# Universal features of Lifshitz Green's functions from holography

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C. Keeler, G.K., J.T. Liu, and K. Sun; arXiv:1505.07830.
C. Keeler, G.K., and J.T. Liu; arXiv:1404.4877.

# Motivation

Challenges of AdS/CMT:

- Holographic dictionary less clear than in relativistic case.
- Holographic models often describe an idealized theory.

$\rightarrow$ Universal predictions of nonrelativistic gauge/gravity duality (e.g. analog of $\eta/s$)?

- Main focus: Holographic Green's functions.
  1. What do we know?
  2. Which features are robust w.r.t. deformations of the theory?
  3. Comparison with field theory

# Constraining holographic observables via symmetries

- Type IIB on $AdS_5 \times S^5 \Leftrightarrow \mathcal{N} = 4$ SYM.
  Consider retarded Green's function of scalar operator $\mathcal{O}_\Delta$:

$$G_R(\omega, \vec{k}) = -i \int d^{d+2}x \, e^{-ik\cdot x} \theta(t) \langle [\mathcal{O}_\Delta(x), \mathcal{O}_\Delta(0)] \rangle$$

  Superconformal symmetry $\Longrightarrow$ $G_R(q^2) = A(-q^2)^{\Delta-2}$,

$$q^2 = \omega^2 - |\vec{k}|^2, \quad A = \mathrm{const.}$$

- What about non-relativistic symmetries?
  - Lifshitz scaling

# Lifshitz Green's functions

Lifshitz scaling symmetry:

$$\vec{x} \to \Lambda^{\frac{1}{z}} \vec{x}, \qquad t \to \Lambda t$$

$z > 1$: dynamical critical exponent

Assuming rotational invariance:

$$G_R(\omega, \vec{k}) = |\vec{k}|^{2\nu z} \mathcal{G}\left(\hat{\omega}\right), \qquad \hat{\omega} = \frac{\omega}{|\vec{k}|^z}$$

Properties of Green's functions in Lifshitz field theories without additional symmetries?

# Lifshitz Green's functions - general properties

$$G_R(\omega, \vec{k}) = |\vec{k}|^{2\nu z} \mathcal{G}(\hat{\omega})$$

Generic features:

1. $\mathcal{G}$ is analytic in the upper half plane (causality)
2. Scaling behavior:
   2.1 $\mathcal{G}(\hat{\omega} \to 0) \sim \mathrm{const.}$ (so that $G_R \sim |\vec{k}|^{2\nu z}$ as $\omega \to 0$)
   2.2 $\mathcal{G}(\hat{\omega} \to \infty) \sim \hat{\omega}^{2\nu}$ (so that $G_R \sim \omega^{2\nu}$ as $|\vec{k}| \to 0$)

No (obvious) further constraints.

"Lifshitz field theory": Family of theories with different dynamics, and thus $\mathcal{G}(\hat{\omega})$.

# Holographic Lifshitz Green's function

However: On the gravity side, there is a "canonical" result.

Lifshitz spacetime:

$$ds^2_{d+2} = \frac{-dt^2 + d\rho^2}{\rho^2} + \frac{d\vec{x}^2}{\rho^{2/z}}$$

$$\text{isometry: } \vec{x} \to \Lambda^{\frac{1}{z}} \vec{x}, \qquad t \to \Lambda t, \qquad \rho \to \Lambda \rho.$$

For $z = 2$:

$$\mathcal{G}(\hat{\omega}) = K_0 \hat{\omega}^{2\nu} \frac{\Gamma\left(\frac{1}{2} + \nu + \frac{i}{4\hat{\omega}}\right)}{\Gamma\left(\frac{1}{2} - \nu + \frac{i}{4\hat{\omega}}\right)}$$

[Kachru, Liu, Mulligan; 0808.1725]

Where is the freedom on the gravity side?

## Review: The two-derivative holographic Green's function

Consider a probe scalar in Lifshitz spacetime:

$$(\Box - m^2)\phi = 0$$

Ansatz: $\phi = e^{i(\vec{k}\cdot\vec{x}-\omega t)}\rho^{d/2z}\psi(\rho) \Longrightarrow$ Schrödinger-like equation:

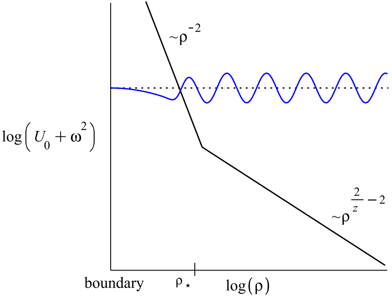
$$-\psi''(\rho) + U_0\psi(\rho) = 0$$

$$U_0 = \frac{\nu^2 - \frac{1}{4}}{\rho^2} + \frac{|\vec{k}|}{\rho^{2-2/z}} - \omega^2, \qquad \nu^2 = m^2 + \left(\frac{d+z}{2z}\right)^2$$

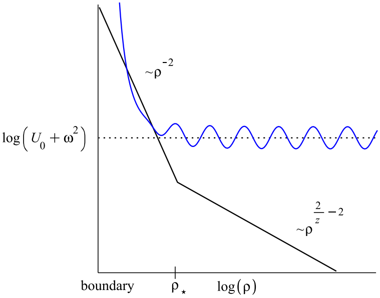
Using scale-invariant coordinates $\hat{\rho} = \rho|\vec{k}|^z$:

$$-\psi''(\hat{\rho}) + \hat{U}_0(\hat{\rho})\psi(\hat{\rho}) = 0, \qquad \hat{U}_0(\hat{\rho}) = \frac{\nu^2 - \frac{1}{4}}{\hat{\rho}^2} + \frac{1}{\hat{\rho}^{2-2/z}} - \hat{\omega}^2$$

normalizable mode:




non-normalizable mode:




Near the horizon:

$$\psi(\hat{\rho} \to \infty) \sim a e^{i\hat{\omega}\hat{\rho}} + b e^{-i\hat{\omega}\hat{\rho}}$$

Near the boundary:

$$\psi(\hat{\rho} \to 0) \sim A\hat{\rho}^{\frac{1}{2}-\nu} + B\hat{\rho}^{\frac{1}{2}+\nu}$$

# Holographic Green's functions

$$\psi(\hat{\rho} \to \infty) \sim a e^{i\hat{\omega}\hat{\rho}} + b e^{-i\hat{\omega}\hat{\rho}}$$

$$\psi(\hat{\rho} \to 0) \sim A\hat{\rho}^{\frac{1}{2}-\nu} + B\hat{\rho}^{\frac{1}{2}+\nu}$$

Choose infalling boundary conditions for retarded Green's function:

$$\mathcal{G}(\hat{\omega}) = \hat{\omega}^{2\nu} \frac{B}{A}\bigg|_{b=0}$$

Find relation between $\{A, B\}$ and $\{a, b\}$ by solving a quantum mechanical scattering problem.

## Introducing higher derivatives

Where does the freedom of finding different $\mathcal{G}$ arise in the gravity theory?

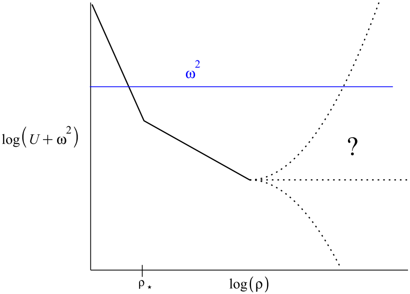
1. Choice of background metric?
2. Dynamics of probe scalar, e.g. higher derivative corrections:
   - $\left(\frac{\partial}{\partial t}\right)^i \mathrel{\widehat{=}} \omega^i, \qquad |\vec{\nabla}|^j \mathrel{\widehat{=}} |\vec{k}|^j$:
     modified Schrödinger potential
     $$\hat{U} = \hat{U}_0(\hat{\rho}) + \frac{1}{\hat{\rho}^2} \sum_{i+j>2} \lambda_{i,j} \hat{\omega}^i \hat{\rho}^{i+j/z}, \qquad \hat{\rho} = \rho|\vec{k}|^z,\ \hat{\omega} = \frac{\omega}{|\vec{k}|^z}$$
     $$= \hat{U}_0(\hat{\rho}) + \frac{1}{\hat{\rho}^2} \sum_{i+j>2} \lambda_{i,j} (\omega\rho)^i \left(|\vec{k}|\rho^{1/z}\right)^j$$
     where $\lambda_{i,j} \sim \left(\frac{\ell}{L}\right)^{i+j-2}$, $\ell \ll L$ (curvature scale).
   - $\left(\frac{\partial}{\partial \rho}\right)^n$:
     $$e.g.: \quad -\psi'' + \hat{U}\psi = \lambda\psi^{(4)}$$
     $$\Rightarrow -\psi'' + U\psi - \lambda(\hat{U}\psi)'' = \mathcal{O}(\lambda^2)$$
     $$\Rightarrow -\tilde{\psi}'' + \tilde{U}\tilde{\psi} \approx 0$$

Higher derivative corrections can be captured by modifying the effective potential.

# Keeping higher derivatives under control

Consider only a single higher derivative term (i temporal, j spatial derivatives):

$$\hat{U}(\hat{\rho}) = \hat{U}_0(\hat{\rho}) + \lambda_{i,j}\hat{\omega}^i \hat{\rho}^{i+j/z-2}$$




Want to compute Green's function perturbatively, i.e.

$$\mathcal{G} = \mathcal{G}_0 + \delta\mathcal{G}.$$

Problem: higher derivatives generically blow up near horizon ($\hat{\rho} \to \infty$).

But: Green's function is given by $B/A$ (near-boundary behavior of $\psi$) and thus only contains information about the effective potential in the tunneling regime.

# WKB approximation

Recall:

$$\mathcal{G}(\hat{\omega}) = \hat{\omega}^{2\nu} \frac{B}{A}\bigg|_{b=0} \qquad \psi(\hat{\rho} \to 0) \sim A\hat{\rho}^{\frac{1}{2}-\nu} + B\hat{\rho}^{\frac{1}{2}+\nu}$$

Infalling boundary conditions can be imposed order by order in $\lambda_{i,j}$.

WKB approximation :

$$\psi_{WKB}(\hat{\rho}) = \begin{cases} \sqrt{\nu}\hat{U}^{-\frac{1}{4}} \left( C_1(a)e^{S(\hat{\rho},\hat{\rho}_0)} + C_2(a)e^{-S(\hat{\rho},\hat{\rho}_0)} \right) & , \hat{\rho} < \hat{\rho}_0 \\ a(-\hat{U})^{-\frac{1}{4}} e^{i\Phi(\hat{\rho},\hat{\rho}_0)} & , \hat{\rho} > \hat{\rho}_0 \end{cases}$$

$$S(\hat{\rho}, \hat{\rho}_0) = \int_{\hat{\rho}}^{\hat{\rho}_0} d\hat{\rho}\sqrt{\hat{U}}, \qquad \Phi(\hat{\rho}, \hat{\rho}_0) = \int_{\hat{\rho}_0}^{\hat{\rho}} d\hat{\rho}\sqrt{-\hat{U}}$$

$\hat{\rho}_0$ : classical turning point.

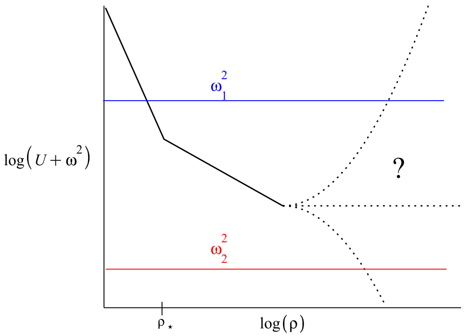
$$\psi_{WKB}(\hat\rho) = \begin{cases} \sqrt{\nu}\hat U^{-\frac{1}{4}}\left(C_1(a)e^{S(\hat\rho,\hat{\rho_0})} + C_2(a)e^{-S(\hat\rho,\hat{\rho_0})}\right) & ,\hat\rho < \hat\rho_0 \\ a(-\hat U)^{-\frac{1}{4}}e^{i\Phi(\hat\rho,\hat{\rho_0})} & ,\hat\rho > \hat\rho_0 \end{cases}$$

Coefficients of non-normalizable/normalizable mode:

$$A = \lim_{\epsilon\to 0} C_1\epsilon^{\nu}e^{S(\epsilon,\rho_0)}, \qquad B = \lim_{\epsilon\to 0} C_2\epsilon^{-\nu}e^{-S(\epsilon,\rho_0)}$$

$$\begin{aligned} S(\epsilon,\hat\rho_0) &= \int_\epsilon^{\hat\rho_0} d\hat\rho\sqrt{\hat U} \\ &= \int_\epsilon^{\hat\rho_0} d\hat\rho\sqrt{\frac{\nu^2}{\hat\rho^2} + \frac{1}{\hat\rho^{2-2/z}} - \hat\omega^2 + \lambda_{\mathrm{i,j}}\hat\omega^{\mathrm{i}}\hat\rho^{\mathrm{i+j/z-2}}} \end{aligned}$$

- Coefficients $A, B$ contain information about $\hat U$ up to the classical turning point $\hat\rho_0$.
  - Can calculate Green's function with higher derivatives, as long as h.d. are subdominant compared to two-derivative terms.




$$S = \int_\epsilon^{\hat\rho_0} d\hat\rho \sqrt{\frac{\nu^2}{\hat\rho^2} + \frac{1}{\hat\rho^{2-2/z}} - \hat\omega^2 + \lambda_{\mathrm{i,j}}\hat\omega^{\mathrm{i}}\hat\rho^{\mathrm{i+j/z}-2}}$$

$$\stackrel{!}{=} S_0 + \delta S + O(\lambda_{i,j}^2)$$

Will have to impose upper bound on $\lambda_{i,j}$ **and** lower cutoff for $\hat\omega$.

# Calculating the spectral function

Calculate $\mathrm{Im}\mathcal{G}$ (spectral function) via

$$\mathrm{Im}\mathcal{G}(\hat{\omega}) = \hat{\omega}^{2\nu}\mathrm{Im}\left(\frac{B}{A}\right) = K_0\hat{\omega}^{2\nu}\lim_{\epsilon\to 0}\epsilon^{-2\nu}e^{-2S(\epsilon,\hat{\rho}_0)}$$

(can find $\mathrm{Re}\mathcal{G}$ via Kramers-Kronig relation)

Can expand

$$\mathrm{Im}\mathcal{G} = \mathrm{Im}\mathcal{G}_0 + \delta\mathrm{Im}\mathcal{G} + \mathcal{O}(\lambda^2)$$

by expanding integral

$$S(\hat{\rho},\hat{\rho}_0) = \int_{\hat{\rho}}^{\hat{\rho}_0} d\hat{\rho}\sqrt{\hat{U}_0 + \lambda_{i,j}\hat{\omega}^i\hat{\rho}^{i+j/z-2}}$$

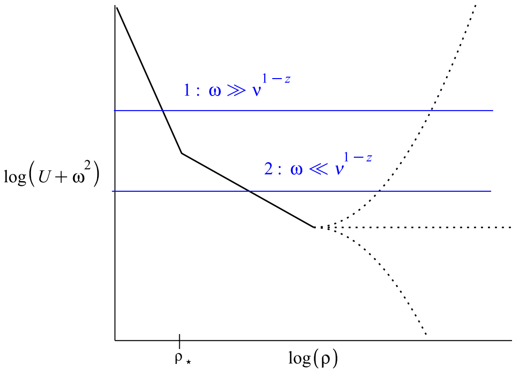
linearly in $\lambda_{i,j}$.

# Results

Recall:

$$\hat{U}(\hat{\rho}) = \frac{\nu^2 - \frac{1}{4}}{\hat{\rho}^2} + \frac{1}{\hat{\rho}^{2-2/z}} - \hat{\omega}^2 + \lambda_{i,j}\hat{\omega}^i\hat{\rho}^{i+j/z-2}$$

Two interesting limits:



# High frequency limit - two derivative result

1. $\hat{\omega} \gg \nu^{1-z}$:

Classical turning point lies at $\hat{\rho}_0 \sim \frac{\nu}{\hat{\omega}}$.

To leading order:

$$S_0 = \int_{\epsilon}^{\hat{\rho}_0} d\hat{\rho}\sqrt{\frac{\nu^2}{\hat{\rho}^2} + \frac{1}{\hat{\rho}^{2-2/z}} - \hat{\omega}^2} \approx \int_{\epsilon}^{\hat{\rho}_0} d\hat{\rho}\sqrt{\frac{\nu^2}{\hat{\rho}^2} - \hat{\omega}^2} \sim -\nu \log(\epsilon) + \text{const}.$$

Gives rise to power law scaling near the boundary:

$$\psi(\epsilon \ll 1) \approx \epsilon^{\frac{1}{2}} \left( C_1 e^{S_0(\epsilon,\hat{\rho}_0)} + C_2 e^{-S_0(\epsilon,\hat{\rho}_0)} \right) \\ \sim \tilde{C}_1 \epsilon^{\frac{1}{2}-\nu} + \tilde{C}_2 \epsilon^{\frac{1}{2}+\nu}$$

2-derivative Green's function:

$$\text{Im}\mathcal{G}(\hat{\omega}) = \hat{\omega}^{2\nu} \text{Im}\left(\frac{B}{A}\right) = \text{Im}\left(\frac{\tilde{C}_2}{\tilde{C}_1}\right)\hat{\omega}^{2\nu}$$

# High frequency limit with higher derivatives

Computing $S$ to linear order in $\lambda$ yields:

$$\mathrm{Im}\mathcal{G}(\hat{\omega}) \approx C\hat{\omega}^{2\nu}, \qquad C = (2\nu)^{-2\nu} \exp\left[2\nu(1 - \delta_{j,0} c_i \lambda_{i,0} \nu^{i-2} + \ldots)\right]$$

- Expected large frequency behavior $G_R = |\vec{k}|^{2\nu z} \mathcal{G}(\hat{\omega}) \sim \omega^{2\nu}$ ("featureless"; similar to AdS)
  - Higher derivatives normalize prefactor. Spatial derivatives derivatives ($j \neq 0$) are subleading in the $\hat{\omega} \to \infty$ limit.
  - H.d. corrections become of order one if

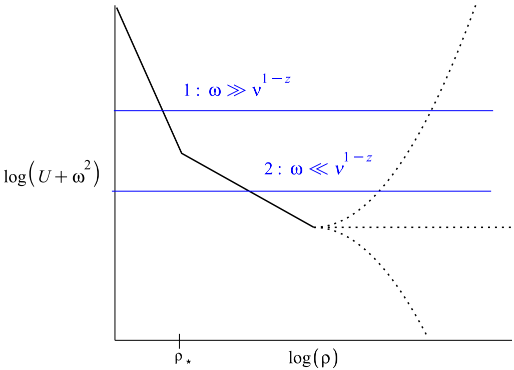
$$\lambda_{i,0}\nu^{i-2} \sim 1 \quad \Leftrightarrow \quad \delta U_{hd}(\hat{\rho}_0) = \lambda_{i,0} \hat{\omega}^i \hat{\rho}_0^{i-2} \sim \hat{\omega}^2$$

    - Size of corrections to spectral function is controlled by relative size of h.d. corrections to the potential up to the classical turning point.
    - Translates into condition on mass: $\lambda_{i,0}\nu^{i-2} \sim \left(\frac{\ell}{L}\nu\right)^{i-2} \sim (m\ell)^{i-2} \ll 1$

# Low frequency limit

Effective potential

$$\hat{U}(\hat{\rho}) = \frac{\nu^2 - \frac{1}{4}}{\hat{\rho}^2} + \frac{1}{\hat{\rho}^{2-2/z}} - \hat{\omega}^2 + \lambda_{i,j}\hat{\omega}^i \hat{\rho}^{i+j/z-2}$$



# Low frequency limit - two derivative result

2. $\hat{\omega} \ll \nu^{1-z}$:

Classical turning point lies at $\hat{\rho}_0 \sim \hat{\omega}^{-\frac{z}{z-1}}$.

To leading order:

$$S_0 = \int_{\epsilon}^{\hat{\rho}_0} d\hat{\rho} \sqrt{\frac{\nu^2}{\hat{\rho}^2} + \frac{1}{\hat{\rho}^{2-2/z}} - \hat{\omega}^2} \sim -\nu \log(\epsilon) + \int^{\hat{\omega}^{-\frac{z}{z-1}}} d\hat{\rho} \frac{1}{\hat{\rho}^{1-1/z}} + \text{const.}$$

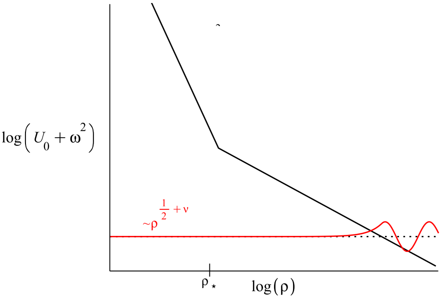
$$\sim -\nu \log(\epsilon) + \hat{\omega}^{-\frac{1}{z-1}} + \text{const.}$$

Wavefunction near the boundary:

$$\psi(\epsilon \ll 1) \approx \epsilon^{\frac{1}{2}} \left( C_1 e^{S_0(\epsilon,\hat{\rho}_0)} + C_2 e^{-S_0(\epsilon,\hat{\rho}_0)} \right)$$

$$\sim \tilde{C}_1 \epsilon^{\frac{1}{2}-\nu} \exp\left( \frac{E_0}{2} \hat{\omega}^{-\frac{1}{z-1}} \right) + \tilde{C}_2 \epsilon^{\frac{1}{2}+\nu} \exp\left( -\frac{E_0}{2} \hat{\omega}^{-\frac{1}{z-1}} \right)$$

e.g. normalizable mode:




$$\psi(\epsilon \ll 1) \sim \tilde{C}_1 \epsilon^{\frac{1}{2}-\nu} \exp\left(\frac{E_0}{2}\hat{\omega}^{-\frac{1}{z-1}}\right) + \tilde{C}_2 \epsilon^{\frac{1}{2}+\nu} \exp\left(-\frac{E_0}{2}\hat{\omega}^{-\frac{1}{z-1}}\right)$$

→ exponential growth/decay due to tunneling under $\sim \hat{\rho}^{2/z-2}$ barrier.

2-derivative Green's function:

$$\mathrm{Im}\mathcal{G}(\hat{\omega}) = D\hat{\omega}^{2\nu} \exp[-E_0\hat{\omega}^{-\frac{1}{z-1}}].$$

## Low frequency limit with higher derivatives

Computing $S$ to linear order in $\lambda$ yields:

$$\mathrm{Im}\mathcal{G}(\hat{\omega}) = D\hat{\omega}^{2\nu}\exp[-\hat{\omega}^{-\frac{1}{z-1}}(E_0 + \delta E(\hat{\omega}))]$$

$$D = (2\nu)^{-2z\nu} z^{2\nu(1-z)} \exp\left[2z\nu(1 + \delta_{i,0} d_j \lambda_{0,j}\nu^{j-2} + \ldots)\right], \qquad d_j = \int^{x=0} dx \frac{x^{\frac{j}{2}-1}}{\sqrt{1+x}}$$

$$E_0 = \frac{\sqrt{\pi}\Gamma\left(\frac{1}{2(z-1)}\right)}{z\Gamma\left(\frac{z}{2(z-1)}\right)}, \qquad \delta E(\hat{\omega}) = e_{i,j}\lambda_{i,j}\hat{\omega}^{-\frac{1}{z-1}(i+j-2)} + \ldots, \qquad e_{i,j} = \int_0^1 dx \frac{x^{i-1+\frac{j-1}{z}}}{\sqrt{1-x^{2-\frac{2}{z}}}}$$

- Prefactor is renormalized by terms $\sim \lambda_{0,j}\nu^{j-2}$.
- Exponent is modified by frequency-dependent terms. Recall:

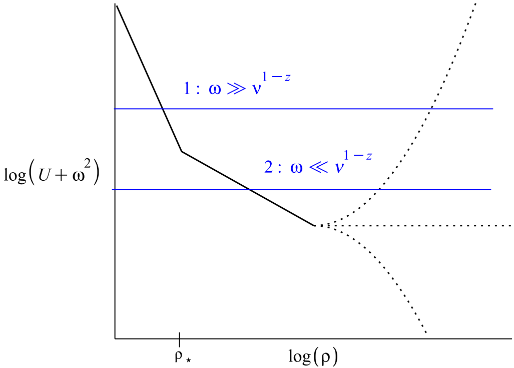
$$\hat{U}(\hat{\rho}) = \frac{\nu^2 - \frac{1}{4}}{\hat{\rho}^2} + \frac{1}{\hat{\rho}^{2-2/z}} - \hat{\omega}^2 + \lambda_{i,j}\hat{\omega}^i\hat{\rho}^{i+j/z-2}$$

Classical turning point lies at

$$\hat{\rho}_0 \approx \hat{\omega}^{-\frac{z}{z-1}}$$

H.d. corrections become of order one if

$$\lambda_{i,j}\hat{\omega}^{-\frac{1}{z-1}(i+j-2)} \sim 1 \quad \Leftrightarrow \quad \lambda_{i,j}\hat{\omega}^i\hat{\rho}_0^{i+j/z-2} \sim \hat{\omega}^2$$



1 : ω ≫ ν^{1−z}
2 : ω ≪ ν^{1−z}
log(U + ω²)
ρ⋆
log(ρ)

# Exponential suppression

At low frequencies:

$$\mathrm{Im}\mathcal{G}(\hat{\omega} \ll \nu^{1-z}) \approx D\hat{\omega}^{2\nu} \exp[-\hat{\omega}^{-\frac{1}{z-1}}(E_0 + \delta E(\hat{\omega}))]$$

$$\delta E(\hat{\omega}) = e_{i,j}\lambda_{i,j}\hat{\omega}^{-\frac{1}{z-1}(i+j-2)} + \ldots$$

Strict low-frequency limit $\hat{\omega} \to 0$ is non-universal (h.d. dominate), but corrections are under control if

$$\lambda_{i,j}\hat{\omega}^{-\frac{1}{z-1}(i+j-2)} \ll 1.$$

Using $\lambda_{i,j} \sim \left(\frac{\ell}{L}\right)^{i+j-2}$:

$$\nu^{1-z} \gg \hat{\omega} \gg \left(\frac{\ell}{L}\right)^{z-1}$$

- Recall: $\nu \ll \frac{L}{\ell} \Leftrightarrow m \ll \frac{1}{\ell}$ $\to$ exponential suppression is a robust feature in a wide window of frequencies.
- Corrections can be computed order-by-order for different models.

# Interpretation of exponential behavior

- Exponential behavior of $\mathrm{Im}\mathcal{G}$ is a robust result at low frequencies.
- Can we find the same behavior of $\mathrm{Im}\mathcal{G}$ on the field theory side?

# Example: Quadratic band crossing model

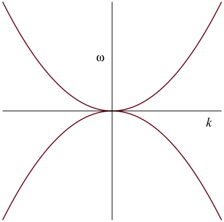
Scale-invariant theory with $z = 2$:

$$S = \int d^2x dt \left\{ \bar{\Psi} \left[ i\gamma_0 \partial_0 + \gamma_1 (\partial_x^2 - \partial_y^2) + 2\gamma_2 \partial_x \partial_y \right] \Psi - g \psi_1^\dagger \psi_2^\dagger \psi_2 \psi_1 \right\}$$

where $\Psi = (\psi_1, \psi_2)^T$.

Dispersion relation:

$$\epsilon_\pm(\vec{k}) = \pm |\vec{k}|^2$$




Can consider bosonic particle-hole operators

$$b_i = \bar{\Psi} \gamma_i \Psi, \qquad i = 0, 1, 2, 3$$

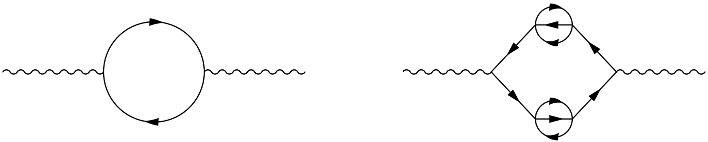
## Calculating self-energy corrections

Want to calculate spectral weight $\mathrm{Im}\, G_{(b)}$.

Optical theorem:

$$\mathrm{Im}\, G_{(b)}(\omega, \vec{k}) \neq 0 \qquad \Leftrightarrow \qquad \Gamma_{b_i \to \ldots}(\omega, \vec{k}) \neq 0$$

1-loop: 5-loop: etc.



At n-loop order, $O(g^{2n})$, $\mathrm{Im}\, G_{(b)}$ is related to decay rate for

$$b\{\omega, \vec{k}\} \to \mathrm{n\ particles}\{\omega_{p_i}, \vec{k}_{p_i}\} + \mathrm{n\ holes}\{\omega_{h_i}, \vec{k}_{h_i}\}.$$

Energy and momentum conservation:

$$\vec{k} = \sum_{i=1}^{n+1} \vec{k}_{p_i} - \sum_{i=1}^{n+1} \vec{k}_{h_i},$$

$$\omega = \sum_{i=1}^{n+1} \omega_{p_i} - \sum_{i=1}^{n+1} \omega_{h_i} = \sum_{i=1}^{n+1} (|\vec{k}_{p_i}|^2 + |\vec{k}_{h_i}|^2) \geq \frac{k^2}{2(n+1)},$$

At each fixed order, spectral weight is zero below a certain threshold.

$\rightarrow$ for fixed $\omega \ll \vec{k}^2$, spectral weight can only arise via process of $O(g^{2n})$, where $n \sim \frac{k^2}{2\omega}$

$$\Rightarrow \mathrm{Im}\, G_{(b)}(\omega \ll \vec{k}) \sim g^{2n} \sim g^{k^2/\omega}$$

For $g \ll 1$ this implies

$$\mathrm{Im}\, G_{(b)}(\omega \ll \vec{k}^2) \sim \exp(-\frac{\text{const.}}{\hat{\omega}}), \qquad \hat{\omega} = \frac{\omega}{|\vec{k}|^2}.$$

Result can be generalized to $z \neq 2$:

$$\mathrm{Im}\, G_{(b)}(\omega \ll \vec{k}^2) \sim g^{\hat{\omega}^{-\frac{1}{z-1}}} \sim \exp(-\mathrm{const.} \cdot \hat{\omega}^{-\frac{1}{z-1}}), \qquad \hat{\omega} = \frac{\omega}{|\vec{k}|^z}.$$

- Agrees with the holographic prediction $\mathrm{Im}\mathcal{G}(\hat{\omega}) \sim \exp[-E_0 \hat{\omega}^{-\frac{1}{z-1}}]$ .
- Exp. behavior is generic for Lifshitz theories with bosonic decay channels.
- Aside: In Dirac theory ($z = 1$), dispersion relation is $\omega = |\vec{k}|$, so energy conservation implies

$$\vec{k} = \sum_{i=1}^{n+1} \vec{k}_{p_i} - \sum_{i=1}^{n+1} \vec{k}_{h_i},$$

$$\omega = \sum_{i=1}^{n+1} \omega_{p_i} - \sum_{i=1}^{n+1} \omega_{h_i} = \sum_{i=1}^{n+1} (|\vec{k}_{p_i}| + |\vec{k}_{h_i}|) \geq |\vec{k}|,$$

$$\Longrightarrow \mathrm{Im}\, G_{(b)}(\omega < |\vec{k}|) = 0$$

# Summary

- Field theory: Family of theories with Lifshitz symmetry.
- Gravity: Family of theories with Lifshitz symmetry; dynamical information encoded in higher derivative couplings $\lambda_{i,j}$.
- Higher derivative corrections can be computed systematically:
  - $\mathrm{Im}\mathcal{G}(\hat{\omega} \gg \nu^{1-z}) \sim C\hat{\omega}^{2\nu}$;
    - $C$ has perturbative expansion in $\lambda\nu^{i-2} \sim (m\ell)^{i-2} \ll 1$.
  - $\mathrm{Im}\mathcal{G}(\hat{\omega} \ll \nu^{1-z}) \sim D\hat{\omega}^{2\nu}\mathrm{exp}[-\mathrm{E}\hat{\omega}^{-\frac{1}{z-1}}]$
    - $D$ has perturbative expansion in $\lambda\nu^{j-2}$.
    - $E$ has perturbative expansion in $\lambda_{i,j}\hat{\omega}^{-\frac{1}{z-1}(i+j-2)}$..
- "Naive" limit $\hat{\omega} \to 0$ is not under perturbative control, but results are robust after imposing the cutoff $\hat{\omega} \gg \left(\frac{\ell}{L}\right)^{z-1}$.
- Can confirm exponential behavior of spectral function in a broad class of field theories.

# Open questions

1. Constraints on sign of $\lambda_{i,j}$ from bulk causality/unitarity?
2. Compute corrections to $\mathrm{Im}\mathcal{G}(\hat{\omega})$ in string embeddings?
3. Measure spectral function experimentally?

Thank you!