
\mathbf{e}_3 &= \frac{\sqrt{\rho^2 + 6a^2}}{\sqrt{6}} \Sigma_1, \\
\mathbf{e}_4 &= \frac{\sqrt{\rho^2 + 6a^2}}{\sqrt{6}} \Sigma_2, \\
\mathbf{e}_5 &= \frac{\rho}{3} \sqrt{\frac{\rho^2 + 9a^2}{\rho^2 + 6a^2}} (\sigma_3 + \Sigma_3), \\
\mathbf{e}_6 &= \sqrt{\frac{\rho^2 + 6a^2}{\rho^2 + 9a^2}} d\rho,
\end{align*}
\]

then the metric remains diagonal with \( \psi = \psi_1 + \psi_2 \).
The corresponding homogeneous holomorphic coordinates in this case read R. Gwyn, A. Kanuf [2008]

\[
\begin{align*}
  x &= \left( 9a^2 \rho^4 + \rho^6 \right)^{1/4} e^{i/2(\psi - \phi_1 - \phi_2)} \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \\
  y &= \left( 9a^2 \rho^4 + \rho^6 \right)^{1/4} e^{i/2(\psi + \phi_1 + \phi_2)} \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \\
  u &= \left( 9a^2 \rho^4 + \rho^6 \right)^{1/4} e^{i/2(\psi + \phi_1 - \phi_2)} \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \\
  v &= \left( 9a^2 \rho^4 + \rho^6 \right)^{1/4} e^{i/2(\psi - \phi_1 + \phi_2)} \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2}.
\end{align*}
\]

They lead to the same holomorphic three–form
The deformed conifold metric is not diagonal in the singular conifold one-forms and we have to define linear combinations of them for the same R. Gwyn, A. Knauf [2000]:

\[
\begin{align*}
    e_1 &= \frac{\sqrt{\hat{\gamma}}}{2} (\alpha \sigma_1 - \beta \Sigma_1), \\
    e_2 &= \frac{\sqrt{\hat{\gamma}}}{2} (\alpha \sigma_2 + \beta \Sigma_2), \\
    e_3 &= \frac{\sqrt{\hat{\gamma}}}{2} (-\beta \sigma_1 + \alpha \Sigma_1), \\
    e_4 &= \frac{\sqrt{\hat{\gamma}}}{2} (\beta \sigma_2 + \alpha \Sigma_2), \\
    e_5 &= \frac{1}{2} \sqrt{(r^2 \hat{\gamma}' - \hat{\gamma}) (1 - \epsilon^4/r^4) + \hat{\gamma}} (\sigma_3 + \Sigma_3), \\
    e_6 &= \frac{\sqrt{(r^2 \hat{\gamma}' - \hat{\gamma}) (1 - \epsilon^4/r^4) + \hat{\gamma}}}{r \sqrt{1 - \epsilon^4/r^4}} dr,
\end{align*}
\]

where \( \alpha^2 + \beta^2 = 1 \). For the metric to also be Ricci flat and Kähler, the coefficients \( \alpha \) and \( \beta \): \( \alpha = \frac{1}{2} \sqrt{1 + \epsilon^2/r^2} + \frac{1}{2} \sqrt{1 - \epsilon^2/r^2}, \) \( \beta = \frac{\epsilon^2}{2r^2 \alpha} \).
For the deformed conifold we can use the same holomorphic coordinates as for the singular conifold, but the three–form is given by:

\[
\Omega = \frac{dz_1 \wedge dz_2 \wedge dz_3}{\sqrt{\epsilon^2 - z_1^2 - z_2^2 - z_3^2}}.
\]
Building up on I. Klebanov, E. Witten [1998]; I. Klabanov, M. Strassler [2000]; P. Ouyang [2003]; A. Buchel [2000]; A. Buchel et al [2001];..., in order to allow the presence of fundamental quarks at finite temperature in the context of type IIB string theory, K. Dasgupta et al [2010] considered \( N \) black \( D3 \)-branes placed at the tip of a conifold, \( M \) \( D5 \) (and \( \overline{D5} \)) - branes wrapping the vanishing two-cycle (distributed along the resolved two-cycle and placed at the outer boundary \( r_0 \) of the IR-UV interpolating region/inner boundary of the UV region to ensure cancelation of UV log divergence in the RG flow, i.e., turning off of three-form fluxes ) and \( N_f \) \( D7 \)(and \( \overline{D7} \)) - branes dripping into the IR (in the IR-UV interpolating region and the UV) via Ouyang embedding, apart from \( 24 - N_f \) in the UV, resulting in a resolved warped deformed conifold. Back-reactions are included in 10-D warp factor.
The charge of $N_1$ $D$-branes corresponding to a vector bundle $E_1$ and $N_2$ $\overline{D}$-branes corresponding to a vector bundle $E_2$, and both wrapping a manifold $X^{(d)}$, $\dim_{\mathbb{C}}(X^{(d)}) = d$, is determined by the Mukai charge vector:

$$(\text{ch}(E_1) - \text{ch}(E_2)) \wedge \sqrt{\hat{A}(X^{(d)})}$$

M.B. Green, J. Harvey, G. Moore [1996]. This can be understood in the language of stability of vector bundles and the triple: $(E_1, E_2, T)$ where the tachyon $T$ can be thought of as the map $T : E_1 \to E_2$.

Y. Oz, T. Pantev, D. Waldram [2000]. Imposing holomorphy of $T$ and gauge fields, the solutions to the low energy EOMs on $X^{(d)}$ were shown to be equivalent to this condition of stability of the triple. So, taking $N_1 = N_2 = 1$ (for simplicity) wrapping the small $S^2$ of a warped resolved deformed conifold and $E_{1,2}$ being $U(1)$ bundles over $S^2$, one sees that that one generates from

$$\int_{\mathbb{R}^{1,3} \times S^2} C_4 \wedge (\text{ch}(E_1) - \text{ch}(E_2)) \wedge \sqrt{\hat{A}}$$

the WZ term for a $D3$-brane: $\int_{\mathbb{R}^{1,3}} C_4$ if $c_1(E_1) - c_1(E_2) = 1$ R. Tatar [2000]. In other words one could turn on a unit flux on the world-volume of the $D5$-brane and none on the $\overline{D5}$ and generate a $D3$-brane. Alternatively, $\overline{D5}$-branes could be absorbed as $D7$-brane world-volume fluxes to generate negative $D5$-brane charge in such a way that there is no net $D5$-brane charge in the UV M. Mia [2013].
One has \( SU(N + M) \times SU(N + M) \) color gauge group (implying gravity dual asymptotically has an \( AdS_5 \)), in the UV: \( r \geq r_0 \). Expect a Higgsing to \( SU(N + M) \times SU(N) \) at \( r = r_0 \) leaving with \( SU(N + M) \times SU(N) \). K. Dasgupta et al [2012]; the two gauge couplings flow oppositely in the IR:

\[
4\pi^2 \left( \frac{1}{g_1^2} + \frac{1}{g_2^2} \right) e^\phi \sim \pi; \quad 4\pi^2 \left( \frac{1}{g_1^2} - \frac{1}{g_2^2} \right) e^\phi \sim \frac{1}{2\pi \alpha'} \int_{C_2} B_2,
\]

with \( g_1 \) flowing to strong coupling. \( SU(N + M)_{\text{strong}} \xrightarrow{\text{Seiberg Dual}} SU(N - (M - N_f))_{\text{weak}} \) in the IR; assume after repeated dualities, \( N \) decreases to 0 and finite \( M \) left so \( SU(M) \) gauge theory + \( N_f \) flavors that confines in the IR - finite temperature version looked at by K. Dasgupta et al [2010].
In M. Dhuria, AM [2013], we considered the following two limits:

(i) weak ($g_s$) coupling – large t’Hooft coupling limit:

$g_s \ll 1, g_s N_f \ll 1, \frac{g_s M^2}{N} \ll 1, g_s M >> 1, g_s N >> 1$

effected by: $g_s \sim \epsilon^d, M \sim (\mathcal{O}(1)\epsilon)^{-\frac{3d}{2}}, N \sim (\mathcal{O}(1)\epsilon)^{-19d}, \epsilon \ll 1, d > 0$

(the limit in the first line though not its realization in the second line, considered in K. Dasgupta et al [2010]);

(ii) MQGP limit: $\frac{g_s M^2}{N} \ll 1, g_s N >> 1$, finite $g_s, M$

effected by: $g_s \sim \epsilon^d, M \sim (\mathcal{O}(1)\epsilon)^{-\frac{3d}{2}}, N \sim (\mathcal{O}(1)\epsilon)^{-39d}, \epsilon \lesssim 1, d > 0.$

[It is believed that QGP being strongly coupled, requires a $g_s \sim 1$ apart from a large t’Hooft coupling M. Natsuume [2007].]
The working metric is given by:

\[
    ds^2 = \frac{1}{\sqrt{h}} \left( -g_1 dt^2 + dx_1^2 + dx_2^2 + dx_3^2 \right) + \sqrt{h} \left[ g_2^{-1} dr^2 + r^2 d\mathcal{M}_5^2 \right].
\]

\( g_i \)'s are black hole functions in modified OKS-BH background and are assumed to be:

\[
    g_{1,2}(r, \theta_1, \theta_2) = 1 - \frac{r_h^4}{r^4} + \mathcal{O} \left( \frac{g_s M^2}{N} \right)
\]

where \( r_h \) is the horizon, and the \((\theta_1, \theta_2)\) dependence come from the \( \mathcal{O} \left( \frac{g_s M^2}{N} \right) \) corrections.

The \( h_i \)'s are expected to receive corrections of \( \mathcal{O} \left( \frac{g_s M^2}{N} \right) \) K. Dasgupta et al [2012]. We assume the same to also be true of the ‘black hole functions’ \( g_{1,2} \). This will have the extremely non-trivial consequence that one can use the same choice of \( h_i \) and \( g_i \) in the weak(\( g_s \))coupling-large t’Hooft couplings as well as the ‘MQGP’ limits, later.

\[
    d\mathcal{M}_5^2 = h_1 (d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)^2 + h_2 (d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2)
    + h_4 (h_3 d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2) + h_5 \cos \psi (d\theta_1 d\theta_2 - \sin \theta_1 \sin \theta_2 d\phi_1 d\phi_2)
    + h_5 \sin \psi (\sin \theta_1 d\theta_2 d\phi_1 + \sin \theta_2 d\theta_1 d\phi_2),
\]

\( r >> a, h_5 \sim \frac{(\text{deformation parameter})^2}{r^3} \ll 1 \forall r \gg \) (deformation parameter) \( \frac{2}{3} \) in the UV.
$h_i$ appearing in internal metric as well as $M, N_f$ are not constant and up to linear order depend on $g_s, M, N_f$ as given below:

$$
\begin{align*}
    h_1 &= \frac{1}{9} + \mathcal{O}\left(\frac{g_s M^2}{N}\right), \\
    h_2 &= \frac{1}{6} + \mathcal{O}\left(\frac{g_s M^2}{N}\right), \\
    h_3 &= 1 + \mathcal{O}\left(\frac{g_s M^2}{N}\right), \\
    h_4 &= h_2 + \frac{4a^2}{r^2}, \\
    h_5 &\neq 0, \\
    L &= (4\pi g_s N)^{\frac{1}{4}}
\end{align*}
$$

The warp factor that includes the back-reactions, in the IR is given as P. Ouyang [2003], K. Dasgupta et al [2010]:

$$
\begin{align*}
    h &= \frac{L^4}{r^4} \left[ 1 + \frac{3g_s M_{\text{eff}}^2}{2\pi N} \log r \left\{ 1 + \frac{3g_s N_{\text{eff}}^\text{c}}{2\pi} \left( \log r + \frac{1}{2} \right) \right. \\
    &\quad + \frac{g_s N_{\text{eff}}^\text{f}}{4\pi} \log \left( \frac{\sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2}}{2} \right) \right\} \right].
\end{align*}
$$
UV behavior requires K. Dasgupta et al [2012]:

\[
\begin{align*}
    h &= \frac{L^4}{r^4} \left[ 1 + \sum_{i=1} h_i \left( \frac{\phi_{1,2}, \theta_{1,2}, \psi}{r^i} \right) \right], \text{ large } r; \\
    h &= \frac{L^4}{r^4} \left[ 1 + \sum_{i,j; (i,j) \neq (0,0)} h_{ij} \left( \frac{\phi_{1,2}, \theta_{1,2}, \psi}{r^j} \log^i r \right) \right], \text{ small } r.
\end{align*}
\]

We will assume that: \( h_i, h_{ij} \sim \mathcal{O} \left( \frac{g_s M^2}{N} \right) \ll 1 \) in the MQGP limit.
In the IR and $h_5 = 0$, the three-forms are as given in P. Ouyang [2003], K. Dasgupta et al [2010]:

\[(a) \tilde{F}_3 = 2MA_1 \left(1 + \frac{3g_s N_f}{2\pi} \log r\right) e_\psi \wedge \frac{1}{2} \left(\sin \theta_1 d\theta_1 \wedge d\phi_1 - B_1 \sin \theta_2 d\theta_2 \wedge d\phi_2\right)\]

\[-\frac{3g_s M N_f}{4\pi} A_2 \frac{dr}{r} \wedge e_\psi \wedge \left(\cot \frac{\theta_2}{2} \sin \theta_2 d\phi_2 - B_2 \cot \frac{\theta_1}{2} \sin \theta_1 d\phi_1\right)\]

\[-\frac{3g_s M N_f}{8\pi} A_3 \sin \theta_1 \sin \theta_2 \left(\cot \frac{\theta_2}{2} d\theta_1 + B_3 \cot \frac{\theta_1}{2} d\theta_2\right) \wedge d\phi_1 \wedge d\phi_2,\]

\[(b) H_3 = 6g_s A_4 M \left(1 + \frac{9g_s N_f}{4\pi} \log r + \frac{g_s N_f}{2\pi} \log \frac{\theta_1}{2} \sin \frac{\theta_2}{2}\right) \frac{dr}{r}\]

\[\wedge \frac{1}{2} \left(\sin \theta_1 d\theta_1 \wedge d\phi_1 - B_4 \sin \theta_2 d\theta_2 \wedge d\phi_2\right) + \frac{3g_s^2 M N_f}{8\pi} A_5 \left(\frac{dr}{r} \wedge e_\psi - \frac{1}{2} de_\psi\right)\]

\[\wedge \left(\cot \frac{\theta_2}{2} d\theta_2 - B_5 \cot \frac{\theta_1}{2} d\theta_1\right),\]

The asymmetry factors

\[A_i = 1 + \mathcal{O} \left(\frac{a^2}{r^2} \text{ or } \frac{a^2 \log r}{r} \text{ or } \frac{a^2 \log r}{r^2}\right) + \mathcal{O} \left(\frac{\text{deformation parameter}^2}{r^3}\right),\]

\[B_i = 1 + \mathcal{O} \left(\frac{a^2 \log r}{r} \text{ or } \frac{a^2 \log r}{r^2} \text{ or } \frac{a^2 \log r}{r^3}\right) + \mathcal{O} \left(\frac{\text{(deformation parameter)}^2}{r^3}\right).\]

As $a^2 = \mathcal{O} \left(\frac{g_s M^2}{N}\right) r_h^2 + \mathcal{O} \left(\frac{g_s M^2}{N} (g_s N_f)\right) r_h^4$ K. Dasgupta et al [2012], taking the MQGP limit, $A_i = B_i = 1$ in the IR/UV.
In the UV and at the boundary of the region interpolating between the IR and the UV, \( r_0, \tilde{F}_{lmn}, H_{lmn} = 0 \) for \( r \geq r_0 \), to ensure conformality in the UV.

Near the \( \theta_1 = \theta_2 = 0 \)-branch, assuming: \( \theta_{1,2} \to 0 \) as \( \epsilon^{\gamma_\theta} > 0 \) and \( r \to r_\Lambda \to \infty \) as \( \epsilon^{\gamma_r} < 0 \), \( \lim_{r \to \infty} \tilde{F}_{lmn} = 0 \) and \( \lim_{r \to \infty} H_{lmn} = 0 \) for all components except \( H_{\theta_1 \theta_2 \phi_{1,2}} \); in the MQGP limit and near \( \theta_{1,2} = \pi/0 \)-branch,

\[
H_{\theta_1 \theta_2 \phi_{1,2}} = 0 / \left( \frac{3g_s^2 M N_f}{8\pi} \right)_{N_f=2,g_s=0.6,M=(\mathcal{O}(1)g_s)^{-\frac{3}{2}}} \ll 1. \text{ So, the UV nature too is captured near } \theta_{1,2} = 0 \text{-branch in the MQGP limit. This mimicks addition of } \overline{D5} \text{-branes in K. Dasgupta et al [2010] to ensure cancellation of } \tilde{F}_3.
\]

As in the UV, \( \left( \frac{\text{(deformation parameter)}}{r^3} \right)^2 \ll \left( \frac{\text{(resolution parameter)}}{r^2} \right)^2 \), assume the same three-form fluxes for \( h_5 \neq 0 \).
The conifold, expressed as a quadric in $\mathbb{CP}^3[2]$: $z_1 z_2 = z_3 z_4$ can be mapped to $\mathbb{CP}^1 \times \mathbb{CP}^1$ via Segré-embedding:

$$\mathbb{CP}^1(A_1, A_2) \times \mathbb{CP}^1(B_1, B_2) \leftrightarrow \mathbb{CP}^3(z_1 = A_1 B_1, z_2 = A_2 B_2, z_3 = A_1 B_2, z_4 = A_2 B_1).$$

Hence, the holomorphic embedding of $D7$-branes $z_1 = 0$ would correspond to two branches $A_1 = 0$ and $B_1 = 0$. Given that there is an $SU(N_f)$ flavor symmetry with each of the two branches, one generates an $SU(N_f) \times SU(N_f)$ symmetry. Cancelation of gauge anomalies requires addition of two flavors of opposite chirality with each of the two branches, let us denote the same by: $q/\bar{q}$ transforming as $(N_f, 1)/(1, N_f)$ under $SU(N_f) \times SU(N_f)$ and $(N + M, 1)/(\bar{N} + \bar{M}, 1)$ under $SU(M + N) \times SU(N)$, and $Q/\bar{Q}$ transforming as $(\bar{N}_f, 1)/(1, \bar{N}_f)$, under $SU(N_f) \times SU(N_f)$ and transforming as $(1, N)/(1, \bar{N})$ under $SU(M + N) \times SU(N)$. 
With $A_i, B_j$ transforming respectively as $(N + M, \overline{N})$ and $(N + M, N)$ the color-invariant and flavor-invariant superpotential will be $W_{\text{flavors}} = \lambda \tilde{q} A_1 Q \tilde{q} A_1 Q + \lambda \tilde{q} B_1 q \tilde{Q} B_1 q$. As $A_i, B_i$ are dimension-$\frac{3}{4}$ fields taking $q_i, \tilde{q}_j, Q_k, \tilde{Q}_k$ to be having the same dimension, they will hence be dimension-$\frac{9}{8}$. The mass terms in the superpotential breaking the $SU(N_f) \times SU(N_f)$ symmetry to the diagonal $SU(N_f)$ are: $W_{\text{masses}} = \lambda \tilde{q} A_1 Q \sqrt{\mu} q \tilde{q} + \lambda \tilde{q} B_1 q \sqrt{\mu} \tilde{Q} Q$. Then, rewrite the total superpotential as:

$$W_{\text{flavors}} + W_{\text{masses}} = \lambda \tilde{q} A_1 Q \tilde{q} A_1 Q + \lambda \tilde{q} B_1 q \tilde{Q} B_1 q + \lambda \tilde{q} A_1 Q \sqrt{\mu} q \tilde{q} + \lambda \tilde{q} B_1 q \sqrt{\mu} \tilde{Q} Q$$

$$= \begin{pmatrix} \tilde{q} \\ \tilde{Q} \end{pmatrix} \begin{pmatrix} \lambda A_1 Q \sqrt{\mu} & \lambda A_1 A_1 \\ \lambda \tilde{q} B_1 q \sqrt{\mu} & \lambda \tilde{q} B_1 q \sqrt{\mu} \end{pmatrix} \begin{pmatrix} q \\ Q \end{pmatrix}.$$
Hence, the 3-7 strings become massless when the $D3$-branes and $D7$-branes intersect. This corresponds to null eigenvalues of the mass matrix

$$
\begin{pmatrix}
\lambda_{hA_1} q Q \sqrt{\mu} & \lambda_{hA_1} q Q A_1 \\
\lambda_{\bar{q}_1 B_1} \bar{Q} B_1 & \lambda_{\bar{q}_1 B_1} \bar{Q} \sqrt{\mu}
\end{pmatrix},
$$
i.e., the Ouyang embedding equation $z_1 = \mu$. 
The values for the axion $C_0$ and the five form $F_5$ are given by P. Ouyang [2004]:

$$C_0 = \frac{N_f}{4\pi} (\psi - \phi_1 - \phi_2) \left[\text{since } \int_{S^1} dC_0 = N_f\right],$$

$$F_5 = \frac{1}{g_s} \left[ d^4x \wedge dh^{-1} + *(d^4x \wedge dh^{-1}) \right].$$

Using $D7$-branes monodromy arguments for Ouyang embedding:

$$z = (9a^2r^4 + r^6)^{1/4} e^{\nu/2(\psi - \phi_1 - \phi_2)} \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} = \mu, \tau \sim \frac{1}{2\pi} ln z,$$

implying, in the IR:

$$e^{-\phi} = \frac{1}{g_s} - \frac{N_f}{8\pi} \log \left( r^6 + 9a^2r^4 \right) - \frac{N_f}{2\pi} \log \left( \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right).$$

Near the $\theta_{1,2} = 0$-branch in the $e^{-\phi}$ written out in the IR, choosing $\gamma_\theta$ and $\gamma_r$ in such a way that in the UV: $\frac{3N_f}{4\pi} \gamma_r = \frac{N_f \gamma_\theta}{\pi}$, then $e^{-\phi}$ in the UV would approach a constant implying a vanishing $\beta$ or conformality in the UV. So, the $\theta_{1,2} = 0$-branch mimicks the required axion-dilaton behavior in the UV.
For simplicity, working in \( \{ z_i \neq 1 = 1 \} \)-patch of the F-theory Weirstrass variety:

\[
y^2 = x^3 + f(\{ z_i \neq 1 = 1, z_1 \}) x + g(\{ z_i \neq 1 = 1, z_1 \})
\]
with

\[
f(\{ z_i \neq 1 = 1, z_1 \}) = f_0 \prod_{i=1}^{8} (z_1 - Z_i), \quad \Delta(z) = \Delta_0 \prod_{j=1}^{24} (z_1 - Z_j), \quad \{ Z_i \} \neq \{ Z_i \}.
\]

For finite \( g_s \), one should in principle consider the entire infinite series for the \( j \)-function:

\[
j(\tau) = \frac{1}{q} + 744 + 19,688q + 21,493,760q^2 + \ldots.
\]

\((q \equiv e^{2i\pi \tau}).\)

Truncating the series at \( O(q^2) \), one obtains for large \( z_1 \):

\[
\tau = \frac{i}{g_s} - i \log \left( -\sqrt{\frac{2123364f_0^6}{\Delta_0^2} - \frac{571392f_0^3}{\Delta_0} - 1625 + \frac{4608f_0^3}{\Delta_0} - 62}{3\sqrt{3646}} \right) + \frac{2,304i}{\pi a} \frac{\sum_{i=1}^{24} Z_i - 3 \sum_{i=1}^{8} Z_i}{\sqrt{\frac{2123364f_0^6}{\Delta_0^2} - \frac{571392f_0^3}{\Delta_0} - 1625}} + \mathcal{O}\left( \frac{1}{z_1^2} \right).
\]

For finite \( g_s \), \( \tau = \frac{i}{g_s} + \frac{i \mathcal{F}(f_0, \Delta_0)}{\pi} + \sum_{m=1}^{\infty} \frac{c_n(\theta_1, 2, \phi_1, 2, \psi; f_0, \Delta_0) + i d_n(\theta_1, 2, \phi_1, 2, \psi; f_0, \Delta_0)}{r^{3n}} \)

in the Ouyang embedding, implying \( \beta \to 0 \) as \( \Lambda \equiv r \to \infty. \)
Despite the mirror set of Hodge data, \( h_{\text{res conf}}^{1,1} = h_{\text{def conf}}^{2,1} = 1, h_{\text{res conf}}^{2,1} = h_{\text{def conf}}^{1,1} = 0 \), the resolved and deformed conifolds are not strict mirrors of each other. The \( \mathcal{O}(-1) \oplus \mathcal{O}(-1) \)-bundle over \( \mathbb{CP}^1 \) is characterized by four \( \chi \) SFs \( X_{1,2,3,4} \) with \( U(1) \) charges: \(-1, -1, 1, 1\) satisfying the \( D \)-term constraint:

\[-|x_1|^2 - |x_2|^2 + |x_3|^2 + |x_4|^2 = r\]

with the LG dual given by four twisted \( \chi \) SFs: \( Y_{1,2,3,4} \) satisfying \(-y_1 - y_2 + y_3 + y_4 = t : |x_i|^2 = Re(y_i)\)

and superpotential \( W_{\text{LG}} = \sum_{i=1}^{4} e^{-Y_i} \). The Hori-Vafa period integral:

\[
\Pi_{\text{HV}}^{\text{LG}} = \int \prod_{i=1}^{4} dY_i e^{-W_{\text{LG}}} \delta \left( \sum_{i=1}^{4} Y_i Q_i - t \right)
\]

\[= \int dY_3 d\hat{Y}_1 d\hat{Y}_2 e^{-Y_3(1+e^{-\hat{Y}_1}+e^{\hat{Y}_2}+e^{-t}e^{-\hat{Y}_1}e^{-\hat{Y}_2})}.
\]

Defining \( \chi_3 \equiv e^{-Y_3}, \hat{\chi}_{1,2} \equiv e^{-\hat{Y}_{1,2}} \in \mathbb{C}^* \) yields:

\[
\int \frac{d\chi_3}{\chi_3} \frac{d\hat{\chi}_1}{\hat{\chi}_1} \frac{d\hat{\chi}_2}{\hat{\chi}_2} e^{-\chi_3(1+\hat{\chi}_1+\hat{\chi}_2+e^{-t}\hat{\chi}_1\hat{\chi}_2)}
\]

\[
\int du dv d\chi_3 \frac{d\hat{\chi}_1}{\hat{\chi}_1} \frac{d\hat{\chi}_2}{\hat{\chi}_2} e^{-\chi_3(1+\hat{\chi}_1+\hat{\chi}_2+e^{-t}\hat{\chi}_1\hat{\chi}_2-uv)}.
\]

The mirror is thus: \( 1 + \hat{\chi}_1 + \hat{\chi}_2 + e^{-t}\hat{\chi}_1\hat{\chi}_2 = uv \) which due to \( \hat{\chi}_{1,2} \in \mathbb{C}^* \), is not a deformed conifold.
To implement mirror symmetry a la SYZ, one needs a special Lagrangian $T^3$. The following gives the embedding equation of a $T^2(\phi_1, \phi_2)$-invariant sLag $C_3(\phi_1, \phi_2, \psi)$ in the deformed conifold $T^*S^3$ M.Ionel and M.Min-OO (2008):

$$K'(r^2) \Im m(z_1 \bar{z}_2) = c_1,$$
$$K'(r^2) \Im m(z_3 \bar{z}_4) = c_2,$$
$$\Im m(z_1^2 + z_2^2) = c_3,$$

which using the same complex structure as that for the singular conifold and $K'(r^2) \xrightarrow{r \gg 1} r^{-3}$:

$$r^7 \left( \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} + \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} \right) \cos(\phi_1 + \phi_2) = c_1,$$
$$r^7 \left( \cos^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} + \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \right) \cos(\phi_1 + \phi_2) = c_2,$$
$$r^3 \sin \theta_1 \sin \theta_2 \cos \psi = c_3.$$
These can be solved to yield:

\[ r = \left( \frac{c_1}{\cos(\phi_1 + \phi_2)} + \frac{c_2}{\cos(\phi_1 - \phi_2)} \right)^{\frac{3}{7}}; \]

In large \( r \) limit: \( c_1 = c_2 \sim (r^<_\Lambda)^{\frac{7}{3}}, c_3 \sim (r^<_\Lambda)^6 \) \{e.g.\( r^<_\Lambda \sim r^4_\Lambda \} : - \)

\[ \cos \theta_{1,2} = \]

\[ F_{1,2}(\sec(\phi_1 \pm \phi_2), \cos(\phi_1 \pm \phi_2 \pm \psi), \cos(2\phi_{1,2}), \cos(2(2\phi_{1,2} + \phi_{2,1})), \sec \psi) \]

M.Dhuria, AM [2014]
From M.Ionel and M.Min-Oo [2008], the following is a $T^2$-invariant special Lagrangian three-cycle in a resolved conifold:

$$
\frac{K'}{2} \left(|x|^2 - |y|^2\right) + 4a^2 \frac{|\lambda_2|^2}{|\lambda_1|^2 + |\lambda_2|^2} = c_1,
$$

$$
\frac{K'}{2} \left(|v|^2 - |u|^2\right) + 4a^2 \frac{|\lambda_2|^2}{|\lambda_1|^2 + |\lambda_2|^2} = c_2,
$$

$$
\Im m(xy) = c_3,
$$

where one uses the complex structure for a resolved conifold Knauf+Gwyn[2007] and $[\lambda_1 : \lambda_2]$ being the homogeneous coordinates of the blown-up $\mathbb{C}P^1 = S^2$; 

$$
\frac{\lambda_2}{\lambda_1} = \frac{x}{-u} = \frac{v}{-y} = -e^{-i\phi_1} \tan \frac{\theta_1}{2}. Also,
$$

$$
\gamma(r^2) \equiv r^2 K'(r^2) = -2a^2 + 4a^4 N^{-\frac{1}{3}}(r^2) + N^{\frac{1}{3}}(r^2),
$$

where

$$
N(r^2) \equiv \frac{1}{2} \left(r^4 - 16a^6 + \sqrt{r^8 - 32a^6 r^4}\right).
$$

Defining $\rho \equiv \sqrt[3]{\frac{3}{2}} \sqrt[4]{\gamma}$, which upon inversion yields: $r \approx \left(\frac{2}{3}\right)^{\frac{3}{4}} \left(3a^2 + \rho^2\right)^{\frac{3}{4}}$ and $K'(r^2) = \frac{\sqrt[3]{\frac{3}{2}} \rho^2}{\left(3a^2 + \rho^2\right)^{\frac{3}{2}}}$.

These can be solved for explicitly: $\rho = \rho(\psi)$, $\theta_{1,2} = \theta_{1,2}(\rho(\psi))$ AM, K. Sil [to appear].
Defining local $T^3$-coordinates, $(\phi_1, \phi_2, \psi) \rightarrow (x, y, z)$:

$$x = \sqrt{h_2 h_4} \sin \langle \theta_1 \rangle \langle r \rangle \phi_1, \quad y = \sqrt{h_4 h_4} \sin \langle \theta_2 \rangle \langle r \rangle \phi_2, \quad z = \sqrt{h_1} \langle r \rangle h_4 \psi,$$

in the MQGP limit, M. Dhuria, AM [2014]; AM, K. Sil [to appear]

$$i^* J_{DC/RC} \approx 0,$$

$$\Im m (i^* \Omega_{DC/RC}) \approx 0,$$

$$\Re e (i^* \Omega_{DC/RC}) \sim \text{volume form } (T^3(x, y, z)).$$
Interestingly, in the ‘delocalized limit’ M. Becker et al. [2004] $\psi = \langle \psi \rangle$, under the coordinate transformation:

$$
\begin{pmatrix}
\sin \theta_2 d\phi_2 \\
d\theta_2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\cos \langle \psi \rangle & \sin \langle \psi \rangle \\
-\sin \langle \psi \rangle & \cos \langle \psi \rangle
\end{pmatrix}
\begin{pmatrix}
\sin \theta_2 d\phi_2 \\
d\theta_2
\end{pmatrix},
$$

and $\psi \rightarrow \psi - \cos \langle \bar{\theta}_2 \rangle \phi_2 + \cos \langle \theta_2 \rangle \phi_2 - \tan \langle \psi \rangle \ln \sin \bar{\theta}_2$, 

$\bar{\theta}_2 \equiv \frac{-\sqrt{6} \sin \langle \psi \rangle}{(4\pi g_s N)^{1/4}} y + \cos \langle \psi \rangle \theta_2 = -\sin \langle \theta_2 \rangle \sin \langle \psi \rangle \phi_2 + \cos \langle \psi \rangle \theta_2$, the $h_5$ term becomes $h_5 [d\theta_1 d\theta_2 - \sin \theta_1 \sin \theta_2 d\phi_1 d\phi_2]$, $e_\psi \rightarrow e_\psi$, i.e., one introduces an isometry along $\psi$ in addition to the isometries along $\phi_{1,2}$. This clearly is not valid globally - the deformed conifold does not possess a third global isometry.
To enable use of SYZ-mirror duality via three T dualities, one also needs to ensure a large base (implying large complex structures of the aforementioned two two-tori) of the $T^3(x, y, z)$ fibration. This is effected via F. Chen et al [2010]:

$$d\psi \rightarrow d\psi + f_1(\theta_1) \cos \theta_1 d\theta_1 + f_2(\theta_2) \cos \theta_2 d\theta_2,$$

$$d\phi_{1,2} \rightarrow d\phi_{1,2} - f_{1,2}(\theta_{1,2}) d\theta_{1,2},$$

for appropriately chosen large values of $f_{1,2}(\theta_{1,2})$. The three-form fluxes remain invariant.

Near $\theta_1 = \theta_2 = 0$ it is possible to obtain a large base for which $f_{1,2}(\theta_{1,2})$ is large. The guiding principle is that one requires that the metric obtained after SYZ-mirror transformation applied to the resolved warped deformed conifold is like a warped resolved conifold at least locally, then $G^\text{IIA}_{\theta_1 \theta_2}$ needs to vanish. M. Dhuria, AM [2013, 2014], AM, K. Sil [2015].
The mirror metric after performing three T-dualities, first along $x$, then along $y$ and finally along $z$, utilizing the results of M. Becker et al [2004] were worked out in M. Dhuria, AM [2013].

We can get a one-form type IIA potential from the triple T-dual (along $x$, $y$, $z$) of the type IIB $F_{1,3,5}$ in M. Dhuria, AM [2013] and using which the following $D = 11$ uplift is obtained:

$$ds_{11}^2 = e^{-\frac{2\phi^{IIA}}{3}} \left[ \frac{1}{\sqrt{h(r, \theta_1, \theta_2)}} \left( -g_1 dt^2 + dx_1^2 + dx_2^2 + dx_3^2 \right) + \sqrt{h(r, \theta_1, \theta_2)} \left( g_2^{-1} dr^2 \right) + ds_{IIA}^2(\theta_{1,2}, \phi_{1,2}, \psi) \right] + e^{\frac{4\phi^{IIA}}{3}} \left( dx_{11} + A^{F_1} + A^{F_3} + A^{F_5} \right)^2.$$
Analogous to the $F_3^{IIIB}(\theta_{1,2})$ (with non-zero components being $F_{\psi \phi_1 \theta_1}, F_{\psi \phi_2 \theta_2}, F_{\phi_1 \phi_2 \theta_1}$ and $F_{\phi_1 \phi_2 \theta_2}$) in Klebanov-Strassler background corresponding to $D5$-branes wrapped around a two-cycle which homologously is given by $S^2(\theta_1, \phi_1) - S^2(\theta_2, \phi_2)$, in the delocalized limit of Becker et al [2004], in M. Dhuria, AM [2014], $\int_{\mathcal{C}_4(\theta_{1,2}, \phi_{1/2}, x_{10})} G_4 \bigg|_{\langle \phi_{2/1}, \langle \psi \rangle, \langle r \rangle}$ was estimated to be very large. The $D = 11$ metric locally and in the UV, can be thought of as black $M5$-branes wrapping a two cycle homologously given by:

$$n_1 S^2(\theta_1, x_{10}) + n_2 S^2(\theta_2, \phi_{1/2}) + m_1 S^2(\theta_1, \phi_{1/2}) + m_2 S^2(\theta_2, x_{10})$$

for some large $n_{1,2}, m_{1,2} \in \mathbb{Z}$, i.e., black $M3$-branes metric.

Globally, one probably would interpret the uplift as $M$ theory on a non-compact seven-fold with $G_2$-structure and fluxes like F. Chen et al [2010]. Locally (schematically), $\tau_{G_2} \in W_2^{14} \oplus W_3^{27} \sim \frac{1}{(g_s N)^{1/4}} \oplus \frac{1}{(g_s N)^{1/4}}$ - More later in this talk.

Hence, the approach of the seven-fold to having a $G_2$ holonomy ($W_1^{G_2} = W_2^{G_2} = W_3^{G_2} = W_4^{G_2} = 0$) is accelerated in the MQGP limit. AM, K. Sil [2015].
$D = 11$ SUGRA EOMs/Bianchi identity M.S. Bremer [1999], are satisfied near the $\theta_{1,2} = 0, \pi$-branches in the MQGP limit:

$$R^M_{MN} = \frac{1}{12} \left( G_{MPQR} G_{N}^{PQR} - \frac{1}{12} G^M_{MN} G_{PQRS} G^{PQRS} \right)$$

$$+ \kappa^2_{11} \left( T_{MN} - \frac{1}{9} G^M_{MN} T^Q_Q \right)$$

$$d *_{11} G_4 + G_4 \wedge G_4 = -2 \kappa^2_{11} T_5 (H_3 - A_3) \wedge *_{11} J_6,$$

$$dG_4 = 2 \kappa^2_{11} T_5 *_{11} J_6,$$

where $M5$-brane current $J_6 \sim \frac{dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge d\theta_1 \wedge d\phi_1}{\sqrt{-G^M}}$, the space-time energy momentum tensor $T_{MN}$ for a single $M5$-brane wrapped around $S^2(\theta_1, \phi_1)$ is given by:

$$T^{MN}(x) = \int_{\mathcal{M}_6} d^6 \xi \sqrt{-\det G_{\mu\nu}^{(M5)}} \partial_\mu X^M \partial_\nu X^N \delta^{11}(x - X(\xi)) \sqrt{-\det G_{MN}^M}$$

where $X = 0, 1, ..., 11$ and $\mu, \nu = 0, 1, 2, 3, \theta_1, \phi_1$. 
A $d$-dimensional Riemannian manifold $\mathcal{M}$, has a $G$-structure if the structure group of the frame bundle can be reduced to $G \subset O(d)$.

In the context of $SU(3)$ structure, there are two invariant tensors. First is the fundamental form $J$ contained as a singlet in the adjoint representation $15$ of $SO(6)$ which decomposes under $SU(3)$ as $15 = 1 + 8 + 3 + \bar{3}$. The second is the invariant complex three-form contained as a pair of singlets in the $20$ of $SO(6)$ which decompose under $SU(3)$ as $20 = 1 + 1 + 3 + \bar{3} + 6 + \bar{6}$ implying the existence of $\Omega = \Omega^+ + i\Omega^-$. There being no singlet in the decomposition of a five-form, one finds that $J \wedge \Omega = 0$. Similarly, a six-form is a singlet of $SU(3)$, so we also must have that $J \wedge J \wedge J = \frac{3i}{4} \Omega \wedge \bar{\Omega}$, Conversely, a non-degenerate $J$ and satisfying $J \wedge \Omega = 0$ and $J \wedge J \wedge J = \frac{3i}{4} \Omega \wedge \bar{\Omega}$ implies that $\mathcal{M}$ has $SU(3)$-structure.
Now, one can define the Riemann curvature tensor $R^q_{mnp}$ and the torsion tensor $T^r_{mn}$ in terms of a metric compatible connection $\nabla'$ as follows:

$$\left[\nabla'_m', \nabla'_n\right] V_p = -R^q_{mnp} V_q - 2T^r_{mn} \nabla'_r V_p,$$

where $V$ is an arbitrary vector field. The Levi-Civita connection $\nabla$ is the unique torsionless connection compatible with the metric and is given by the usual expression in terms of Christoffel symbols $\Gamma^p_{mn} = \Gamma^p_{nm}$. Any metric-compatible connection can be written in terms of the Levi-Civita connection $\nabla^{(T)} = \nabla + \kappa$, where $\kappa^p_{mn}$ is the contorsion tensor. Metric compatibility implies $\kappa_{mnp} = -\kappa_{mpn}$, where $\kappa_{mnp} = \kappa^r_{mn} g_{rp}$.

The torsion $T^p_{mn}$ and the contorsion $\kappa^p_{mn}$ are related via: $T^p_{mn} = \kappa^p_{[mn]}$. 
In general, the Levi-Civita connection does not preserve the $G$-invariant tensors (or spinor) $\xi$, i.e., $\nabla \xi \neq 0$. However, one can show that there always exists a connection $\nabla^{(T)}$ which is compatible with the $G$ structure so that $\nabla^{(T)} \xi = 0$. On a manifold with $SU(3)$-structure, we can always find $\nabla^{(T)} : \nabla^{(T)} J = 0$, $\nabla^{(T)} \Omega = 0$.

If $\kappa$ is the contorsion tensor corresponding to $\nabla^{(T)}$, then symmetries of $\kappa_{mnp}$ imply $\kappa \in \Lambda^1 \otimes \Lambda^2$ where $\Lambda^n$ is the space of $n$-forms. Alternatively, since $\Lambda^2 \cong so(d)$, $\kappa_{mn}^p$ can also be thought of as a one-form valued in the Lie-algebra $so(d)$, i.e., $\Lambda^1 \otimes so(d)$. 
Given the existence of a $G$-structure, we can decompose $so(d)$ into a part in the Lie algebra $g$ of $G \subset SO(d)$ and its orthogonal complement $g^\perp = so(d)/g$. The contorsion $\kappa$ splits accordingly into $\kappa = \kappa^0 + \kappa^g$, where $\kappa^0$ is the part in $\Lambda^1 \otimes g^\perp$.

Since an invariant tensor (or spinor) $\xi$ is fixed under $G$ rotations, the action of $g$ on $\xi$ vanishes and one has: $\nabla^{(T)}\xi = (\nabla + \kappa^0 + \kappa^g)\xi = (\nabla + \kappa^0)\xi = 0$. Thus, any two $G$-compatible connections must differ by a piece proportional to $\kappa^g$ and they have a common term $\kappa^0$ in $\Lambda^1 \otimes g^\perp$ called the "intrinsic contorsion".
One can decompose $\kappa^0$ into irreducible $G$ representations providing a classification of $G$-structures in terms of which representations appear in the decomposition. In the special case when $\kappa^0$ vanishes so that $\nabla \xi = 0$, one says that the structure is “torsion-free”. For an almost Hermitian structure this is equivalent to requiring that the manifold is complex and Kähler. In particular, it implies that the holonomy of the Levi-Civita connection is contained in $G$.

Let us consider the decomposition of $T^0 (T^0_{mn} \equiv \kappa^0_{mn} \in \Lambda^1 \otimes g^\perp)$ in the case of $SU(3)$-structure. The relevant representations are:

$\Lambda^1 \sim 3 \oplus \bar{3}$, $g \sim 8$, $g^\perp \sim 1 \oplus 3 \oplus \bar{3}$. Thus the intrinsic torsion can be decomposed into the following $SU(3)$ representations:

$$\Lambda^1 \otimes su(3)^\perp = (3 \oplus \bar{3}) \otimes (1 \oplus 3 \oplus \bar{3})$$
$$= (1 \oplus 1) \oplus (8 \oplus 8) \oplus (6 \oplus \bar{6}) \oplus (3 \oplus \bar{3}) \oplus (3 \oplus \bar{3})'$$
$$\equiv W_1 \oplus W_2 \oplus W_3 \oplus W_4 \oplus W_5.$$
The $SU(3)$ structure torsion classes can be defined in terms of $J$, $\Omega$, $dJ$, $d\Omega$ and the contraction operator $\wr : \Lambda^k T^* \otimes \Lambda^n T^* \to \Lambda^{n-k} T^*$, $J$ being given by:

$$J = e^1 \wedge e^2 + e^3 \wedge e^4 + e^5 \wedge e^6,$$

and the $(3,0)$-form $\Omega$ being given by

$$\Omega = (e^1 + i e^2) \wedge (e^3 + i e^4) \wedge (e^5 + i e^6).$$

The torsion classes are defined in the following way:

1. $W_1 \leftrightarrow [dJ]^{(3,0)}$, given by real numbers $W_1 = W_1^+ + W_1^-$ with $d\Omega_+ \wedge J = \Omega_+ \wedge dJ = W_1^+ J \wedge J \wedge J$ and $d\Omega_- \wedge J = \Omega_- \wedge dJ = W_1^- J \wedge J \wedge J$;

2. $W_2 \leftrightarrow [d\Omega]^{(2,2)}_0 : (d\Omega_+)^{(2,2)} = W_1^+ J \wedge J + W_2^+ \wedge J$ and $(d\Omega_-)^{(2,2)} = W_1^- J \wedge J + W_2^- \wedge J$;

3. $W_3 \leftrightarrow [dJ]^{(2,1)}_0$ is defined as $W_3 = dJ^{(2,1)} - [J \wedge W_4]^{(2,1)}$;

4. $W_4 \leftrightarrow J \wedge dJ : W_4 = \frac{1}{2} J \wr dJ$;

5. $W_5 \leftrightarrow [d\Omega]^{(3,1)}_0 : W_5 = \frac{1}{2} \Omega_+ \wr d\Omega_+$ (the subscript 0 indicative of the primitivity of the respective forms).
The resolved warped deformed conifold can be written in the form of the Papadopoulos-Tseytlin [2001] ansatz in the string frame:

\[ ds^2 = h^{-1/2} ds_{\mathbb{R}^{1,3}}^2 + e^x ds^2_{\mathcal{M}} = h^{-1/2} dx_{1,3}^2 + \sum_{i=1}^{6} G_i^2 , \]

where Butti et al [2004], M.K. Bena, I.Klebanov [2008]:

\[ G_1 \equiv e^{(x(\tau) + g(\tau))/2} e_1 , \]
\[ G_2 \equiv A e^{(x(\tau) + g(\tau))/2} e_2 + B(\tau) e^{(x(\tau) - g(\tau))/2} (e_2 - ae_2) , \]
\[ G_3 \equiv e^{(x(\tau) - g(\tau))/2} (e_1 - ae_1) , \]
\[ G_4 \equiv B(\tau) e^{(x(\tau) + g(\tau))/2} e_2 - A e^{(x(\tau) - g(\tau))/2} (e_2 - ae_2) , \]
\[ G_5 \equiv e^{x(\tau)/2} v^{-1/2(\tau)} d\tau , \]
\[ G_6 \equiv e^{x(\tau)/2} v^{-1/2(\tau)}(d\psi + \cos \theta_2 d\phi_2 + \cos \theta_1 d\phi_1) , \]

wherein \[ A \equiv \frac{\cosh \tau + a(\tau)}{\sinh \tau} , B(\tau) \equiv \frac{e^{g(\tau)}}{\sinh \tau} . \] The \( e_i \)s are one-forms on \( S^2 \)
\[ e_1 \equiv d\theta_1 , \quad e_2 \equiv -\sin \theta_1 d\phi_1 , \]

and the \( \epsilon_i \)s a set of one-forms on \( S^3 \)
\[ \epsilon_1 \equiv \sin \psi \sin \theta_2 d\phi_2 + \cos \psi d\theta_2 , \]
\[ \epsilon_2 \equiv \cos \psi \sin \theta_2 d\phi_2 - \sin \psi d\theta_2 , \]
\[ \epsilon_3 \equiv d\psi + \cos \theta_2 d\phi_2 . \]
As $r \sim e^{\frac{\tau}{3}}$, in the MQGP limit, the metric matches the RWDC metric with the identifications:

$$e^x(\tau) \sim \frac{\sqrt{4\pi g_s N}}{v(\tau)} \left( 1 + \mathcal{O}(r_h^2 e^{-\frac{2\tau}{3}}) \right);$$

$$v(\tau) \sim \frac{3}{2} \left[ 1 + \mathcal{O} \left( \left\{ \frac{g_s M^2}{N} a_{\text{res}}^2, r_h^2 \right\} e^{-\frac{2\tau}{3}} \right) \right];$$

$$e^x(\tau) \sim \frac{\sqrt{4\pi g_s N}}{6} \left[ 1 + \mathcal{O} \left( \frac{g_s M^2}{N} a_{\text{res}}^2 e^{-\frac{2\tau}{3}} \right) \right];$$

$$g(\tau) \sim -2e^{-2\tau}; \ a(\tau) \sim -2e^{-\tau}.$$
In the MQGP limit (schematically) M.Dhuria, AM [2014]:
\[
T^{\text{IIB}}_{SU(3)} \in W_1 \oplus W_2 \oplus W_3 \oplus W_4 \oplus W_5 \sim \frac{e^{-3\tau}}{\sqrt{g_s N}} \oplus (g_s N)^{\frac{1}{4}} e^{-3\tau} \oplus \sqrt{g_s N} e^{-3\tau} \oplus -\frac{2}{3} \oplus -\frac{1}{2}(r \sim e^\frac{\tau}{3}) \text{ such that } \frac{2}{3} W_5^3 \approx W_4^5 \text{ in the UV, implying a Klebanov-Strassler-like supersymmetry Butti et al [2004].}
\]
Locally, the type IIA torsion classes after a fine tuning of the delocalized SYZ type IIA mirror metric, are:
\[
\tau^{\text{IIA}}_{SU(3)} \in W_2 \oplus W_3 \oplus W_4 \oplus W_5 \sim \gamma_2 g_s^{-\frac{1}{4}} N^{\frac{3}{10}} \oplus g_s^{-\frac{1}{4}} N^{-\frac{1}{20}} \oplus g_s^{-\frac{1}{4}} N^{\frac{3}{10}} \oplus g_s^{-\frac{1}{4}} N^{\frac{3}{10}} \sim W_2 \oplus W_4 \oplus W_5 : \Re e(W_5) \sim W_4
\]
So, generically, the large-$N$ suppression of the deviation of the type IIB resolved warped deformed conifold from being a complex manifold, is lost on being duality-chased to type IIA. However, it is possible by one further fine tuning to ensure that $\gamma_2 = 0$ in $W_2^{\text{IIA}}$ thereby ensuring that the local type IIA mirror is complex. Also, before and after application of delocalized SYZ mirror symmetry, locally and in the UV, one has supersymmetry. AM, K. Sil [2015].
If $V$ is a seven-dimensional real vector space, then a three-form $\varphi$ is said to be positive if it lies in the $GL(7, \mathbb{R})$ orbit of $\varphi_0$, where $\varphi_0$ is a three-form on $\mathbb{R}^7$ which is preserved by $G_2$-subgroup of $GL(7, \mathbb{R})$. The pair $(\varphi, g)$ for a positive 3-form $\varphi$ and corresponding metric $g$ constitute a $G_2$-structure. The space of $p$-forms decompose as following irreps of $G_2$:

\[
\begin{align*}
\Lambda^1 & = \Lambda_7^1 \\
\Lambda^2 & = \Lambda_7^2 \oplus \Lambda_{14}^2 \\
\Lambda^3 & = \Lambda_1^3 \oplus \Lambda_7^3 \oplus \Lambda_{27}^3 \\
\Lambda^4 & = \Lambda_1^4 \oplus \Lambda_7^4 \oplus \Lambda_{27}^4 \\
\Lambda^5 & = \Lambda_7^5 \oplus \Lambda_{14}^5 \\
\Lambda^6 & = \Lambda_7^6
\end{align*}
\]

The subscripts denote the dimension of representation and components of same representation/dimensionality, are isomorphic to each other.
Let $M$ be a 7-manifold with a $G_2$-structure $(\varphi, g)$. Then, defining $\psi \equiv *\varphi$, the components of spaces of 2-, 3-, 4-, and 5-forms are:

\[
\begin{align*}
\Lambda^2_7 &= \{\alpha \wedge \varphi : \alpha \in \Lambda^1_7\} \\
\Lambda^2_{14} &= \{\omega \in \Lambda^2_7 : (\omega_{AB}) \in g_2\} = \{\omega \in \Lambda^2_7 : \omega \wedge \varphi = 0\} \\
\Lambda^3_1 &= \{f \varphi : f \in C^\infty(M)\} \\
\Lambda^3_7 &= \{\alpha \wedge \psi : \alpha \in \Lambda^1_7\} \\
\Lambda^3_{27} &= \{\chi \in \Lambda^3_7 : \chi_{ABC} = h^D_{[A} \varphi_{BC]D} \text{ for } h_{AB} \text{ traceless, symmetric}\} \\
\Lambda^4_1 &= \{f \psi : f \in C^\infty(M)\} \\
\Lambda^4_7 &= \{\alpha \wedge \varphi : \alpha \in \Lambda^1_7\} \\
\Lambda^4_{27} &= \{\chi \in \Lambda^4_7 : \chi_{ABCD} = h^E_{[A} \psi_{BCD]E} \text{ for } h_{AB} \text{ traceless, symmetric}\} \\
\Lambda^5_7 &= \{\alpha \wedge \psi : \alpha \in \Lambda^1_7\} \\
\Lambda^5_{14} &= \{\omega \wedge \varphi : \omega \in \Lambda^2_{14}\}.
\end{align*}
\]
The metric $g$ defines a reduction of the frame bundle $F$ to a principal $SO(7)$-sub-bundle $Q$, that is, a sub-bundle of oriented orthonormal frames. Now, $g$ also defines a Levi-Civita connection $\nabla$ on the tangent bundle $TM$, and hence on $F$. However, the $G_2$-invariant 3-form $\varphi$ reduces the orthonormal bundle further to a principal $G_2$-subbundle $Q$. The Levi-Civita connection can be pulled back to $Q$. On $Q$, $\nabla$ can be uniquely decomposed as

$$\nabla = \bar{\nabla} + T$$

where $\bar{\nabla}$ is a $G_2$-compatible canonical connection on $P$, taking values in the sub-algebra $g_2 \subset so(7)$, while $T$ is a 1-form taking values in $g_2^\perp \subset so(7)$; $T$ is known as the intrinsic torsion of the $G_2$-structure - the obstruction to the Levi-Civita connection being $G_2$-compatible.
Now \( s_0 (7) \) splits under \( G_2 \) as

\[
\mathfrak{s}_0 (7) \cong \Lambda^2 V \cong \Lambda^2_7 \oplus \Lambda^2_{14}.
\]

But \( \Lambda^2_{14} \cong g_2 \), so the orthogonal complement \( g_2^\perp \cong \Lambda^2_7 \cong V \). Hence \( T \) can be represented by a tensor \( T_{ab} \) which lies in \( W \cong V \otimes V \). Now, since \( \varphi \) is \( G_2 \)-invariant, it is \( \bar{\nabla} \)-parallel. So, the torsion is determined by \( \nabla \varphi \).

Following Karigiannis [2007], consider the 3-form \( \nabla_X \varphi \) for some vector field \( X \) from where:

\[
\nabla_X \varphi \in \Lambda^3_7
\]

and:

\[
\nabla \varphi \in \Lambda^1_7 \otimes \Lambda^3_7 \cong W.
\]

equivalent to:

\[
\nabla_A \varphi_{BCD} = T_A \,^E \psi_{EBCD}
\]

\( (T_{AB} \equiv \text{full torsion tensor}) \); can be inverted to yield:

\[
T_A \,^M = \frac{1}{24} (\nabla_A \varphi_{BCD}) \, ^{MBCD}.
\]

Due to the isomorphism between the \( \Lambda^a_7 = 1, \ldots, 5 \)s, \( \nabla \varphi \) lies in the same space as \( T_{AB} \) and thus completely determines it.
The tensor $T_A^M$, like the space $W$, possesses 49 components and hence fully defines $\nabla \varphi$. In general $T_{AB}$ can be split into torsion components as

$$T = \tau_1 g + \tau_7 \varphi + \tau_{14} + \tau_{27}$$

where $\tau_1$ is a function and gives the 1 component of $T$. We also have $\tau_7$, which is a 1-form and hence gives the 7 component, and, $\tau_{14} \in \Lambda^2_{14}$ gives the 14 component. Further, $\tau_{27}$ is traceless symmetric, and gives the 27 component. Writing $\tau_i$ as $W_i$, we can split $W$ as

$$W = W_1 \oplus W_7 \oplus W_{14} \oplus W_{27}.$$  

From P. Kaste et al [2002], we see that a $G_2$ structure can be defined as:

$$\varphi_0 = \frac{1}{3!} f_{ABC} e^{ABC} = e^{-\phi_{IIA}} f_{abc} e^{abc} + e^{-\frac{2\phi_{IIA}}{3}} J \wedge e^{x_{10}},$$

where $A, B, C = 1, \ldots, 6, 10; a, b, c = 1, \ldots, 6$ and $f_{ABC}$ are the structure constants of the imaginary octonions.
As Karigiannis [2007]:

\[ d\varphi_0 = 4W_1 \star_7 \varphi_0 - 3W_7 \wedge \varphi_0 - \star_7 W_{27} \]
\[ d\star_7 \varphi_0 = -4W_7 \wedge \star_7 \varphi_0 - 2\star_7 W_{14}, \]

where \( W_{27} \) corresponds to the symmetric traceless rank-two tensor \( h_{AB} \) contracted with the \( \varphi_{0ABC} \) to give a rank-three \( \chi_{ABC} \) valued in \( W_{27} \) via \( \chi_{ABC} = h^d [A \varphi_{BC}]^D \), and \( W_{14} \) corresponds to the anti-symmetric rank \( \omega_{AB} \) satisying \( \omega \llcorner \varphi_0 = 0 \).

For our local M-theory uplift, \( \tau_{G_2} \in W_{14} \oplus W_{27} \) AM, K. Sil [2015]. The non-zero \( G_2 \) torsion classes are large-\( N \) suppressed. If all torsion classes of a \( G \) structure become trivial the manifold is supposed to possess a holonomy given by \( G \). So, the MQGP limit accelerates the approach of the seven-fold relevant to the eleven-dimensional uplift, locally, to being a \( G_2 \)-holonomy manifold.
Temperature

In the near-horizon limit \( r = r_h + \epsilon' \chi \), which implies \( 1 - \frac{r_h^4}{r^4} = \frac{4\epsilon' \chi}{r_h} + \mathcal{O}(\epsilon' ^2) \). In the MQGP limit, \( G_{00}^M \sim \epsilon \frac{55d}{3} r^2 \left( 1 - \frac{r_h^4}{r^4} \right) \), \( G_{rr}^M \sim \frac{\epsilon \frac{59d}{3}}{r^2 \left( 1 - \frac{r_h^4}{r^4} \right)} \). One can rewrite \( G_{rr} dr^2 = \xi' \frac{d\omega^2}{\omega} \)

where \( \xi' \sim \frac{r_h \epsilon'}{\epsilon \frac{59d}{3}} \). Writing \( \xi' \frac{d\omega^2}{\omega} = dv^2 \) or \( \omega = \frac{v^2}{4\xi_T} \), one sees that near \( r = r_h \), \( G_{tt}^M dt^2 \sim r_h^2 \epsilon^{38d} u^2 dt^2 \equiv 4\pi^2 T^2 dt^2 \), implying \( T \sim \frac{r_h}{\sqrt{g_s N}} \).

Now, in both limits, \( G_{00}^M \), \( G_{rr}^M \) have no angular dependence and hence the temperature \( T = \frac{\partial_r G_{00}}{4\pi \sqrt{G_{00} G_{rr}}} \) of the black \( M3 \)-brane then turns out to be given by:

\[
T = \frac{\sqrt{2}}{r_h \sqrt{\pi} \sqrt{\frac{g_s \left( 18g_s^2N_f \ln^2(r_h)M_{\text{eff}}^2 + 3g_s (4\pi - g_s N_f (-3 + \ln(2))) \ln(r_h)M_{\text{eff}}^2 + 8N\pi^2 \right)}{r_h^4}}}
\]

Both limits \( \rightarrow \frac{r_h}{\pi L^2} \).
In Ouyang [2003], a basis of one-forms consisting of the following holomorphic forms and their complex conjugates, was constructed:

\[
\lambda = 3 \frac{dr}{r} + ie_\psi,
\]

\[
\sigma_1 = \cot \frac{\theta_1}{2} (d\theta_1 - i \sin \theta_1 d\phi_1),
\]

\[
\sigma_2 = \cot \frac{\theta_2}{2} (d\theta_2 - i \sin \theta_2 d\phi_2).
\]

The following basis of imaginary self-dual (2,1) forms were constructed for the conifold:

\[
\eta_1 = \lambda \wedge \omega_2
\]

\[
\eta_2 = \frac{1}{2} \lambda \wedge (\sigma_1 \wedge \bar{\sigma}_2 - \sigma_2 \wedge \bar{\sigma}_1)
\]

\[
= \cot(\theta_1/2) \cot(\theta_2/2) \lambda \wedge (d\theta_1 \wedge d\theta_2 + \sin(\theta_1) d\phi_1 \wedge \sin(\theta_2) d\phi_2)
\]

\[
\eta_3 = \left( \frac{dr}{r} \wedge e_\psi + \frac{1}{2} \Omega_{22} \right) \wedge \sigma_1 = \left( \frac{i}{6} \lambda \wedge \bar{\lambda} - \frac{i}{2} d\sigma_2 \right) \wedge \sigma_1,
\]

\[
\eta_4 = \left( \frac{i}{6} \lambda \wedge \bar{\lambda} - \frac{i}{2} d\sigma_1 \right) \wedge \sigma_2,
\]

\[
\eta_5 = \bar{\lambda} \wedge \sigma_1 \wedge \sigma_2
\]

\[
= \bar{\lambda} \wedge (d\theta_1 \wedge d\theta_2 - \sin(\theta_1) d\phi_1 \wedge \sin(\theta_2) d\phi_2 - i(\Omega_{12} - \Omega_{21}))
\]

where \( \Omega_{ij} \equiv d\theta_i \wedge \sin \theta_j d\phi_j \).
In the $r \gg a$, $(\text{deformation parameter})^{\frac{2}{3}}$-limit of the asymmetry factors one obtains Ouyang[2003], AM, K. Sil[2015]:

$$G_3 = \frac{2M}{i} \left[ \left( 1 + \frac{3g_sN_f}{2\pi} \log r \right) \eta_1 + \frac{3g_sN_f}{8\pi} (\eta_4 - \eta_3) \right]$$

$$+ \mathcal{O} \left( (g_sN_f)^2; \left( \frac{a^2}{r^2}, \frac{a^2 \log r}{r}, \frac{a^2 \log r}{r^2}, \frac{a^2 \log r}{r^3} \right); \left( \text{deformation parameter}^2 \right) \right),$$

where the $\mathcal{O} \left( (g_sN_f)^2 \right)$ terms are:

$$\frac{3M(g_sN_f)^2}{4r} \left( -\frac{3}{4\pi} \log r - \frac{1}{2\pi} \log \left( \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right) \right) \left( 1 + \frac{9}{4\pi} \log r + \frac{1}{2\pi} \log \left( \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right) \right) \, dr \wedge (\Omega_{11} - \Omega_{22}).$$
In other words, $G_3$ is of the (2,1) type in the UV and near $\theta_1 \sim \frac{1}{N^{5}}, \theta_2 \sim \frac{1}{N^{10}}$. This is related to our result that the type IIB $SU(3)$-structure torsion classes are given by $T \in W_4 \oplus W_5 : \frac{2}{3} \text{Re} W_{5}^{3} = W_{4}^{3}$ (column “(B)”, “Table 2” of Butti et al [2004]).

Equivalently, the deviation from $G_3$ being imaginary self dual: $|iG_3 - *G_3|^2 \propto \frac{a^4}{r^4}$; assuming a negligible bare resolution parameter, $a$ in turn is related to the horizon radius $r_h$ via: $a^2 = \mathcal{O} \left( \frac{g_s M^2}{N} \right) r_h^2 + \mathcal{O} \left( \frac{g_s M^2}{N} (g_s N_f) \right) r_h^4$ K. Dasgupta et al [2012]: very small in MQGP limit.

The amount of near-horizon supersymmetry will be determined by solving for the killing spinor $\epsilon$ by the vanishing supersymmetric variation of the gravitino in $D = 11$. The near-horizon black $M3$-brane solution, near $\theta_{1,2} = 0, \pi$, possesses 1/8 supersymmetry M. Dhuria, AM [2013] reminiscent of M. Cvetic et al [2001]. Similar argument for MQGP limit.
UV Limit Before $\theta$ Integration in Evaluating $U(1)_{\mu C}$

- Assuming $\mu(\neq 0) \in \mathbb{R}/\mathbb{C}$ in Ouyang's embedding H.-Y. Chen et al [2008]:
  
  \[
  r_2^3 e^{\frac{i}{2} (\psi - \phi_1 - \phi_2)} \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} = |\mu|, \text{ which could be satisfied for } \psi = \phi_1 + \phi_2(+\pi) \text{ and } \\
  r_2^3 \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} = |\mu|.
  \]

- Include a $U(1)(of \quad U(N_f) = U(1) \times SU(N_f))$ field strength $F = \partial_r A_t dr \wedge dt$ in addition to $B$ in the $D7$-brane DBI action.

\[
S_{\text{DBI}} \sim N_f \int_{r_h}^{\infty} dr \int_{\theta_2=0}^{\pi} d\theta_2 
\]

\[
\left[ \left( F_{rt}^2 - 1 \right) \cot^2 \left( \frac{\theta_2}{2} \right) \csc^4 \left( \frac{\theta_2}{2} \right) \left( 2 \left( 5|\mu|^2 - 2r^3 \right) \cos(\theta_2) + 14|\mu|^2 \\
+ 3r^3 \cos(2\theta_2) + r^3 \right) \left( 8|\mu|^2 - 4r^3 \right) \cos(\theta_2) + r^3(\cos(2\theta_2) + 3) \right]^{\frac{1}{2}} \\
+ O \left( \left[ 1, h_5, a^2, \frac{|\mu|^2}{r} \right] \left[ \frac{1}{\sqrt{g_s N}}, \frac{g_s M^2}{N} \right] \right).
\]
Taking $r > |\mu|^{\frac{2}{3}} > 1$ - limit before $r$ integration, we see:

$$\int_0^{2\pi} \int_0^{2\pi} d\phi_1 d\phi_2 \int_0^{\pi} d\theta_2 \sqrt{\det(i^*(g + B) + F)} \sim r^3 \sqrt{1 - F_{rt}^2}$$

$$+ \mathcal{O}\left(\left[1, h_5, a^2, \frac{|\mu|^{\frac{2}{3}}}{r}\right] \left[\frac{1}{\sqrt{g sN}}, \frac{g s M^2}{N}\right]\right).$$

Hence:

$$N_f \int dr e^{-\phi(r)} r^3 \sqrt{1 - F_{rt}^2},$$

where $e^{-\phi(r)} = \frac{1}{g s} - \frac{N_f}{8\pi} \ln\left(r^6 + a^2 r^4\right) - \frac{N_f}{2\pi} \ln\left(|\mu| r^{-\frac{3}{2}}\right) \xrightarrow{r >> a} \frac{1}{g s} - \frac{N_f}{2\pi} \ln|\mu| - \frac{N_f a^2}{8\pi r^2}$. The Euler-Lagrange eom's solution:

$$\partial_r A_t(r) = \frac{C e^{\phi(r)}}{\sqrt{C^2 e^{2\phi(r)} + r^6}} \text{ implying,}$$

$$\mu_C = \int_{r_h}^{\infty} F_{rt} dr = \frac{\pi C g s 2 F_1\left(\frac{1}{3}, \frac{1}{2}, \frac{4}{3}; \frac{C^2}{r h^6\left(\frac{1}{g s s} - \frac{N_f \log(|\mu|)}{2\pi}\right)^2}\right)}{r h^2 (2\pi - g s N_f \log(|\mu|))}.$$
\[
\frac{\partial \mu_C}{\partial T} \bigg|_{N_f} = \pi \sqrt{4\pi g_s N} \left. \frac{\partial \mu_C}{\partial r_h} \right|_{N_f} - \frac{2\pi C r_h g_s}{r_h^3(2\pi - r_h g_s r_h N_f \log(|\mu|))} \sqrt{\frac{C^2}{r_h^6 \left( \frac{1}{g_s} - \frac{r_h N_f \log(|\mu|)}{2\pi} \right)^2 + 1} < 0
\]

\[
\frac{\partial \mu_C}{\partial N_f} \bigg|_{T} = \left( \pi C g_s^2 \log(|\mu|) \right) \left( \sqrt{\frac{C^2}{r_h^6 \left( \frac{1}{g_s} - \frac{N_f \log(|\mu|)}{2\pi} \right)^2 + 1} + \frac{4C^2 g_s^2 \pi^2}{r_h^6 (g_s N_f \log(|\mu|) - 2\pi)^2} \right) + \frac{3r_h^2 (g_s N_f \log(|\mu|) - 2\pi)^2}{\sqrt{\frac{C^2}{r_h^6 \left( \frac{1}{g_s} - \frac{N_f \log(|\mu|)}{2\pi} \right)^2 + 1}} > 0
\]

M.Dhuria, AM [2013] ⇒ Thermodynamical Stability
The $\theta_2$ integral is expressed in terms of elliptic integrals of the first kind and incomplete elliptic integrals.

In the MQGP and large-$r$ limit, after angular integration one obtains:

$$S \sim \int_{r=r_h}^\infty dr \left[ \sqrt{|\mu|} r^9 \sqrt{1 - F_{rt}^2} + O \left( r^{3/2}, (1, h_5, a^2) \left[ \frac{1}{\sqrt{g_s N}}, \frac{g_s M^2}{N} \right] \right) \right].$$

With $e^{-\phi} \approx \frac{1}{g_s} - \frac{N_f}{2\pi} \ln|\mu|$ in the MQGP limit, one obtains:

$$A_t = r \, _2F_1 \left( \frac{2}{9}, \frac{1}{2}, \frac{11}{9}, -r^9 \left( \frac{1}{g_s} - \frac{N_f \ln|\mu|}{2\pi} \right)^2 \frac{C^2}{C^2} \right) - \gamma$$

$$\approx \frac{72\pi^3 C^3 g_s^3 \left( \frac{1}{r} \right)^{23/4} \Gamma \left( \frac{11}{9} \right)}{23\Gamma \left( \frac{2}{9} \right) (g_s N_f \log(|\mu|) - 2\pi)^3} - \frac{36\pi C g_s \left( \frac{1}{r} \right)^{5/4} \Gamma \left( \frac{11}{9} \right)}{5\Gamma \left( \frac{2}{9} \right) (g_s N_f \log(|\mu|) - 2\pi)} + 2^{4/9} \Gamma \left( \frac{5}{18} \right) \Gamma \left( \frac{11}{9} \right) (C g_s)^{4/9}$$

$$+ \frac{2^{4/9}}{18\pi (g_s N_f \log(|\mu|) - 2\pi)^{4/9}} - \gamma \equiv \gamma_1 - \frac{\gamma_2}{r^4} + \frac{\gamma_3}{r^{23/4}} - \gamma.$$
The \( \gamma \) is chosen such that:
\[
\int_{r_0}^{r_1} \sqrt{g} (A_t - \gamma)^2 \sim \int_{r_0}^{r_1} r^3 (A_t - \gamma)^2 < \infty, \text{ i.e.,}
\]
\[
\frac{8}{11} \gamma_2 r_\Lambda^{11/4} (\gamma - \gamma_1) + \frac{1}{4} r_\Lambda^4 (\gamma - \gamma_1)^2 + \frac{2}{3} \gamma_2^2 r_\Lambda^{3/2} = 0, \text{ i.e.}:
\]
\[
\gamma = \frac{33 \gamma_1 r_\Lambda^{5/4} \pm 10i\sqrt{6} \gamma_2 - 48 \gamma_2}{33 r_\Lambda^{5/4}},
\]

implying that the square-integrable gauge field (choosing minus sign in \( \pm \)) is given by:
\[
A_t - \gamma = \frac{\gamma_3}{r^{23/4}} + \frac{1}{33} \gamma_2 \left( -\frac{33}{r^{5/4}} + \frac{2}{r^{5/4}} \right). 
\]

Utilizing that dimensionally \([C] = [r^{9/4}]\), and impose a Dirichlet boundary condition at a cut-off \( r_0 \): \( A_t(r_0) - \gamma = 0 \) where the cut-off is given by:
\[
\frac{C g_s \pi}{r_0^{9/4} (-2\pi + g_s N_f \log|\mu|)} = \pm \sqrt{\frac{23}{10}}.
\]

As \( e^{-\phi} \approx \frac{1}{g_s} - \frac{N_f \log|\mu|}{2\pi} > 0 \) we choose the minus sign.

Writing \( C \equiv m_\rho^{9/4} \) on dimensional grounds, where \( m_\rho \) provides the mass scale of the lightest vector boson, one obtains:
\[
m_\rho = \left( \frac{23}{10} \right)^{2/9} r_0 \left( \frac{2\pi - g_s N_f \log(|\mu|)}{g_s} \right)^{4/9}. 
\]
If \( m \rho = 760 \text{ MeV} \), the cut-off is:

\[
 r_0 = 760 \left( \frac{10^{-23}}{2 \pi - g s N_f \log(|\mu|)} \right)^{4/9} \pi^{4/9} (2 \pi - g s \kappa)^{4/9}.
\]

Hawking-Page phase transition is equivalent to the quark-gluon confinement and deconfinement phase transition. Gravitational action is of the form:

\[
 I \approx - \int \sqrt{g} (R - 2 \Lambda) d^5 x
\]

putting \( R = -(20/L^2) \) and \( \Lambda = -6/L^2 \) we get,

\[
 I \approx \int (8/L^2) \sqrt{g} d^5 x
\]

For type IIB metric without a black hole function in the MQGP limit, \( \sqrt{g} = r_h^4/(u^5 L^3) \). Now using an IR/UV cut off \( r = r_0/r_\Lambda \) E. Witten [1998], C. Herzog [2006], the regularized action density for thermal AdS background is given by

\[
 V_1 \approx (-8/L^5) \int_0^\beta dt \int_{r_0}^{r_\Lambda} r^3 dr.
\]
For the type IIB metric with a black hole function, the same is given by

\[ V_2 \approx \left( -\frac{8}{L^5} \right) \int_0^{\pi L^2/r_h} dt \int_{\min(r_0, r_h)}^{r_\Lambda} r^3 dr. \]

Comparing the two energy densities at \( r = r_\Lambda \) and using \( \beta = (\pi L^2/r_h) \sqrt{g_1(r_\Lambda)} \) we get

\[ \Delta V = \lim_{r_\Lambda \to \infty} (V_2(r_\Lambda) - V_1(r_\Lambda)) = \left( \frac{2\pi}{L^3 r_h} \right) (r_h^4/2), \quad r_0 > r_h \]
\[ = \left( \frac{2\pi}{L^3 r_h} \right) (r_0^4 - r_h^4/2), \quad r_0 < r_h \]

The Hawking-Page phase transition occurs when \( \Delta V \) is equal to zero giving \( r_h = 2^{1/4} r_0 \) which gives the transition temperature

\[ T_c = 2^{\frac{1}{4}} r_0 / L^2 \pi \]
So, one obtains:

\[
N_f = \frac{46\pi}{g_s} \pm \frac{288802^{9/16}5^{3/4}4\sqrt{19}\sqrt{23}8\sqrt{g_sN}}{\pi^{19/8}g_s^{5/4}N^{5/4}T_c^{9/4}} / 23\log(|\mu|),
\]

and for \( g_s \sim 1 \), the embedding parameter \( \mu \sim 13.6 \) and the number of light flavors \( N_f = 3 \), one obtains \( T_c = 175 \) MeV consistent with lattice data in the MQGP limit from the type IIB background.

Now, dimensionally, \([\mu] = [r^{3/2}]\) and using the AdS/CFT dictionary, hence mass dimensions of 3/2. Curiously, if one were to write \( m_q : \sqrt{|\mu|} = m_q^{3/4} \), one would obtain, in units of MeV, \( m_q \approx 5.6 \) - exactly the mass scale of the first generation light quarks!
\[ \mu_C = \int_{r_h}^{\infty} F_{rt} \, dr = \frac{2^{4/9} C g_s \Gamma \left( \frac{5}{18} \right) \Gamma \left( \frac{11}{9} \right) \left( \frac{g_s N_f \log(|\mu|) - 2\pi}{C^2 g_s^2} \right)^{5/18}}{18 \sqrt{\pi (2\pi - g_s N_f \log(|\mu|))}} \]

\[ -r_h^2 F_1 \left( \frac{2}{9}, \frac{1}{2}, \frac{11}{9}; -\frac{r_h^{9/2} (g_s N_f \log(|\mu|) - 2\pi)^2}{4 C^2 g_s^2 \pi^2} \right). \]

\[ \left. \frac{\partial \mu_C}{\partial T} \right|_{N_f} = \pi \sqrt{4\pi g_s N} \left. \frac{\partial \mu_C}{\partial r_h} \right|_{N_f} = \pi \sqrt{4\pi g_s N} \left( -\frac{1}{\sqrt{r_h^{9/2} (g_s N_f \log(|\mu|) - 2\pi)^2 + 4\pi^2 C^2 g_s^2}} \right) < 0. \]
\[
\left. \frac{\partial \mu C}{\partial N_f} \right|_T = \frac{4 \cdot 2^{4/9} \Gamma \left( \frac{5}{18} \right) \Gamma \left( \frac{11}{9} \right) \log(|\mu|)}{9^{18/\pi} C \left( \frac{(g_s N_f \log(|\mu|) - 2\pi)^2}{C^2 g_s^2} \right)^{13/18}} - \frac{4 g_s r_h \log(|\mu|)}{9 (g_s N_f \log(|\mu|) - 2\pi)} \times \left[ \frac{1}{\sqrt{\frac{r_h^{9/2} (g_s N_f \log(|\mu|) - 2\pi)^2}{4\pi^2 C^2 g_s^2}}} - 1 \right]^{-1} \\
2 \, \text{F}_1 \left( \frac{2}{9}, \frac{11}{2}; \frac{11}{9}; - \frac{r_h^{9/2} (g_s N_f \log(|\mu|) - 2\pi)^2}{4 C^2 g_s^2 \pi^2} \right)_{g_s=0.8, N_f=3, |\mu|=13.6, C=(760)^{9/4}} > 0.
\]
\[ S_E = \frac{1}{16\pi} \int_{\mathcal{M}} d^{11}x \sqrt{G} R + \frac{1}{8\pi} \int_{\partial\mathcal{M}} d^{10}x K \sqrt{\hat{h}} - \frac{1}{4} \int_{\mathcal{M}} (G_4 \wedge * G_4 - C_3 \wedge G_4 \wedge G_4) + \frac{T_2}{2\pi^4 32^2 2^{13}} \int_{\mathcal{M}} d^{11}x \sqrt{G} (J - \frac{1}{2} E_8) + T_2 \int C_3 \wedge X_8 - S^{\text{ct}}, \]

where \( T_2 \equiv M2\)-brane tension, and:

\[
J = 3.2^8 \left( R^{mijn} R_{p[ij} R^{rs,p} R^{q}_{rsn} + \frac{1}{2} R^{mnij} R_{pqij} R^{rs,p} R^{q}_{rsn} \right),
\]

\[
E_8 = \epsilon_{abcm_1 n_1 \ldots m_4 n_4} \epsilon_{abcm'_1 n_1' \ldots m'_4 n'_4} R^{m'_1 n'_1 \ldots m_4 n_4}_{m_1 n_1 \ldots m'_4 n'_4},
\]

\[
X_8 = \frac{1}{192 \cdot (2\pi^2)^4} \left[ \text{tr}(R^4) - (\text{tr} R^2)^2 \right],
\]

where \( \mathcal{M} \) is a volume of spacetime defined by \( r < r_\Lambda \), where the counter-term \( S^{\text{ct}} \) is added such that the action \( S_E \) is finite \( \text{R. Monteiro et al} [2009], \text{R. B. Mann, R. Mcnees} [2009] \).
The action, apart from being divergent (as $r \to \infty$) also possesses pole-singularities near $\theta_{1,2} = 0, \pi$. We will regulate the second divergence by taking a small $\theta_{1,2}$-cutoff $\epsilon_{\theta}$, $\theta_{1,2} \in [\epsilon_{\theta}, \pi - \epsilon_{\theta}]$, and demanding $\epsilon_{\theta} \sim \epsilon^\gamma$, for an appropriate $\gamma$. We will then explicitly check that the finite part of the action turns out to be independent of this cut-off $\epsilon/\epsilon_{\theta}$. 

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In the MQGP Limit M. Dhuria, AM [2013]

\[ S_{EH} = a_{EH} \left( \alpha_{1,3,4,5} : g_s = \alpha_1 \epsilon, N = \alpha_2 \epsilon^{-39}, M = \alpha_3 \epsilon^{-\frac{3}{2}}, \theta_{1,2} \sim 0 \sim \alpha_{4,5} \epsilon^{\frac{15}{6}} \right) \frac{r_4^A}{\epsilon^5 r_h}; \]

\[ S_{GHY} = (+\text{ive}) \left( \frac{r_4^A - r_h^4}{r_h} \right); \Rightarrow S_{GHY}^{\text{finite}} \sim -r_h^3 \equiv \text{independent of cut-off } \epsilon. \]

\[ C_{\mu \nu \rho}^{\mathcal{M}} = B_{\mu \nu}, \quad C_{\mu \nu \rho}^{\mathcal{M}} = C_{\mu \nu \rho}^{IIA}; \quad F_4^{IIA} = dC_3^{IIA} \text{ can not be obtained via SYZ mirror duality applied to } F_1^{IIB} \Rightarrow G_4 = H_3 \wedge dx_{10} + \left( A_1^{F_1} + A_1^{F_3} + A_1^{F_5} \right) \wedge H_3 \]

\[ C_3^{\mathcal{M}} = B_2 \wedge dx_{10}; \quad \int C_3 \wedge G_4 \wedge G_4 = 0. \]

\[ S_{\text{Flux}} = -\epsilon^{19} \left( \frac{r_4^\Lambda}{r_h} + \frac{r_h^3}{\ln(r_\Lambda)} \right). \]

\[ S_{R^4} \left( J - \frac{1}{2} E_8 \right) = O(10^{-15}) \epsilon^{42} \frac{r_4^A}{r_h} + O(10^6) \epsilon^{81.3} \left( \frac{1}{r_\Lambda^2 r_h} \right). \]
Locally, $M_8 \simeq S^1(x^0) \times \mathbb{R}^3_{\text{conf}} \times M_4(r, \beta_i, x_{a_1}, x_{a_2})$ where $\beta_i \equiv \theta_1$ or $\theta_2$, $x_{a_1,2} \equiv (y, z)$ or $(x, z)$ or $(x, y)$;

$p_1(TM_8) = p_2(TM_8) = 0 \Rightarrow X_8 \sim 4p_2 - p_1^2 = 0 \Rightarrow \int C_3 \wedge X_8 = 0.$

One can show that:

$$\int_{r=r_{\Lambda}} \sqrt{h} R^M \sim \epsilon^{\kappa_{\text{EH-surface}}} \frac{r^4}{r_{h}}$$

$$\int_{r=r_{\Lambda}} \sqrt{h} \sim \epsilon^{\kappa_{\text{cosmo}}} \frac{r^4}{r_{h}},$$

$$\int_{r=r_{\Lambda}} \sqrt{h}|G_4|^2 \sim \epsilon^{\kappa_{\text{flux}}} \frac{r^4}{r_{h}},$$

$S_{\text{finite}}^{\text{EH+GHY+}|G_4|^2+O(R^4)} \sim -r_h^3.$
In the MQGP limit the divergent contribution is given by:

\[
\frac{r^4_\Lambda}{r_h} \left( \frac{a_{EH}}{\epsilon^5} + a_{GHY-\text{boundary}} - \epsilon^{19} a_{G4} + a_{R4} \epsilon^{42} \right).
\]

Counter terms: \( \int \left( \epsilon^{\frac{\kappa^{(ii)}}{\text{EH-surface}}} \sqrt{h^M R^M}, \epsilon^{\frac{\kappa^{(ii)}}{\text{cosmo}}} \sqrt{h}, \epsilon^{\frac{\kappa^{(ii)}}{\text{flux}}} \sqrt{h} |G_4|^2 \right) \bigg|_{r=r_\Lambda} \times \left( \frac{a_{EH}}{\epsilon^5} + a_{GHY-\text{boundary}} - \epsilon^{19} a_{G4} + a_{R4} \epsilon^{42} \right) \) M. Dhuria, AM [2013]

The entropy density \( s \sim r_h^3 \). Using the same one can show that \( C > 0 \) - implying a stable uplift!
Freezing the angular dependence on $\theta_{1,2}$ (there being no dependence on $\phi_{1,2}, \psi, x_{10}$ in the MQGP limit), noting that $G^{\text{IIA}/M}_{00,rr,R^3}$ are independent of the angular coordinates (additionally possible to tune the chemical potential $\mu_C$ to a small value M.Dhuria, AM [2013]), using the result of Kovtun Son Starinets [2003]:

$$\frac{\eta}{s} = T \frac{\sqrt{|G^{\text{IIA}/M}|}}{\sqrt{|G^{\text{IIA}/M}_{tt} G^{\text{IIA}/M}_{rr}|}} \bigg|_{r=r_h} \int_{r_h}^{\infty} dr \frac{|G^{\text{IIA}/M}_{00} G^{\text{IIA}/M}_{rr}|}{G^{\text{IIA}/M}_{R^3} \sqrt{|G^{\text{IIA}/M}|}} = \frac{1}{4\pi}.$$
In the notations of Kovtun Son Starinets [2003] one can pull out a common $Z(r)$ in the angular-part of the metrics as: $Z(r)K_{mn}(y)dy^i dy^j$, (which for the type IIB/IIA backgrounds, is $\sqrt{r}r^2$) in terms of which:

$$D = \frac{\sqrt{|G^{IIB/IIA}|} Z^{IIB/IIA}(r)}{G^{IIB/IIA} \sqrt{|G_{00}^{IIB/IIA} G_{rr}^{IIB/IIA}|}} \bigg|_{r=r_h} \int_{r_h}^{\infty} dr \frac{|G^{IIB/IIA}_{00} G^{IIB/IIA}_{rr}|}{\sqrt{|G^{IIB/IIA}|} Z^{IIB/IIA}(r)}$$

$$= \frac{1}{2\pi T}$$
A solution of the linearized field equation for any field $\phi(u, x)$ choosing $q^\mu = (w, q, 0, 0)$:

$$
\phi(u, x) = \int \frac{d^4 q}{(2\pi)^4} e^{-iwt+iqx} f_q(u) \phi_0(q)
$$

where $f_q(u)$ is normalized to 1 at the boundary and satisfies the incoming wave boundary condition at $u = 1$, and $\phi_0(q)$ is defined via:

$$
\phi(u = 0, x) = \int \frac{d^4 q}{(2\pi)^4} e^{-iwt+iqx} \phi_0(q).
$$

If the kinetic term for $\phi(u, x)$ is given by: $\frac{1}{2} \int d^4 x du A(u) (\partial_u \phi(x, u))^2$, then using the equation of motion for $\phi$ it is possible to reduce an on-shell action to the surface terms as,

$$
S = \int \frac{d^4 q}{(2\pi)^4} \phi_0(-q) F(q, u) \phi_0(q)|_{u=0}^{u=1}
$$

where the function

$$
F(q, u) = A(u)f_{\pm q}(u)\partial_u f_{\pm q}(u).
$$

Finally, the retarded Green's function is given by the formula proposed in D.T.Son, A.O.Starinets [202]:

$$
G_R(q) = -2F(q, u)|_{u=0}.
$$
We consider the metric and gauge field fluctuations of the background. The retarded Green's functions are defined as

\[ G_{\mu\nu,\rho\sigma}^{R_T}(q) = -i \int d^4 x e^{-iwt+iq\cdot x} \theta(t) \langle [T_{\mu\nu}(x), T_{\rho\sigma}(0)] \rangle, \]

with \( \langle [T_{\mu\nu}, T_{\rho\sigma}] \rangle \sim \frac{\delta^2 S}{\delta h_{\mu\nu} \delta h_{\rho\sigma}} \) and

\[ G_{\mu\nu}^{R_J}(q) = -i \int d^4 x e^{-iwt+iq\cdot x} \theta(t) \langle [J_\mu(x), J_\nu(0)] \rangle \]

with \( \langle [J_\mu(x), J_\nu(0)] \rangle \sim \frac{\delta^2 S}{\delta A_\mu \delta A_\nu} \).

The Kubo formulae: \( \eta = -\lim_{w \to 0} \frac{1}{w} \left( \lim_{q \to 0} \Im mG_{xy,xy}^{R_T}(w, q) \right) \) corresponding to vector-mode of metric fluctuation \( h_{xy} \) or \( \eta = -\lim_{w \to 0} \frac{1}{w} \left( \lim_{q \to 0} \Im mG_{yz,yz}(w, q) \right) \) corresponding to tensor-mode metric fluctuation. Similarly, the DC electrical conductivity is given by \( \sigma = \lim_{w \to 0} \frac{\Im mG_{xx}^{R_J}(w,0)}{w} \).
The spectral functions in the presence of non-zero baryon density are obtained by computing correlator functions of gauge field fluctuations about a non-zero temporal component of gauge field background: \( \hat{A}_\mu(u, \vec{x}) = \delta_\mu^0 A_t(u) + \tilde{A}_\mu(\vec{x}, u) \).

The action for \( U(1) \) gauge field in the presence of \( N_f \) flavor branes is

\[
I_{D7} = T_7 N_f \int d^8 \xi e^{-\phi(r)} \sqrt{\det[i^*(g + B) + F]}.
\]

Introducing fluctuations around \( A_t(u) \), \( \hat{A}_\mu = A_t + \tilde{A}_\mu \). Defining \( G \equiv i^*(g + B) + F \), the DBI Action will be given by

\[
I_{D7} = T_7 \int d^8 \xi e^{-\phi(r)} \sqrt{\det(G + \tilde{F})}.
\]
The EOM for components of $\tilde{A}_\mu$ are:

$$
\partial_\nu \left[ \sqrt{|\det G|} \times \left( G^{\mu\nu} G^{\sigma\gamma} - G^{\mu\sigma} G^{\nu\gamma} - G^{[\nu\sigma]} G^{\gamma\nu} \right) \partial_\gamma \tilde{A}_\mu \right] = 0
$$

and the on-shell DBI action can be written as Erdmenger et al [2007]:

$$
\mathcal{I}_{D7}^{(2)} = T_7 \int d^8 \xi \ e^{-\phi(r)} \sqrt{|\det G|} \times \left( (G^{tu})^2 \tilde{A}_t \partial_u \tilde{A}_t - G^{rr} G^{ik} \tilde{A}_i \partial_u \tilde{A}_k - A_0 G^{ut} \text{tr}(G^{-1} \tilde{F}) \right) \bigg|_{u=0}^{u=1}.
$$

Using: $G_{ut} \to 0$ at $u = 0$, the action near $u = 0$ gets simplified and is given as:

$$
\mathcal{I}_{D7}^{(2)} = T_7 \int d^8 \xi \ e^{-\phi(u)} \sqrt{|\det G|} \times \left( -G^{uu} G^{ii} \tilde{A}_i \partial_u \tilde{A}_i \right) \bigg|_{u=0}
$$

where $i \in \mathbb{R}^{1,3}(t, x, y, z)$. 
The coordinates in Minkowski directions are chosen such that four-vector exhibits only one spatial component i.e. $q^\mu = (w, q, 0, 0)$. Writing $\tilde{A}_\mu(x, u) = \int \frac{d^4q}{(2\pi)^4} e^{-i\omega t + iqx} \tilde{A}_\mu(q, u)$, in terms of the gauge invariant field components $E_x = w\tilde{A}_x + q\tilde{A}_t$, $E_\alpha = w\tilde{A}_\alpha$, the DBI Action in terms of these co-ordinates will be given as:

$$\mathcal{I}^{(2)}_{D7} = T_7 \int \frac{dwd^3q}{(2\pi)^4} \left[ \frac{e^{-\phi(u)} r^2_h}{u} \left( \frac{E_x \partial_u E_x}{q^2 - w^2/g_1} - \frac{1}{w^2} E_y \partial_u E_y \right) \right] \Bigg|_{u=0} \bigg|_{u=1}.$$ 

Defining $E_x(q, u) = E_0(q) \frac{E_q(u)}{E_q(u=0)}$, the flux factor Son, Starinets [2002]:

$$\mathcal{F}(q, u) = -\frac{e^{-\phi(u)} r^2_h}{w^2 u} \frac{E_{-q}(u) \partial_u E_q(u)}{E_{-q}(u=0) E_q(u=0)},$$

and the retarded Green's function for $E_x$ is:

$$\mathcal{G}_{xx} = -2\mathcal{F}(q, u) = \frac{2e^{-\phi(u)} r^2_h}{u} \frac{\partial_u E_q(u)}{E_q(u)} \bigg|_{u=0}.$$ 

The spectral functions in zero momentum limit will be given as:

$$\chi_{xx}(w, q = 0) = -2\text{Im}\mathcal{G}_{xx}(w, 0) = e^{-\phi(u)} r^2_h \text{Im} \left[ \frac{1}{u} \frac{\partial_u E_q(u)}{E_q(u)} \right]_{u=0}.$$
In new set of gauge invariant coordinates, in the zero-momentum limit ($q = 0$):

$$E''(u) + \left( -\frac{4u^3}{1 - u^4} - \frac{1}{u} + \sqrt{\frac{3C^2u^5}{g_s^2r_h^6 + C^2u^6}} \right) E'(u)$$

$$+ \frac{w_3^2}{(1 - u^4)^2} \sqrt{\frac{g_s^2r_h^6}{g_s^2r_h^6 + C^2u^6}} E(u) = 0,$$

where $E(u) = E_x(u) = E_y(u)$. One sees that $u = 1$ is a regular singular point and the roots of the indicial equation about the same are given by: $\pm \frac{iw_3g_5r_h}{4\sqrt{c^2 + g_s^2r_h^6}}$; choosing 'incoming wave' solution, solutions sought will be of the form:

$$E(u) = (1 - u) \frac{-iw_3g_5r_h}{4\sqrt{c^2 + g_s^2r_h^6}} \mathcal{E}(u) \equiv (1 - u)^{-\frac{iw_3\beta}{4}} \mathcal{E}(u),$$

where $\mathcal{E}(u)$ is analytic in $u$. 
Performing a perturbation theory in powers of $w_3$, one obtains a solution of the form:

$$\mathcal{E}^{(0)}(u \sim 0) = \frac{C_1^{(0)}}{2} u^2 + C_2^{(0)};$$

$$\mathcal{E}^{(1)}(u \sim 0) = \frac{C_1^{(0)}}{2} u^2 + C_2^{(0)} + \frac{i\beta}{4} \left( \frac{C_1^{(1)}}{2} u^3 - \frac{C_2^{(1)}}{2} u^2 \right).$$

To get non-zero conductivity, we will require $C_1^{(0)} \in \mathbb{C}$.

The conductivity will be given as:

$$\sigma = \lim_{w \to 0} \frac{\chi_{xx}(w, q = 0)}{w} = \frac{r_h^2 C_1^{(0)}}{g_s \pi T} = \frac{\pi g_s N T}{g_s}.$$

The charge susceptibility is given by Mas, Shock, Tarrio [2009]:

$$\chi = \left( \int_{r_h}^{r_b} \frac{dF_{rt}}{dn_q} \right)^{-1}, \text{ charge density } n_q = \frac{\delta S_{DBI}}{\delta F_{rt}}.$$
From M.Dhuria, AM [2013]:

\[ n_q \sim \frac{r^3 F_{rt} \left( \frac{1}{g_s} - \frac{N_f \ln \mu}{2\pi} \right)}{\sqrt{1 - F_{rt}^2}}, \quad \mu \equiv \text{embedding parameter}. \]

So:

\[
\frac{1}{\chi} \sim \frac{1}{\left( \frac{1}{g_s} - \frac{N_f \ln \mu}{2\pi} \right)} \int_{r_h}^{\infty} \frac{dr}{r^3} \left( 1 - F_{rt}^2 \right)^{\frac{3}{2}}
\]

\[
\frac{1}{6r_h^2 \frac{(g_s N_f \log(\mu) - 1)}{Cg_s}} \left( C^2 g_s^2 + g_s^2 N_f^2 r_h^6 \log^2(\mu) - 2g_s N_f r_h^6 \log(\mu) + r_h^6 \right)
\]

\[
\times \left[ \left( C^2 g_s^2 - 2g_s^2 N_f^2 r_h^6 \log^2(\mu) + 4g_s N_f r_h^6 \log(\mu) - 2r_h^6 \right)
\]

\[
\times _2 F_1 \left( \frac{1}{3}, \frac{1}{2}; \frac{4}{3}; -\frac{C^2 g_s^2}{r_h^6 (g_s N_f \log(\mu) - 1)^2} \right)
\]

\[ +5r_h^6 (g_s N_f \log(\mu) - 1)^2 _2 F_1 \left( -\frac{1}{2}, \frac{1}{3}; \frac{4}{3}; -\frac{C^2 g_s^2}{r_h^6 (g_s N_f \log(\mu) - 1)^2} \right) \right].
\]

In the MQGP limit M.Dhuria, AM [2013], one sees: \( \chi \sim r_h^2 \sim g_s N T^2. \)
As given in J. Mas et al [2008], the diffusion constant is:

\[ D = e^{-\phi} \sqrt{G G^{00} G^{uu} G^{ii}} \bigg|_{u=1} \int_{u=1}^{u=0} du \frac{du}{e^{-\phi} \sqrt{G G^{00} G^{uu}}} \]

\[ = \frac{L^2}{r_h} + O(C^2 g_s^2 / r_h^8) \sim \frac{1}{T}. \]

Thus, \( \frac{\sigma}{\chi} \sim \frac{1}{T} \sim D \), verifying the Einstein relation: \( D = \frac{\sigma}{\chi} \).
Considering a chemical potential with $SU(2)$ flavour structure the general action is given by:

$$S = -T_r T_{D7} \int d^8 \xi \sqrt{\det(g + \hat{F})}$$

where the group-theoretic factor $T_r = \frac{1}{2}$ for $SU(2)$ and the field strength tensor is given as:

$$\hat{F}_{\mu \nu} = \sigma^a (2 \partial_{[\mu} \hat{A}^a_{\nu]} + \frac{r_h^2}{2\pi \alpha'} f^{abc} \hat{A}^b_{\mu} \hat{A}^c_{\nu}),$$

$\sigma^a$ are the Pauli matrices and $\hat{A}$ is given by

$$\hat{A}_\mu = \delta^0_\mu \tilde{A}_0 + A_\mu$$

with the $SU(2)$ background gauge field

$$\tilde{A}_0^3 \sigma^3 = \tilde{A}_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
Similar to $E^3(u)$, the $U(1)$ EOM corresponding to gauge-invariant $E(u)$ is AM, K. Sil [2015]:

$$\frac{\partial^2 E}{\partial u^2} + \Sigma(u) \frac{\partial E}{\partial u} + \left[ \frac{w^2}{\pi^2 T^2(u^4 - 1)^2} \right] E = 0$$

where

$$\Sigma(u) \equiv \frac{1}{4(u^4 - 1)\sqrt{\frac{r_h}{u}} \left( r_h^4 \sqrt{\frac{r_h}{u}} + C^2 e^{2\phi} u^5 \right)^3}$$

$$\times \left\{ 16 C^6 e^{6\phi} \sqrt{\frac{r_h}{u}} u^{14} (2u^4 - 1) + 6 C^2 e^{2\phi} r_h^9 \sqrt{\frac{r_h}{u}} u^5 (13u^4 - 5) + r_h^{14} (23u^4 - 7) + 3 C^4 \exp 4\phi r_h^5 u^9 (29u^4 - 13) \right\},$$

one realizes that $u = 1$ is a regular singular point with solutions to the indicial equation given by: $\pm i \frac{w_3}{4}$ and we choose the minus sign for incoming-wave solutions: $Z(u) = (1 - u)^{-\frac{iw_3}{4}} Z(u)$. Use a perturbative ansatz:

$$Z(u) = Z^{(0)}(u) + w_3 Z^{(1)}(u) + O(w_3^2).$$
We notice that the only distinction between the $SU(2)$ and $U(1)$ EOMs is the shift in the roots of the indicial equation corresponding to the horizon being a regular singular point; the incoming plane-wave root of the former (in $\alpha' = \frac{1}{2\pi}$-units) is given by AM, K. Sil [2015]:

$$-\frac{i}{4} \left( w_3 + \overline{A_0^3}(u = 1) \right) = -\frac{i}{4} \left( w_3 + \left[ \frac{2^{4/9} \Gamma \left( \frac{5}{18} \right) \Gamma \left( \frac{11}{9} \right)}{\sqrt[18]{\pi}} \left( \frac{g_s N_f \log(\mu) - 2\pi^2}{C^2 g_s^2} \right)^{2/9} - 1 \right] r_h \right).$$
Defining the longitudinal electric field as $E_x(q, u) = E_0(q) \frac{E_q(u)}{E_q(u=0)}$, the “flux factor” in the zero momentum limit, will hence be given as:

$$\mathcal{F}(q, u) = -\frac{e^{-\phi(u)} r_h^{\frac{1}{4}} u^{\frac{7}{4}}}{w^2} \frac{E_{-q}(u) \partial_u E_q(u)}{E_{-q}(u=0)E_q(u=0)},$$

and the retarded Green’s function for $E_x$, using the prescription of D.T.Son, A. Starinets [2002], will be given by: $\mathcal{G}(q, u) = -2\mathcal{F}(q, u)$. The retarded Green function for $A_x$ is $w^2$ times above expression and for $q = 0$, it gives

$$\mathcal{G}_{xx} = 2e^{-\phi(u)} r_h^{\frac{1}{4}} u^{\frac{7}{4}} \frac{\partial_u E_q(u)}{E_q(u)} \bigg|_{u=0}.$$

The spectral functions in zero momentum limit will be given as:

$$\mathcal{X}_{xx}(w, q = 0) = -2\text{Im}\mathcal{G}_{xx}(w, 0) = e^{-\phi(u)} r_h^{\frac{1}{4}} \text{Im} \left[ u^{\frac{7}{4}} \frac{\partial_u E_q(u)}{E_q(u)} \right]_{u=0}.$$


$$\sigma = \lim_{w \to 0} \frac{\mathcal{X}_{xx}(w, q = 0)}{w} = \lim_{u \to 0, w \to 0} \frac{r_h^{\frac{1}{4}} u^{\frac{7}{4}} \text{Im} \left( \frac{E'(u)}{E(u)} \right)}{w}.$$
The final result for the DC conductivity $\sigma$ is given as under:

$$\sigma = \frac{r_h^{\frac{1}{4}}}{\pi T} \Im \Im \left( \frac{c_2}{c_2^2} \left( \frac{i}{16} \left( -\frac{3}{4} c_1 + \frac{\gamma_0}{4} c_2 \right) - c_3 \frac{c_1 \gamma_0}{4} \right) \right) \sim (g_s N)^{\frac{1}{8}} T^{-\frac{3}{4}} \frac{c_1}{c_2}.$$}

Interestingly, this mimicks a one-dimensional interacting system - Luttinger liquid - for appropriately tuned Luttinger parameter T. Giamarchi [1991].

Another physically relevant quantity is the charge susceptibility $\chi$, which is thermodynamically defined as response of the charge density to the change in chemical potential, is given by the following expression J. Mas et al [2008]:

$$\chi = \frac{\partial n_q}{\partial \mu_C} \bigg|_T,$$

where $n_q = \frac{\delta S_{DBI}}{\delta F_{rt}}$, and the chemical potential $\mu_C$ is defined as $\mu_C = \int_{r_h}^{r_B} F_{rt} dr$. The charge density will be given as:

$$n_q = \frac{\delta S_{DBI}}{\delta F_{rt}} \sim \frac{F_{rt} \sqrt{|\mu|} r_{rt}^{\frac{9}{4}}}{\sqrt{1 - F_{rt}^2}},$$
One gets the following charge susceptibility AM, K. Sil [2015]:

\[
\frac{1}{\chi} = \int_{r_h}^{\infty} dr \frac{dF_{rt}}{dn_q} = \int_{r_h}^{\infty} dr \frac{r_h^9}{\sqrt{|\mu|} \left( \frac{C^2}{\left( \frac{1}{g_s} - \frac{N_f}{2\pi} \log |\mu| \right)^2 + r_h^9/2} \right)^{3/2}}
\]

\[
= \frac{1}{45 \sqrt{\mu} r_h^{5/4} \left( \frac{C^2}{\left( \frac{1}{g_s} - \frac{N_f}{2\pi} \log |\mu| \right)^2 + r_h^{9/2}} \right)^{3/2}}
\]

\[
\times \left\{ 414 r_h^{9/2} \, _2F_1 \left( -\frac{1}{2}, \frac{5}{18}; \frac{23}{18}; -\frac{C^2}{\left( \frac{1}{g_s} - \frac{N_f}{2\pi} \log |\mu| \right)^2 / r_h^{9/2}} \right) \right. \\
\left. + \left( 4 \frac{C^2}{\left( \frac{1}{g_s} - \frac{N_f}{2\pi} \log |\mu| \right)^2} - 5 r_h^{9/2} \right) \, _2F_1 \left( \frac{5}{18}, \frac{1}{2}; \frac{23}{18}; -\frac{C^2}{\left( \frac{1}{g_s} - \frac{N_f}{2\pi} \log |\mu| \right)^2 / r_h^{9/2}} \right) \right\} 
\]

\[
= \frac{4}{5 \sqrt{|\mu|} (4\pi g_s N)^{5/8} T^{5/4}} + \mathcal{O} \left( \frac{1}{(g_s N)^{23/8}} \right).
\]
Hence, the charge susceptibility is given by:

\[ \chi \sim \sqrt{|\mu|} (g_s N)^{5/8} T^{5/4}. \]

Given that one is in the regime of linear response theory, one expects the Einstein’s relation: \( \frac{\sigma}{\chi} = D \sim \frac{1}{T} \), to hold. However, a naive application yields \( \frac{\sigma}{\chi} \sim \frac{c_1}{c_2} \frac{1}{\sqrt{|\mu| g_s N}} \frac{1}{T^2} \). One expects the Ouyang embedding parameter to be related to the deformation parameter if there were supersymmetry. In the MQGP limit, there is approximate supersymmetry. The resolution parameter possesses an \( r_h \)-dependence. If one assumes that \( |\mu| \sim \frac{1}{r_h^2} \) (in \( \alpha' = 1 \)-units), then the Einstein’s relation is preserved.
The $U(1)_R$-charges are defined in the bulk gravitational background dual to the isometry group corresponding to the spherical directions transverse to the AdS space. As the black $M3$-branes can asymptotically be expressed as $M5$-branes wrapped around two-cycles defined homologously as integer sum of two-spheres, there will be a rotational (R)- symmetry group dual to the isometry group $U(1) \times U(1)$ corresponding to $\phi_{1/2}$ and $\psi$ in the directions transverse to $M5$-branes wrapped around $n_1 S^2(\theta_1, \phi_{(2/1)}) + n_2 S^2(\theta_2, \chi_{10})$.

To determine the diffusion coefficient due to the $R$-charge, one needs to evaluate the two-point correlation function of $\tilde{A}_\mu$ which basically will be a metric perturbation of the form $h_{M\mu}$ where $M$ is a spherical direction and $\mu$ is an asymptotically AdS direction. As a first step, the $\tilde{A}_\mu$ EOM is:

$$\partial_\beta \left[ g^{\mu\nu} g^{\alpha\beta} \sqrt{g} \tilde{F}_{\nu\alpha} \right] = 0.$$
Writing $\tilde{A}_\mu(x_1) = \int \frac{d^4 q}{(2\pi)^4} e^{-i \omega t + ix_1} \tilde{A}_\mu(q, u)$ and working in the $A_u = 0$ gauge, by setting $\mu = u, x_1, t, \alpha(\in \mathbb{R}^3)$, one ends up with the following equations:

\[
\begin{align*}
&w_3 \tilde{A}_t' + g_1 q_3 \tilde{A}_{x_1} = 0, \\
&\tilde{A}_{x_1}'' - \frac{1}{u} \tilde{A}_{x_1}' + \frac{g_1'}{g_1} \tilde{A}_{x_1}' - \frac{1}{g_1^2} \left( w_3 q_3 \tilde{A}_t + w_3^2 \tilde{A}_{x_1} \right) = 0, \\
&\tilde{A}_t'' - \frac{1}{u} \tilde{A}_t' - \frac{1}{g_1} \left( w_3 q_3 \tilde{A}_t + q_3^2 \tilde{A}_{x_1} \right) = 0, \\
&\tilde{A}_\alpha'' - \frac{1}{u} \tilde{A}_\alpha' + \frac{g_1'}{g_1} \tilde{A}_\alpha' + \frac{1}{g_1^2} \left( w_3^2 - g_1 q_3^2 \right) \tilde{A}_\alpha = 0.
\end{align*}
\]

where $\alpha = (x_2, x_3)$.

Make a double perturbative ansatz:

\[
\tilde{A}_{x_1}(u) = \tilde{A}_{x_1}^{(0,0)}(u) + q_3 \tilde{A}_{x_1}^{(1,0)}(u) + w_3 \tilde{A}_{x_1}^{(0,1)}(u) + q_3^2 \tilde{A}_{x_1}^{(2,0)}(u) + \ldots.
\]
The kinetic terms relevant to the evaluation of two-point correlators of $\tilde{A}_{u,t,\alpha}$ in the MQGP limit are:

$$S \sim g_s^{-\frac{4}{3}} r_h^2 L^2 \int d^4 x du \frac{1}{u} \left[ (1 - u^4) \left( \tilde{A}'_\alpha \right)^2 + (1 - u^4) \left( \tilde{A}'_x \right)^2 - \left( \tilde{A}'_t \right)^2 \right]$$

Hence, the retarded Green's function $G^{R}_{\alpha\alpha}$ will be given as:

$$G^{R}_{\alpha\alpha}^{finite} \sim \frac{1}{u} \frac{\tilde{A}_\alpha, -q(u)}{\tilde{A}_\alpha, -q(0)} \partial_u \left( \frac{\tilde{A}_\alpha, q(u)}{\tilde{A}_\alpha, q(0)} \right) \bigg|_{u=0}$$

$$\sim iw + 2D_R q^2 \text{ where } D_R^{\alpha} \sim \frac{1}{\pi T}.$$
One can decouple $\tilde{A}_{x_1}$ and $\tilde{A}_t$, and obtain, e.g., the following third order differential equation for $\tilde{A}_{x_1}$:

$$
(1 - u^4)^2 \tilde{A}_{x_1}'''(u) - (1 - u^4) \left\{(1 - u^4) + 12u^3 \right\} \tilde{A}_{x_1}''(u)
- \left[ 4u^2(1 - u^4) - 16u^6 - q_3^2(1 - u^4) + w_3^2 \right] \tilde{A}_{x_1}' = 0.
$$

$u = 1$ is a regular singular point. The exponents of the indicial equation are $-1 \pm \frac{iw_3}{4}$; choosing the incoming boundary condition, we write $\tilde{A}_{x_1}' = (1 - u)^{-1} - \frac{iw_3}{4} \tilde{A}_{x_1}'$ and assume a double perturbative series for solution to $\tilde{A}_{x_1}'(u)$ of the form:

$$
\tilde{A}_{x_1}'(u) = \tilde{A}_{x_1}^{(0,0)}(u) + w_3 \tilde{A}_{x_1}^{(0,1)}(u) + q_3^2 \tilde{A}_{x_1}^{(2,0)}(u) + O(w_3^2, w_3 q_3^2).
$$
\[ G_{x_1x_1}^R \sim \lim_{u \to 0} \frac{1}{u} \left( \frac{\tilde{A}_{x_1,q}(u)}{\tilde{A}_{x_1,q}(u=0)} \right) \partial_u \left( \frac{\tilde{A}_{x_1,-q}(u)}{\tilde{A}_{x_1,-q}(u=0)} \right) \]

\[ \sim \left( \frac{w^2}{iw - D_R q^2} \right) \quad \text{where } D_R^\alpha \sim \frac{1}{\pi T}.\]
Under small perturbations of the five-dimensional metric, \( g_{\mu\nu} = g_{\mu\nu}(0) + h_{\mu\nu} \), the first order Einstein equation: \( \mathcal{R}^{(1)}_{\mu\nu} = \frac{2}{3} \Lambda h_{\mu\nu} \).

We assume the perturbation of metric of \( M^3 \)-branes to be dependent on \( x_1 \) and \( t \) only i.e after Fourier decomposing the same, we have \( h_{\mu\nu}(\vec{x}, t, u) = \int \frac{d^4 q}{(2\pi)^4} e^{-iwt+iq_1} h_{\mu\nu}(q, w, u) \) and choose the gauge where \( h_{\mu u} = 0 \). In case of \( M^3 \)-branes, there will be rotation group \( SO(2) \) acting on the directions transverse to \( u, t, \) and \( x_1 \). Based on the the spin of different metric perturbations under this group, the same can be classified into groups as follows:

(i) **vector modes**: \( h_{x_1 x_2}, h_{tx_2} \neq 0 \) or \( h_{x_1 x_3}, h_{tx_3} \neq 0 \), with all other \( h_{\mu\nu} = 0 \).

(ii) **Scalar modes**: \( h_{x_1 x_1} = h_{x_2 x_2} = h_{x_3 x_3} = h_{tt} \neq 0 \), \( h_{x_1 t} \neq 0 \), with all other \( h_{\mu\nu} = 0 \).

(iii) **Tensor modes**: \( h_{x_2 x_3} \neq 0 \), with all other \( h_{\mu\nu} = 0 \).

We are interested to calculate shear viscosity in the context of \( M^3 \)-brane by obtaining correlator functions corresponding to vector and tensor modes.
Vector Mode Fluctuations

- $h_{tx_2}, h_{x_1 x_2} \neq 0$ with all other $h_{\mu\nu} = 0$. Since the D=11 metric is conformally flat near $u = 0$, one can make a Fourier decomposition such that

$$h_t^{x_2} = e^{-iwt+iqx_1} H_t(u), \quad h_{x_1}^{x_2} = e^{-iwt+iqx_1} H_{x_1}(u).$$

$$\Lambda \sim \int_{M_6(\theta_1, 2, \phi_1, 2, \psi, x_{10})} \frac{G_4 \wedge *G_4}{\sqrt{G^M}} \sim c_1 \frac{2g_s \epsilon^{15}}{3L^2}.$$ 

Analogous to a partial cancelation in Herzog [2002] of the coefficient of $H_x$ originating from $R^{(1)}_{\mu\nu}$ and the cosmological constant contribution from the $\Lambda h_{\mu\nu}$ term, assuming that

$$\alpha = \frac{2\tilde{c}_1 \epsilon^{15}}{3} - 8 = 0$$

in MQGP limit after incorporating value of $\Lambda$, we assume that the $\frac{1}{u^2g_1}$-term will be canceled out with remaining terms appearing in the coefficient of $H_{x, t}$ in $H_x(x, t)$'s EOM. Hence the linearized set of EOM will be given as

$$w_3 H'_t + g_1 q_3 H'_{x_1} = 0,$$

$$H''_{x_1} - \frac{4 - \frac{5}{2} g_1}{ug_1} H'_{x_1} + \frac{1}{g_1^2} \left( w_3 q_3 H_t + w_3^2 H_{x_1} \right) = 0,$$

$$H''_t - \frac{3}{u} H'_t - \frac{1}{g_1} \left( w_3 q_3 H_{x_1} + q_3^2 H_t \right) = 0.$$
To obtain two-point function at boundary, one needs to determine the kinetic term for vector modes $H_x$ and $H_t$. The same can be calculated using the Einstein-Hilbert Action upto quadratic order in $h_{\mu\nu}$ given as:

$$S = \frac{1}{2\kappa_{11}^2} \int \sqrt{g} g^{\mu\nu} \left( \Gamma^{(1)\alpha}_{\mu\beta} \Gamma^{(1)\beta}_{\mu\nu} - \Gamma^{(1)\alpha}_{\mu\nu} \Gamma^{(1)\beta}_{\alpha\beta} \right) + \ldots \sim \frac{r_h^4}{g_s^2} \int du \ d^4x \frac{1}{u^3} \left[ (H'_x)^2 - g_1 (H'_t)^2 \right].$$

According to the Kubo’s formula, shear viscosity is defined as:

$$\eta = -\lim_{w \to 0} \left[ \frac{1}{w} \Im m G_{x_1 x_2, x_1 x_2} \right].$$

In the $q_3 \to 0$ limit, the $H_t$ and $H_{x_1}$ decouple and the EOM for ‘$H_{x_1}$’ becomes:

$$H''_{x_1} (u) - \frac{[4 - \frac{5}{2} (1 - u^4)]}{u(1 - u^4)} H'_{x_1} (u) + \left[ \frac{\omega_3^2}{(1 - u^4)^2} + \frac{\alpha}{u^2(1 - u^4)} \right] H_{x_1} (u) = 0,$$

where $u = 1$ is thus seen to be a regular singular point with exponents of the corresponding indicial equation given by: $\pm \frac{i \omega_3}{4}$. Choosing the ‘incoming boundary condition’ exponent, we will look for solutions of the form $H_{x_1} (u) = (1 - u)^{-\frac{i \omega_3}{4}} \mathcal{H}_{x_1} (u)$, $\mathcal{H}_{x_1} (u)$ being analytic in $u$. 

Assuming a perturbative ansatz for
\[ H_{x_1}(u) = H^{(0)}_{x_1}(u) + w_3 H^{(1)}_{x_1}(u) + w_3^2 H^{(2)}_{x_1} + O(w_3^3), \]
one notices that
\[ H^{(0)}_{x_1}(u = 0, \alpha) \neq 0 \text{ for } \alpha = 0. \]

In the MQGP limit:
\[ g_s = \alpha g_s \epsilon, \quad N = \alpha N \epsilon^{-39}, \quad \theta \to 0 \text{ as } \alpha \theta \epsilon^{\frac{3}{2}}, \alpha \theta << 1. \]
As \( \epsilon \sim 1 \), for appropriate choices of \( \alpha g_s, N, \theta \) and constants of integration in the solutions of
\[ H^{(0), (1)}_{x_1} \]
one obtains:
\[ \frac{n}{s} = \frac{1}{4\pi}. \]
Tensor Mode Fluctuations

$h_{x_2 x_3} \neq 0$ with all other $h_{\mu \nu} = 0$ Herzog [2002]. By Fourier decomposing $h_{x_3}^{x_2}(u, \vec{x}) = e^{-i \omega t + i q x_1} \phi(u)$, the linearized Einstein EOM for $\phi(u)$ by will be given as:

$$\phi''(u) - \frac{(3 + u^2)}{u(1 - u^4)} \phi'(u) + \frac{1}{(1 - u^4)} \left[ w_3^2 - (1 - u^4) q_3^2 \right] \phi(u) = 0.$$ 

The horizon $u = 1$ is a regular singular point and the roots of the indicial equation around this are $\pm \frac{i w_3}{4}$; choosing the incoming-wave boundary condition, $\phi(u) = (1 - u)^{-\frac{i w_3}{4}} \Phi(u)$ write a double perturbative ansatz: $\Phi(u) = \Phi^{(0,0)}(u) + w_3 \Phi^{(0,1)}(u) + q_3^2 \Phi^{(2,0)}(u) + ...$,
The kinetic term for $\phi$ is given by:

$$\left( \frac{r_h^4}{g_s^2 \kappa_{11}^2} \right) \int d^4u g_1(u) (\phi')^2 \frac{1}{u^3}.$$ 

By fine tuning of $\alpha_{g_s,N,\theta}$ and constants of integration in the solutions of $\Phi^{(0,0)}$ and $\Phi^{(0,1)}$, one obtains: $\frac{\eta}{s} = \frac{1}{4\pi}$. 
Demanding that infinitesimal diffeomorphism:
\[ x^\mu \rightarrow x^\mu + \xi^\mu, \ g_{\mu\nu} \rightarrow g_{\mu\nu} - \nabla (\mu \xi_\nu) \] preserves the gauge condition \( h_{\mu u} = 0 \) implies imposing G. Policastro et al [2002]:

\[ \partial (\mu \xi_u) - 2 \Gamma^\rho_{\mu u} \xi_\rho = 0, \]

wherein \( \Gamma^\rho_{\mu u} \) is calculated w.r.t. \( g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu} \).

Three gauge transformations that preserve \( h_{\mu u} = 0 \), for the black M3-brane metric having integrated out the \( M_6 \) in the (asymptotic) \( AdS_5 \times M_6 \) in the MQGP limit.
Set I: The Gauge transformations are generated by

\[ \xi_{x_1} = \frac{C_{x_1}(t, x_1)}{u^2} + \xi_{x_1}^{(1)}(u, t, x_1); \quad \xi_t = \xi_t^{(1)}(u, t, x_1) \]

Set II: The Gauge transformations are generated by

\[ \xi_t = -\frac{g_1 C_t(t, x_1)}{u^2} + \xi_t^{(1)}(u, t, x_1); \quad \xi_{x_1} = \xi_{x_1}^{(1)}(u, t, x_1) \]

Set III: Writing \( \xi_u^{(0)} = \frac{C_u(t, x_1)}{u \sqrt{g}}, \xi_t^{(0)} = -\partial_t C_u(t, x_1) \psi(u), \xi_{x_1}^{(0)} = -\partial_{x_1} C_u(t, x_1) \chi(u) \), and demanding the solutions to be well behave at \( u = 0 \), one obtains:

\[ \xi_u^{(0)} = \frac{C_u(t, x_1)}{u \sqrt{g}}; \quad \xi_t^{(0)} = -\left( \frac{1}{2} - \frac{u^4}{3} \right) \sqrt{g} \partial_t C_u(t, x_1); \]

\[ \xi_{x_1}^{(0)} = -\partial_{x_1} C_u(t, x_1) \frac{F(\sin^{-1} u | 1)}{u} \]

\[ = -\partial_{x_1} C_u(t, x_1) \left( 1 + \frac{u^4}{10} + O(u^8) \right). \]

- Choosing \( C_u, \tilde{C}_{x_1}, u : \left( C_u, \frac{\tilde{C}_{t, x_1}}{i} \right) \frac{g_s^{\frac{2}{3}}}{L^2} = 1 \), the non-zero solutions gauge equivalent to
\( H_{ab} = 0 (H_{ab} = 0, \xi_a = 0), \) near \( u = 0, \) are given by:

\[
\begin{align*}
H^{(I)}_{xx}(0) &= -2q_3, \\
H^{(III)}_{xx}(0) &= 2; \\
H^{(II)}_{tt}(0) &= 2w_3; \\
H^{(I)}_{xt}(0) &= w_3, \\
H^{(II)}_{xt}(0) &= q_3.
\end{align*}
\]
\[ H_{ab}(u) = aH_{ab}^{(i)}(u) + bH_{ab}^{(ii)}(u) + cH_{ab}^{(iii)}(u) + dH_{ab}^{\text{inc}}(u), \]

to determine the \( H_{ab}^{\text{inc}}(u) \), the following are the relevant EOMs of the scalar modes of the metric perturbations:

\[ H''_{tt}(u) + (1/u) \left( -\frac{6}{g(u)} + 5 \right) H'_{tt}(u) + H''_{s}(u) + (1/u) \left( -\frac{2}{g(u)} + 1 \right) H'_{s}(u) = 0, \]

\[ H''_{tt}(u) + (2/u) \left( -\frac{3}{g(u)} + 1 \right) H'_{tt}(u) + (1/u) \left( -\frac{2}{g(u)} + 1 \right) H'_{s}(u) - \frac{q_{3}^{2}}{g(u)} H_{tt}(u) \]
\[ + \frac{w_{3}^{2}}{g^{2}(u)} H_{s}(u) + 2 \frac{q_{3} w_{3}}{g^{2}(u)} H_{xt}(u) = 0, \]

\[ H''_{s}(u) - \frac{2}{u} H'_{tt}(u) - \frac{1}{u} \left( 1 + \frac{4}{g(u)} \right) H'_{s}(u) - \frac{q_{3}^{2}}{g(u)} H_{tt}(u) + \frac{w_{3}^{2}}{g(u)} H_{s}(u) \]
\[ - \frac{4q_{3}^{2}}{(g(u))} H_{yy}(u) + \frac{2w_{3} q_{3}}{g^{2}(u)} H_{xt}(u) = 0, \]

\[ H''_{yy}(u) - \frac{H'_{tt}(u)}{u} - \frac{H'_{s}(u)}{u} + \frac{1}{u} \left( -\frac{4}{g(u)} + 1 \right) H_{yy}(u) + \frac{1}{g^{2}(u)} \left( w_{3}^{2} - g(u) q_{3}^{2} \right) H_{yy}(u) = 0, \]

\[ H''_{xt}(u) - \frac{3}{u} H'_{xt}(u) + \frac{2q_{3} w_{3}}{g(u)} H_{yy}(u) = 0, \]

\[ q_{3} \left( -g(u) H'_{tt}(u) + 2u^{3} H_{tt}(u) \right) - 2q_{3} g(u) H'_{yy}(u) + w_{3} H'_{xt}(u) = 0, \]

\[ w_{3} \left( g(u) H'_{s}(u) + 2w_{3} u^{3} H_{s}(u) \right) + q_{3} \left( g(u) H'_{xt}(u) + 4u^{3} H_{xt}(u) \right) = 0. \]
Solving near the horizon $u = 1$, one can see that the same is a regular singular point with exponent of the indicial equation corresponding to the incoming-wave solution given by $-\frac{iw_3}{4}$, implying that $H_{ab}(u) = (1 - u)^{-\frac{iw_3}{4}} H_{ab}(u)$. Make a double perturbative ansatze:

$$H_{ab}(u) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \mathcal{H}_{ab}^{(m,n)}(u) q_3^m w_3^n.$$
Upon using $H_{tt}(0) = H_{t}^{(0)}$, $H_{xt}(0) = H_{xt}^{(0)}$, $H_{s}(0) = H_{s}^{(0)}$ and solving for $a, b, c$ and $d$, the following is the common denominator:

$$
\Omega(w_3, q_3) \equiv \alpha_{yy}^{(0,0)} + \alpha_{yy}^{(1,0)} q_3 + C_{1yy}^{(1,0)} q_3^2 + \alpha_{yy}^{(1,0)} w_3 + (-i/4 + C_{2yy}^{(1,1)} - (2C_{1yy}^{(1,1)} e^3)/9) q_3 w_3
$$

$$
+ \left( C_{1yy}^{(0,2)} + C_{2yy}^{(0,2)} + \frac{\Sigma_{2yy}^{(0,1)}}{4} \right) w_3^2.
$$

Now, one can solve for $w_3$ and the solution is given by:

$$
w_3 = - \frac{2 \left( \alpha_{yy}^{(1,0)} \pm \sqrt{\alpha_{yy}^{(1,0)^2} - \alpha_{yy}^{(0,0)} (4C_{1yy}^{(0,2)} + 4C_{2yy}^{(0,2)} + i\Sigma_{2yy}^{(0,1)})} \right)}{4C_{1yy}^{(0,2)} + 4C_{2yy}^{(0,2)} + i\Sigma_{2yy}^{(0,1)}}
$$

$$
q_3 \left( \pm \frac{\alpha_{yy}^{(1,0)} \left( 72C_{1yy}^{(0,2)} + 8e^3 C_{1yy}^{(1,1)} + 72C_{2yy}^{(0,2)} - 36C_{2yy}^{(1,1)} + 18i\Sigma_{2yy}^{(0,1)} + 9i \right)}{\sqrt{\alpha_{yy}^{(1,0)^2} - \alpha_{yy}^{(0,0)} (4C_{1yy}^{(0,2)} + 4C_{2yy}^{(0,2)} + i\Sigma_{2yy}^{(0,1)})}} + 8e^3 C_{1yy}^{(1,1)} - 36C_{2yy}^{(1,1)} + 9i \right) + \mathcal{O}(q_3^2)
$$

$$
+ \mathcal{O}(q_3^2)
$$
Assuming the constants of integration in solutions of $H_{xx,tt,s}$ are fine tuned such that:

$$\alpha_{yy}^{(0,0)} << 1, |\Sigma_{2,yy}^{(0,1)}| >> 1 (i \Sigma_{2,yy}^{(0,1)} \in \mathbb{R}) : \alpha_{yy}^{(0,0)} \Sigma_{2,yy}^{(0,1)} < 1; \alpha_{yy}^{(1,0)} = -|\alpha_{yy}^{(1,0)}|,$$

$$w_3 \approx \pm q_3 \left(1 + i \frac{\alpha_{yy}^{(0,0)} \Sigma_{2,yy}^{(0,1)}}{\Sigma_{2,yy}^{(1,0)} (\alpha_{yy}^{(1,0)})^2} \right) \equiv \pm v_s q_3.$$ 

Experience with non-conformal backgrounds dictates that:

$$i \alpha_{yy}^{(0,0)} \Sigma_{2,yy}^{(0,1)} < 0; \frac{\alpha_{yy}^{(0,0)} i \Sigma_{2,yy}^{(0,1)}}{(\alpha_{yy}^{(1,0)})^2} < -\frac{2}{3}.$$
To put the previous result on a ‘sound’ footing, we will now look at the evaluation of the two-point correlation function $\langle T_{00} T_{00} \rangle$ from the on-shell action having dimensionally reduced $M$ theory on $M_5 \times M_6$ in the MQGP limit to $M_5$, which asymptotically is $AdS_5$.

On-shellness dictates that: $R^{(0)} = \frac{10}{3} \Lambda$ under metric perturbation $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$.

Including the boundary Gibbons-Hawking-York surface term: $\int_{\partial M_5:u=0} \sqrt{-g} \partial M_5 K$ ($K$ being the extrinsic scalar curvature) $= -\sqrt{G^{uu}} \frac{\partial}{\partial u} \int \prod_{i=0}^3 dx_i \sqrt{-g}|_{u=0}$ H. Liu, A. Tseytlin [1998].

The complete $D = 5$ action is:

$$S_{D=5} \sim \int_0^1 du \int d^4 x \sqrt{-g} (R - 2\Lambda) + a \int_{u=\text{constant}} d^4 x \sqrt{-g_{u=\text{constant}}},$$

where $a$, the perturbation-independent constant divergence-cancelling term, turns out to be $-\frac{6g_3^1}{L}$. The on-shell action, in the $q_3 \rightarrow 0$, $w_3 \rightarrow 0$ limit of the action, reduces to:

$$S = \frac{r_h^4}{(g^7/4 N^3/4)} \lim_{\epsilon \rightarrow 0} \left\{ \int d^4 x \frac{1}{80} \left[ 51H_s^2 + 40H_s H_{tt} + 90H_{tt}^2 - 10H_s H_{xx} + 15H_{xx}^2 \right]|_{u=\epsilon} \right\} - \int d^4 x \left[ \frac{1}{4u^3} \left( -\frac{3}{4} H_s^2 - H_s H_{tt} - H_{tt}^2 + \frac{H_s H_{xx}}{2} - \frac{3}{4} H_{xx}^2 \right)' - \frac{1}{3u^3} \left( H_{xt}^2 \right)' \right]|_{u=\epsilon}$$
The equations of motion imply that \( H''_{tt}(u = 0) = -H'_{s}(u = 0) \), from where we will assume that \( H_{t0} = -H_{s0} \). So, the relevant two-point correlation function involving \( T_{00} \) will require finding out the coefficient of \( H_{t0}^2 \). As the generic form of this two-point function in the hydrodynamical limit - P. Benincasa [2005]: \( w_3 \rightarrow 0, q_3 \rightarrow 0 : \frac{w_3}{q_3} = \alpha \equiv \text{constant} - \) is expected to be of the form: \( \frac{q_3^2}{w_3^2 - v_s^2 q_3^2} \), we isolate these terms and work up to leading order in \( \Sigma_{2yy}^{(0,1)} \).
We find:

\[ H^2 \text{ terms : } \frac{3i \left(140\alpha^6 - 20\pi\alpha^4 - 200\alpha^4 + 17\pi^2\alpha^2 - 116\pi\alpha^2 + 332\alpha^2\right)}{1280(\alpha^2 - 1)^2(w_3^2 - v_s^2q_3^2)} q_3^2\Sigma_{2yy}^0; \]

\[
\left. \frac{(HH')^O(u^0)}{u^3} \right|_{u=\epsilon} = \frac{1}{\epsilon} \frac{3i\alpha^2 \left(-5\alpha^2 + \pi + 1\right)}{32(\alpha^2 - 1)(w_3^2 - v_s^2q_3^2)} q_3^2\Sigma_{2yy}^0; \]

\[
\left. \frac{(HH')^O(u)}{u^3} \right|_{u=\epsilon} = i\alpha^2 \times \]

\[
\left. \frac{(HH')^O(u^2)}{u^3} \right|_{u=\epsilon} = \frac{1}{\epsilon} \frac{i\alpha^2 \left(36\alpha^2 + 3\pi\alpha - 12\alpha + \pi - 40\right)}{64(\alpha^2 - 1)(w_3^2 - v_s^2q_3^2)} q_3^2\Sigma_{2yy}^0; \]

\[
\left. \frac{(HH')^O(u^3)}{u^3} \right|_{u=\epsilon} = \]

\[
i\alpha^2 \left(16\alpha^4 + 8\pi\alpha^3 - 32\alpha^3 + 3\pi^2\alpha^2 + 20\pi\alpha^2 - 160\alpha^2 - 8\pi\alpha + 32\alpha - 8\pi^2 + 20\pi + 64\right) q_3^2\Sigma_{2yy}^0 \]

\[ 256(\alpha^2 - 1)^2(w_3^2 - v_s^2q_3^2) \]

One can show that for \(\alpha \sim -0.9\) the second, third and fourth lines vanish and one obtains the required correlation function of the type \(\frac{q_3^2}{w_3^2 - v_s^2q_3^2}\) with the same \(v_s\).
SU(3) Structure Torsion Classes of Resolved Warped Deformed Conifold $M_{RWDC}$

Type IIB: black D3s, D5s wrapped around 2-cycle, D7s wrapped around 4-cycle via OKS embedding

SYZ mirror ['delocalized' limit] along local $T^3(x,y,z)$ in the MQGP limit

Type IIA: Black D6s wrapped around three cycles; torsion classes $\Rightarrow$ not complex but after a fine tuning, approx. so

$U(1) \in U(N_f)$

Therm. stability $\left. \frac{\partial^2 V}{\partial N_f^2} \right|_{N_f} > 0$; $T_c$ w/lattice and 1st gen q mass scale

Black M3-Branes $(x^0, x^{1,2,3}) \equiv$ Black M5 branes wrapped around a two-cycle $\equiv \sum_{i=1}^{4} n_i L_i^2$ in AdS5 x M9 satisfying D=11 SUGRA EOMs; after fine tuning, $G_2$ structure, torsion classes $\Rightarrow G_2$ structure $\Rightarrow G_2$ holonomy in large-N limit

$U(1) \in U(N_f)$ gauge field fluctuations

R-Charge Correlators via metric perturbations $h_{m(angular)}(eAdS_5) = A_{\mu}$

Metric Perturbations $h_{\mu\nu eR^{1,3}}$

$\sigma \sim T, \chi \sim T^2, D \sim \frac{1}{T}$

$D_R$ from $G_{\alpha\alpha}^\text{ret} \sim i \omega + 2 D_R q^2$

Vector Modes: $h_{x^1 x^2, t x^3} \neq 0$ or

$\eta \sim T^2 (s \sim T^3)$,

$\frac{\eta}{s} = \frac{1}{4\pi}$ possible

Scalar Modes: $h_{tt}, h_{x^1 t}, h_{x^2}$,

$\nu_s < \frac{1}{\sqrt{3}}$

Flow Chart of the Various Portions/Results in MQGP Limit
In the large-$r$ limit, the $D = 11$ space-time is a warped product of $\text{AdS}_5(\mathbb{R}^{1,3} \times \mathbb{R}_{>0})$ and $\mathcal{M}_6(\theta_{1,2}, \phi_{1,2}, \psi, x_{10})$

\[
\begin{align*}
\mathcal{M}_6(\theta_{1,2}, \phi_{1,2}, \psi, x_{10}) &\leftarrow S^1(x_{10}) \\
\downarrow & \\
\mathcal{M}_3(\phi_1, \phi_2, \psi) &\longrightarrow \mathcal{M}_5(\theta_{1,2}, \phi_{1,2}, \psi) \\
\downarrow & \\
\mathcal{B}_2(\theta_1, \theta_2) &\leftarrow [0, 1]_{\theta_1} \\
\downarrow & \\
[0, 1]_{\theta_2}
\end{align*}
\]
Thank you.