Tales from the Edge:
Boundary Terms and Entanglement Entropy

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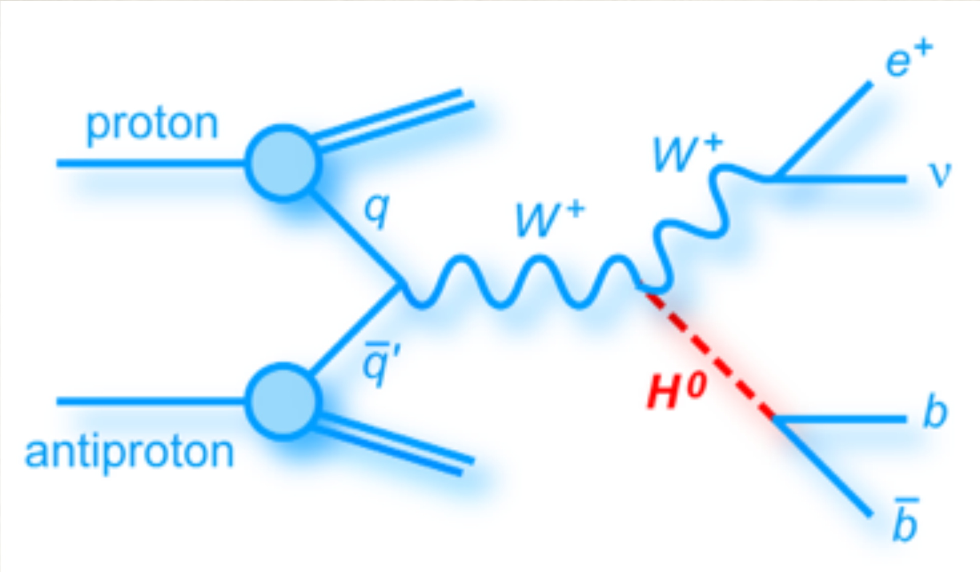
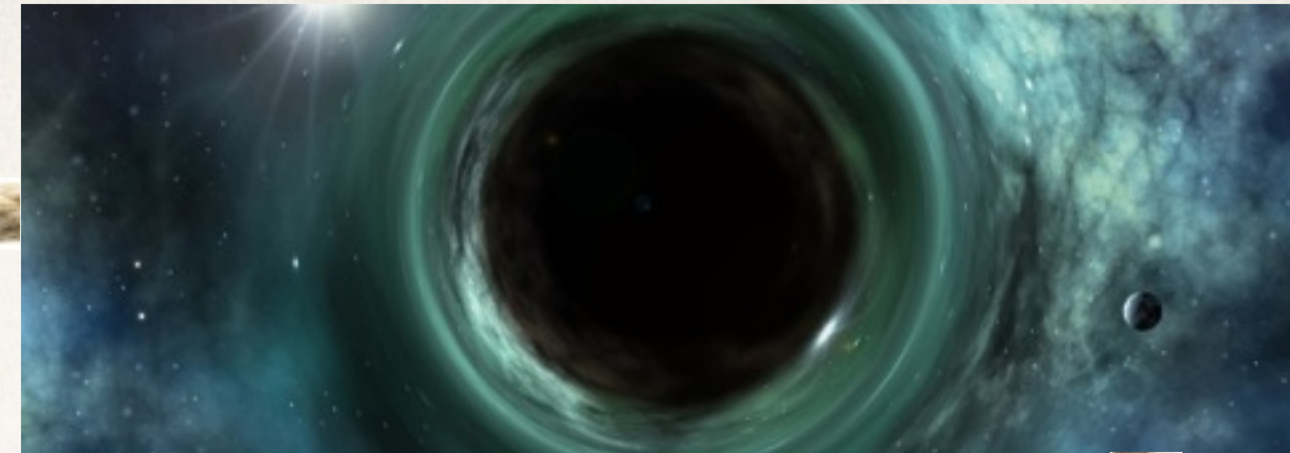
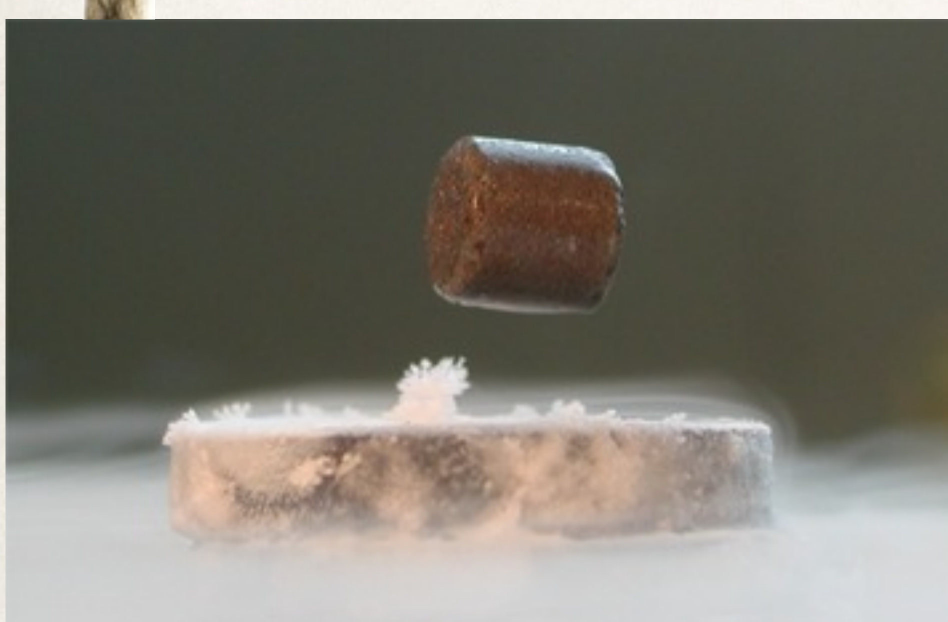
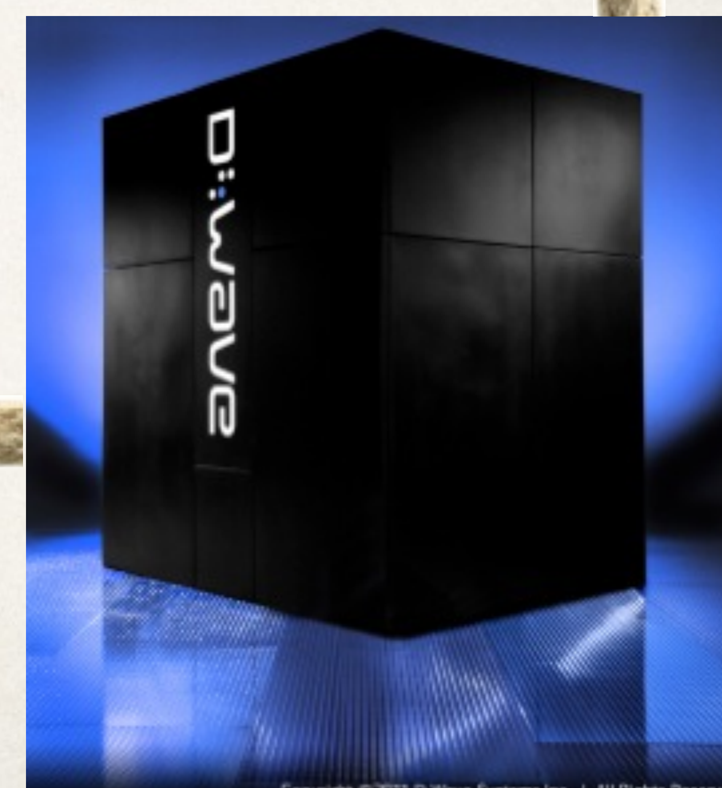




For a nonlocal, nonobservable, ultraviolet cut-off dependent quantity, entanglement entropy has become surprisingly important in theoretical physics today.



A Unifying Theme



Why is It Important?

- Quantum information, communication and computation — measure of entanglement in quantum systems
- Condensed matter physics — order parameter for exotic phase transitions (Osborne-Nielsen 2002, Vidal et al. 2003)
- Quantum field theory (QFT) — measure of renormalization group flow (a and c theorems) (Casini-Huerta 2006, 2012)
- Gravity — relations to black hole entropy (Bombelli et al. 1986, Srednicki 1993); Bekenstein bound (Casini 2008)
- String theory — Ryu-Takayanagi (2006) formula and AdS/CFT ties QFT and gravity aspects together.

Entanglement Entropy

- Consider a state $|\psi\rangle \in \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ in a factorizable Hilbert space. (*A* and *B* spatial.)

- Form density matrix: $\rho = |\psi\rangle\langle\psi|$

- Perform the partial trace: $\rho_A = \mathrm{tr}_B\, \rho$

- Compute the von Neumann entropy of ρ_A

$$S_E \equiv -\,\mathrm{tr}(\rho_A \log \rho_A)$$

For the EPR pair

$$\rho_A = \frac{1}{2}\left(|\downarrow\rangle\langle\downarrow| + |\uparrow\rangle\langle\uparrow|\right)$$

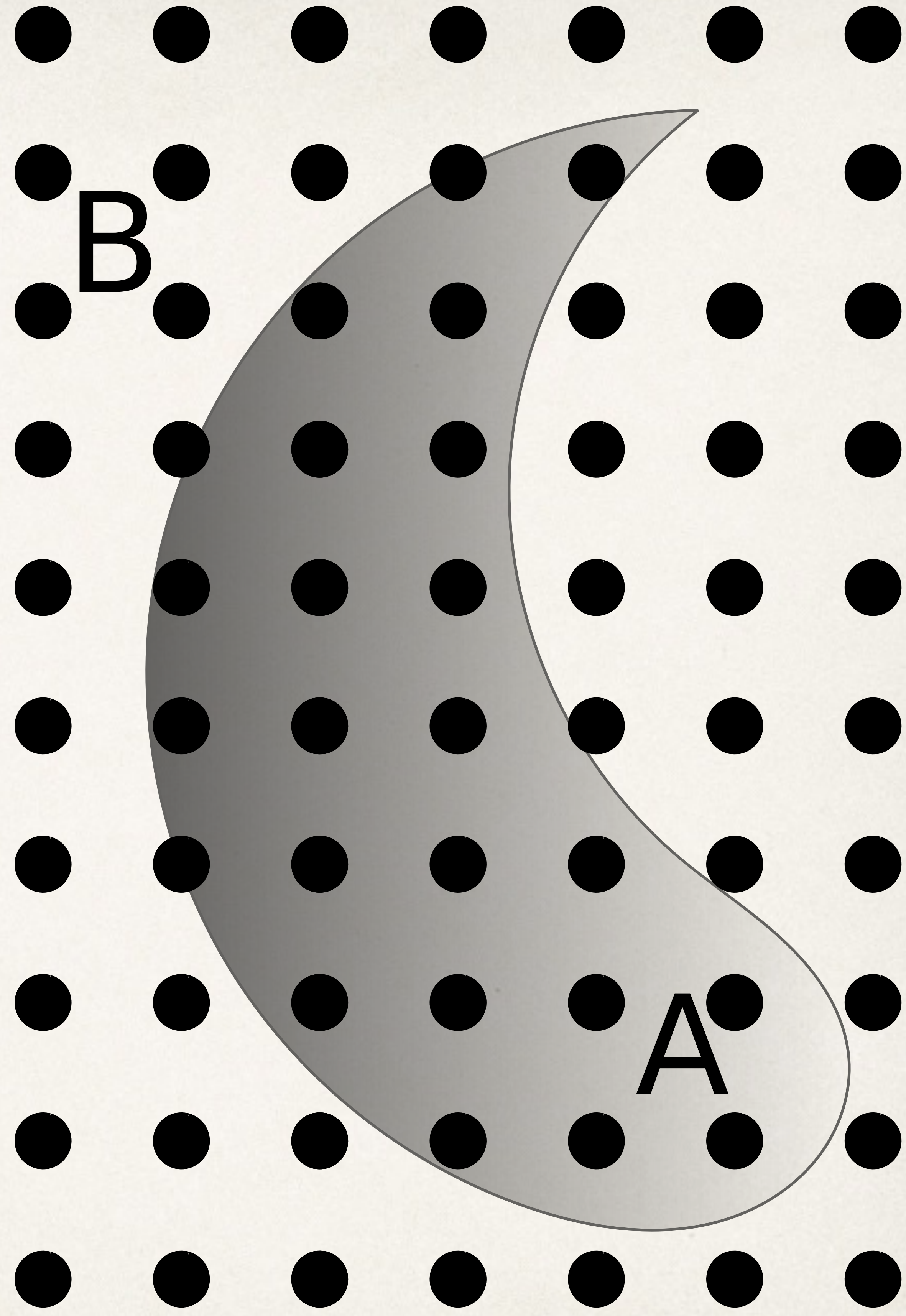
$$S_E = \log 2$$

The Challenges in QFT

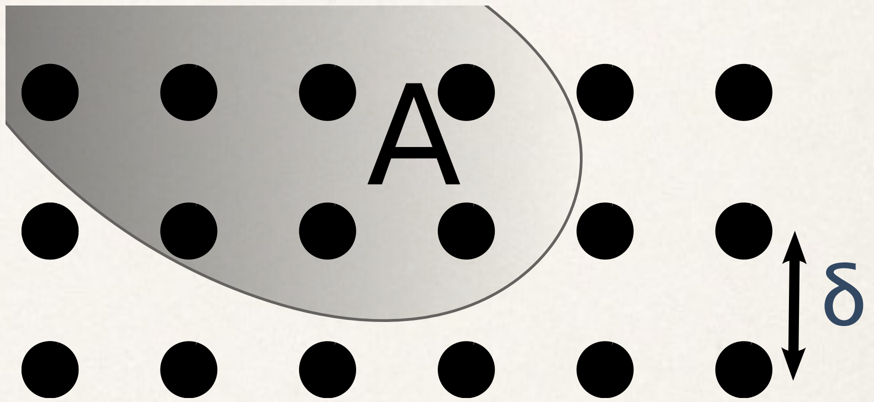
- The assumption that the Hilbert space can be factorized wrt to A and B is often problematic.
- The infinite number of degrees of freedom means entanglement entropy is badly divergent.
- That the density matrix grows exponentially with the size of the Hilbert space means entanglement entropy is difficult to compute.

Use a Lattice

- Provides a natural UV cut-off — the lattice spacing.
- In many lattice models, the Hilbert space factors lattice site by lattice site.



Boundaries and the UV Cutoff



Most of the interesting EE results involve a UV regulator

E.g., for a quantum field theory in the ground state

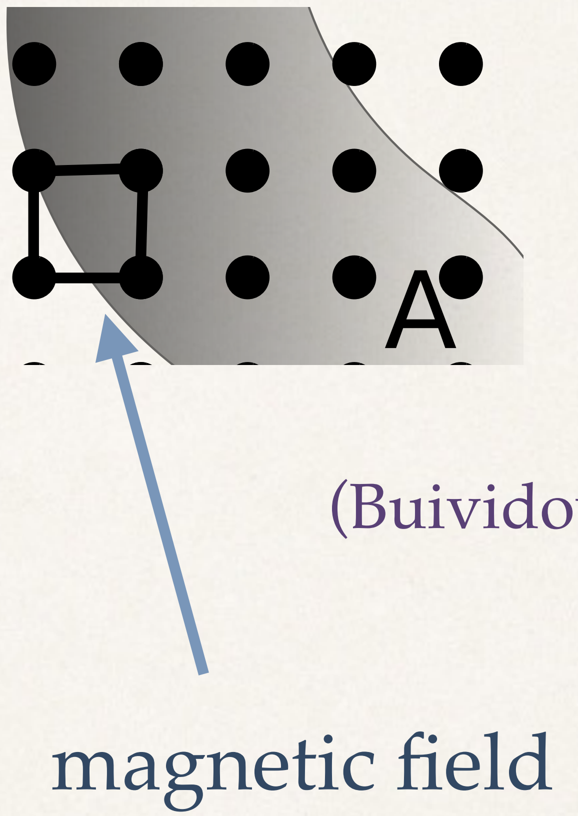
$$S_E \sim \frac{\text{Area}(\partial A)}{\delta^{d-2}}$$ (Srednicki 1993)

Lesson here that most of the correlations are local.

Corollary: Better treat the boundary of *A* correctly.

Factorizability and Boundaries

- Difficult to get chiral fermions on the lattice. (Iqbal-Wall, Nishioka-Yarom, Hellerman)
- For a lattice version of E&M, observables are loops. (Buividovich-Polikarpov, Casini-Huerta, Radicevic, Donnelly, et al.)
- There may indeed be issues in general.



(Buividovich-Polikarpov 2008)

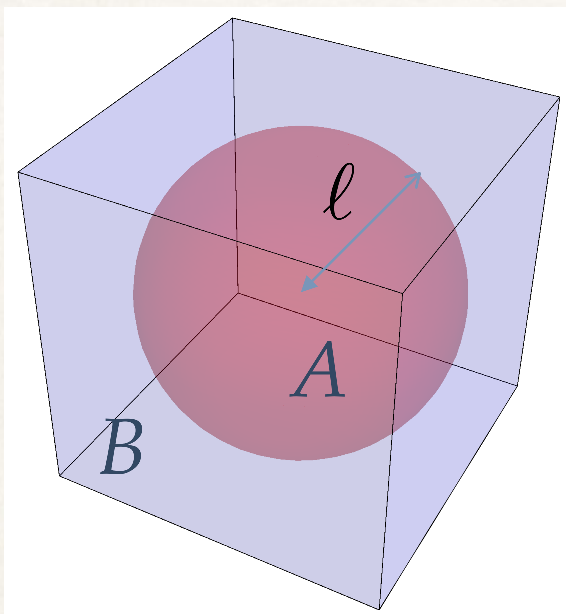
Two Tales from the Edge

For conformal field theories (CFTs)

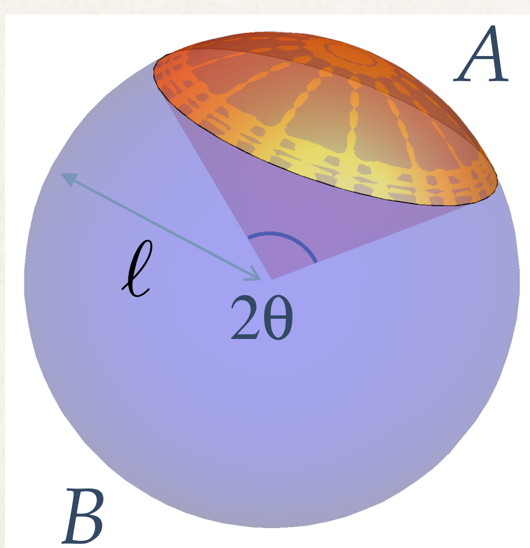
- Universal contributions to entanglement entropy at zero temperature (work with K.-W. Huang and K. Jensen).
- Thermal corrections to entanglement entropy (work with M. Spillane, J. Nian, R. Vaz, and J. Cardy).

Moral: The importance of boundary terms.

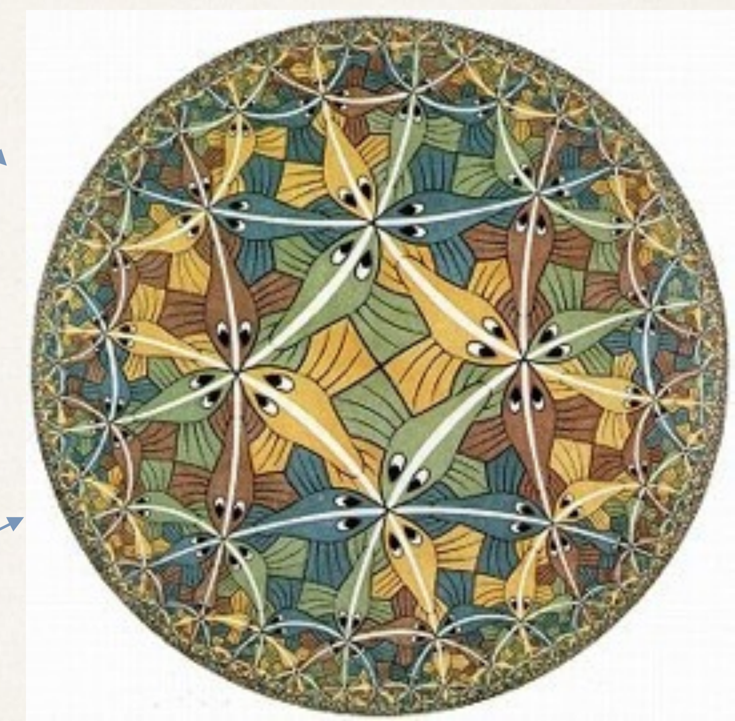
Trick for Calculating EE of CFTs



conformal
transformation



Has $T = 1/2\pi\ell$



For caps on spheres and balls in flat space,
"A" gets mapped to all of hyperbolic space.

Map to Hyperbolic Space

- Density matrix on hyperbolic space is thermal: $\beta = 2\pi\ell$

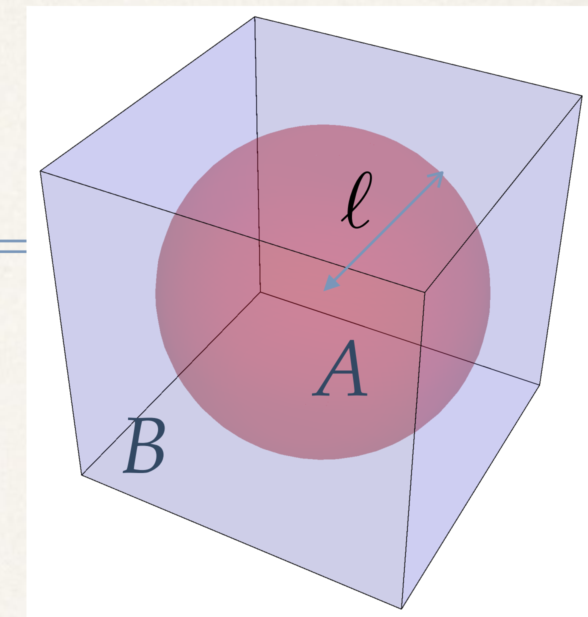
$$\rho = \frac{e^{-\beta H}}{\operatorname{tr} e^{-\beta H}}$$

H called the modular Hamiltonian

- $\rho_A = U^{-1} \rho U$ for some unitary operator U.

- EE invariant under U implies thermal entropy of hyperbolic space is EE. (see e.g. Casini-Huerta-Myers 2011)

Universal contributions to EE at zero T



There is a "universal" contribution to EE that is proportional to "a" anomaly coefficient in $\langle T^\mu_\mu \rangle$.

$$\langle T^\mu{}_\mu \rangle = \sum_j c_j I_j - (-1)^{d/2} \frac{4a}{d!\,\mathrm{Vol}(S^d)} E_d + \mathrm{D}_\mu J^\mu$$

Weyl curvature invariants

Euler density

UV cutoff

$$S_E = \alpha \frac{\mathrm{Area}(\partial A)}{\delta^{d-2}} + \ldots + 4a(-1)^{d/2} \ln \frac{\delta}{\ell} + \ldots$$

2×Euler character of sphere.

(Solodukhin 2008; Casini-Huerta-Myers 2011)

A Puzzle

$$S_E = \alpha \frac{\text{Area}(\partial A)}{\delta^{d-2}} + \ldots + 4a(-1)^{d/2} \ln \frac{\delta}{\ell} + \ldots$$

- Casini-Huerta-Myers (2011) try and fail to get this log from the hyperbolic space map.

- They succeed using a sphere (Euclidean de Sitter) — no boundary; they succeed also using the RT formula.

- The result is consistent (predicted) by earlier work using the replica method and squashed cones Solodukhin (2008).

Can we succeed where CHM failed?

$$\rho = \frac{e^{-\beta H}}{\mathrm{tr}\, e^{-\beta H}} \qquad \Rightarrow \qquad S_E = -\,\mathrm{tr}(\rho \ln \rho) = \beta\langle H\rangle + \ln \mathrm{tr}(e^{-\beta H})$$

Casimir energy

log of partition function; call it -*W*

Guess: universal contributions encoded in an effective action that reproduces the "a" part of the trace anomaly.

Warm-Up: 2D Case

We want to deduce an effective action $W[g_{\mu\nu}]$ from the trace anomaly

$$\langle T^{\mu}_{\mu} \rangle = \frac{c}{24\pi} R$$

According to Polchinski, in the presence of a boundary, the most general form for the anomalous variation is

$$\delta_\sigma W = -\frac{c}{24\pi} \left[\int_M \mathrm{d}^2 x \sqrt{g} R \, \delta\sigma + 2 \int_{\partial M} \mathrm{d}y \sqrt{\gamma} K \, \delta\sigma \right]$$

K here is the trace of the extrinsic curvature.

The Euler characteristic for a 2d manifold with boundary!

The 2d effective action.

We want to integrate $\delta_\sigma W$.
In fact the best we can do is determine a difference:

$$\mathcal{W}[g_{\mu\nu}, e^{-2\tau} g_{\mu\nu}] \equiv W[g_{\mu\nu}] - W[e^{-2\tau} g_{\mu\nu}]$$

The answer is

$$\mathcal{W} = -\frac{c}{24\pi}\left[\int_M \mathrm{d}^2x\sqrt{g}\left(R[g_{\mu\nu}]\tau - (\partial\tau)^2\right) + 2\int_{\partial M} \mathrm{d}y\sqrt{\gamma}\,K[g_{\mu\nu}]\tau\right]$$

Various methods:
1) guess work
2) dimensional regularization
3) integral formula

Dimensional Regularization

Define $\widetilde{W}[g_{\mu\nu}]$ in $n = 2 + \epsilon$ dimensions.

$$\widetilde{W}[g_{\mu\nu}] \equiv -\frac{c}{24\pi(n-2)}\left[\int_M \mathrm{d}^n x\,\sqrt{g}\,R + 2\int_{\partial M} \mathrm{d}^{n-1}y\,\sqrt{\gamma}\,K\right]$$

Then

$$\mathcal{W}[g_{\mu\nu}, e^{-2\tau}g_{\mu\nu}] = \lim_{n\to 2}\left(\widetilde{W}[g_{\mu\nu}] - \widetilde{W}[e^{-2\tau}g_{\mu\nu}]\right)$$

Trick employed by Brown and Cassidy (1977).
Relies on nice transformation properties of R under Weyl scaling.

under $g_{\mu\nu} \to e^{-2\tau}g_{\mu\nu}$, $\sqrt{g}R \to e^{(2-n)\tau}\sqrt{g}R + \text{total derivative}$

Entanglement of an Interval

- Consider an interval with endpoints u and v on the z plane along with the following map to the cylinder with coordinate w:

$$e^{2\pi w/\beta} = \frac{z-u}{z-v}$$

$$\Rightarrow \tau = -\frac{1}{2} \ln\left[\frac{\beta}{2\pi}\left(\frac{1}{v-z} - \frac{1}{u-z}\right)\right] + c.c.$$

- The cylinder has a periodic Euclidean time coordinate.

- The reduced density matrix on the interval is mapped to the thermal density matrix on the cylinder with inverse temperature β.

Plan of Attack

$$S_E = \beta\langle H\rangle - W_{\text{cyl}}$$

Can be obtained from Schwarzian derivative which in turn can be derived from varying

$$\mathcal{W}[g_{\mu\nu}, e^{-2\tau}g_{\mu\nu}]$$

with respect to the metric.

Think of this term as

$$\mathcal{W}[\delta_{\mu\nu}, e^{-2\tau}\delta_{\mu\nu}] - \widetilde{W}[\delta_{\mu\nu}]$$

Assembling the Pieces

$$\beta\langle H\rangle \sim \frac{c}{6}\ln\frac{|v-u|}{\delta}$$

$$\mathcal{W}[\delta_{\mu\nu}, e^{-2\tau}\delta_{\mu\nu}]|_{\text{bulk}} \sim \frac{c}{6}\ln\frac{|v-u|}{\delta}$$

Comes from regulating infinite volume of the cylinder

$$\mathcal{W}[\delta_{\mu\nu}, e^{-2\tau}\delta_{\mu\nu}]|_{\text{boundary}} \sim -\frac{c}{3}\ln\frac{|v-u|}{\delta}$$

τ multiplying *K* in the effective action

$$-\widetilde{W}[\delta_{\mu\nu}] \sim \frac{c}{3}\ln\frac{|v-u|}{\delta}$$

Dim reg of extrinsic curvature

$$S_E \sim \frac{c}{3}\ln\frac{|v-u|}{\delta}$$

Holzhey, Larsen, Wilczek (1994)

Remarks about 2d

- Two ways of picking apart the answer.
  - EE comes from bulk terms on the cylinder.
  - EE comes purely from $\widetilde{W}[\delta_{\mu\nu}]$
- One can use $\mathcal{W}[\delta_{\mu\nu}, e^{-2\tau}\delta_{\mu\nu}]$ for three purposes:
  - to derive Schwarzian derivative
  - to compute the EE
  - to compute the Rényi entropies $S_n \sim \frac{c}{6}\left(n+\frac{1}{n}\right)\ln\frac{|v-u|}{\delta}$

Anomaly Action in General

"a" contribution to trace anomaly comes from the Euler character χ

$$\delta_\sigma W = (-1)^{d/2} 2a\chi(M) + \dots$$
$$= (-1)^{d/2} \frac{4a}{d!\,\mathrm{Vol}(S^d)} \left(\int_M \mathcal{E}_d \delta\sigma - \int_{\partial M} \mathcal{Q}_d \delta\sigma \right) + \dots$$

Euler density

CS like term

Then for dim reg, define

$$\widetilde{W}[g_{\mu\nu}] = (-1)^{d/2} \frac{4a}{(n-d)d!\,\mathrm{Vol}(S^d)} \left(\int_M \mathcal{E}_{n,d} - \int_{\partial M} \mathcal{Q}_{n,d} \right)$$

and $\quad \mathcal{W}[g_{\mu\nu}, e^{-2\tau} g_{\mu\nu}] = \lim_{n\to d} \left(\widetilde{W}[g_{\mu\nu}] - \widetilde{W}[e^{-2\tau} g_{\mu\nu}] \right)$

4d effective action

Euler density Einstein tensor

$$\mathcal{W}[g_{\mu\nu}, e^{-2\tau} g_{\mu\nu}] = \frac{a}{(4\pi)^2} \int_M \mathrm{d}^4x \sqrt{g} \left[\tau E_4 + 4E^{\mu\nu}(\partial_\mu \tau)(\partial_\nu \tau) + 8(\mathrm{D}_\mu \partial_\nu \tau)(\partial^\mu \tau)(\partial^\nu \tau) + 2(\partial\tau)^4\right]$$
$$- \frac{a}{(4\pi)^2} \int_{\partial M} \mathrm{d}^3y \sqrt{\gamma} \left[\tau Q_4 + 4(K\gamma^{\alpha\beta} - K^{\alpha\beta})(\partial_\alpha \tau)(\partial_\beta \tau) + \frac{8}{3}\tau_n^3\right]$$

CS like term: only place τ appears w/out a derivative in the bry

normal derivative of τ

Bulk term figured in Komargodski-Schwimmer proof of the "a"-theorem

Boundary term is a new result.

6d effective action (bulk)

$$
\mathcal{W}[g_{\mu\nu}, e^{-2\tau} g_{\mu\nu}]_{(\mathrm{Bulk})} =
$$

$$
\frac{a}{3(4\pi)^3} \int_M \mathrm{d}^6 x \sqrt{g} \Big\{ -\tau E_6 + 3 E^{(2)}_{\mu\nu} \partial^\mu \tau \partial^\nu \tau + 16 C_{\mu\nu\rho\sigma} (\mathrm{D}^\mu \partial^\rho \tau)(\partial^\nu \tau)(\partial^\sigma \tau)
$$

$$
+ 16 E_{\mu\nu} \left[(\partial^\mu \tau)(\partial^\rho \tau)(\mathrm{D}_\rho \partial^\nu \tau) - (\partial^\mu \tau)(\partial^\nu \tau) \Box \tau \right] - 6 R (\partial \tau)^4
$$

$$
-24 (\partial \tau)^2 (\mathrm{D} \partial \tau)^2 + 24 (\partial \tau)^2 (\Box \tau)^2 - 36 (\Box \tau)(\partial \tau)^4 + 24 (\partial \tau)^6 \Big\} \ .
$$

where

$$
E^{(2)\mu\nu} \equiv g^{\mu\nu} E_4 + 8 R^\mu_\rho R^{\rho\nu} - 4 R^{\mu\nu} R + 8 R_{\rho\sigma} R^{\mu\rho\nu\sigma} - 4 R^\mu{}_{\rho\sigma\tau} R^{\nu\rho\sigma\tau} \ ,
$$

$$
C_{\mu\nu\rho\sigma} \equiv R_{\mu\nu\rho\sigma} - g_{\mu\rho} R_{\nu\sigma} + g_{\mu\sigma} R_{\nu\rho}
$$

Reproduces a result from Elvang, Freedman, Hung, Kiermaier, Myers, Theisen (2012).

6d effective action (conformally flat)

$$\mathcal{W}[\delta_{\mu\nu}, e^{-2\tau}\delta_{\mu\nu}] = -\frac{a}{16\pi^3}\int_M d^6x\sqrt{g}\left\{2(\partial\tau)^2(\partial_\mu\partial_\nu\tau)^2 - 2(\partial\tau)^2(\Box\tau)^2 + 3\Box\tau(\partial\tau)^4 - 2(\partial\tau)^6\right\}$$

$$\begin{aligned}-\frac{a}{3(4\pi)^3}\int_{\partial M} d^5y\sqrt{\gamma}\Big[& -\tau Q_6[\delta_{\mu\nu}] + 48P^\alpha_\beta(\partial_\alpha\tau)(\partial^\beta\tau) + 3Q_4[\delta_{\mu\nu}](\mathring{D}\tau)^2 \\ & + 48K^{\alpha\beta}(\mathring{\Box}\tau)(\mathring{D}_\alpha\partial_\beta\tau) + 24K(\mathring{D}_\alpha\partial_\beta\tau)^2 - 48K_{\alpha\gamma}(\mathring{D}^\beta\partial^\alpha\tau)(\mathring{D}^\gamma\partial_\beta\tau) \\ & - 24K(\mathring{\Box}\tau)^2 - 32K(\mathring{D}\tau)^2\mathring{\Box}\tau - 16K(\partial^\alpha\tau)(\partial^\beta\tau)(\mathring{D}_\alpha\partial_\beta\tau) \\ & + 16K_{\alpha\beta}(\partial^\alpha\tau)(\partial^\beta\tau)\mathring{\Box}\tau + 32K_{\alpha\beta}(\mathring{D}^\alpha\partial^\beta\tau)(\mathring{D}\tau)^2 + 12K\tau_n^4 \\ & + 12K(\mathring{D}\tau)^4 + 24K(\mathring{D}\tau)^2\tau_n^2 + 48(\mathring{\Box}\tau)(\mathring{D}\tau)^2(\tau_n) + 16(\mathring{\Box}\tau)(\tau_n^3) \\ & - 24(\mathring{D}\tau)^2\tau_n^3 - 36\tau_n(\mathring{D}\tau)^4 - \frac{36}{5}\tau_n^5\Big]\end{aligned}$$

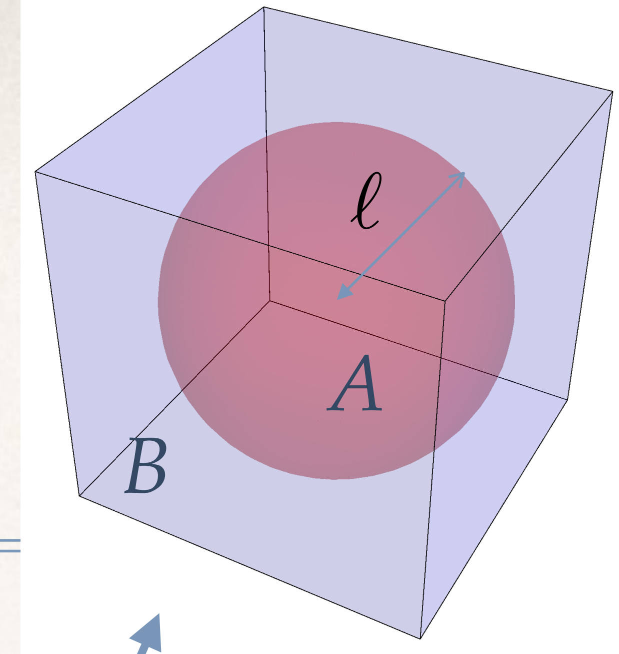
only τ
in the bry

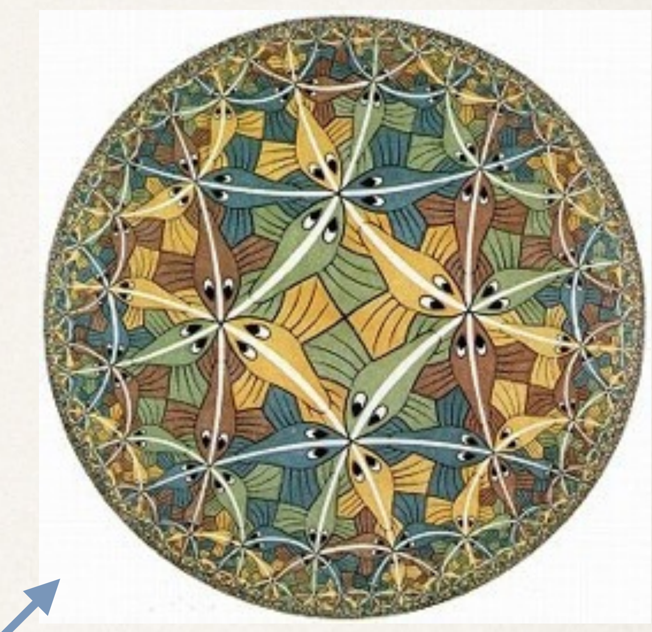
where

$$P^\alpha_\beta \equiv \left(K^2 - \mathrm{tr}(K^2)\right)K^\alpha_\beta - 2KK^{\alpha\gamma}K_{\beta\gamma} + 2K_{\gamma\delta}K^{\alpha\gamma}K^\delta_\beta$$

The boundary term is a new result.

EE of the Ball





flat space

$$
\begin{aligned}
\mathrm{d}s^2 &= -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2_{d-2} \ , \\
&= e^{2\sigma} \left[-\mathrm{d}T^2 + \ell^2 (\mathrm{d}u^2 + \sinh^2 u \, \mathrm{d}\Omega^2_{d-2}) \right]
\end{aligned}
$$

$S^1 \times H^{d-1}$

where

$$
e^{-\sigma} = \cosh u + \cosh T/\ell
$$

$$
S_E = \beta \langle H \rangle + \mathcal{W}[\delta_{\mu\nu}, e^{-2\sigma}\delta_{\mu\nu}] - \widetilde{W}[\delta_{\mu\nu}]
$$

Assembling the Pieces: 4d

$$\beta\langle H\rangle \sim -\frac{3}{2}a\ln\frac{\ell}{\delta}$$

$$\mathcal{W}[\delta_{\mu\nu}, e^{-2\sigma}\delta_{\mu\nu}]|_{\text{bulk}} \sim \left(\frac{3}{2}-4\right)a\ln\frac{\ell}{\delta}$$

$$\mathcal{W}[\delta_{\mu\nu}, e^{-2\sigma}\delta_{\mu\nu}]|_{\text{boundary}} \sim 4a\ln\frac{\ell}{\delta}$$

$$-\widetilde{W}[\delta_{\mu\nu}] \sim -4a\ln\frac{\ell}{\delta}$$

$$S_E \sim -4a\ln\frac{\ell}{\delta}$$

Assembling the Pieces: 6d

$$\beta\langle H\rangle \sim \frac{5}{4}a\ln\frac{\ell}{\delta}$$

$$\mathcal{W}[\delta_{\mu\nu}, e^{-2\sigma}\delta_{\mu\nu}]|_{\text{bulk}} \sim \left(-\frac{5}{4}+4\right)a\ln\frac{\ell}{\delta}$$

$$\mathcal{W}[\delta_{\mu\nu}, e^{-2\sigma}\delta_{\mu\nu}]|_{\text{boundary}} \sim -4a\ln\frac{\ell}{\delta}$$

$$-\widetilde{W}[\delta_{\mu\nu}] \sim 4a\ln\frac{\ell}{\delta}$$

$$S_E \sim 4a\ln\frac{\ell}{\delta}$$

Technical Problem

Why can't I give you the story in general dimension?

Order of limits issue
(fixing the metric before or after
taking the n to d limit)

We have not been able to evaluate $\widetilde{W}[g_{\mu\nu}]$
for $S^1 \times H^{d-1}$ reliably.

Computing $\mathcal{W}[g_{\mu\nu}, e^{-2\sigma} g_{\mu\nu}]$ becomes harder as dimension increases.

Point of View #1

We can make an invariant distinction between $\mathcal{W}[\delta_{\mu\nu}, e^{-2\tau}\delta_{\mu\nu}]|_{\text{boundary}}$ and $\mathcal{W}[\delta_{\mu\nu}, e^{-2\tau}\delta_{\mu\nu}]|_{\text{bulk}}$.

Then $\beta\langle H\rangle + \mathcal{W}[\delta_{\mu\nu}, e^{-2\tau}\delta_{\mu\nu}]|_{\text{bulk}}$ computes the EE

while $\mathcal{W}[\delta_{\mu\nu}, e^{-2\tau}\delta_{\mu\nu}]|_{\text{boundary}} - \widetilde{W}[\delta_{\mu\nu}]$ comes purely from flat space and vanishes

Somewhat nicer — clean separation:
Maps a problem in flat space to a problem in hyperbolic space.

Point of View #2

$-\widetilde{W}[\delta_{\mu\nu}]$ computes the EE and all the other terms cancel.

Consistent with Solodukhin's result in 4d
that the "a" contribution to the EE is proportional
to χ of the entangling surface.

$$S_E \sim \ldots + (-1)^{d/2} 2a\, \chi(\partial A)\, \ln \frac{\delta}{\ell} + \ldots$$

Somewhat discouraging:
We tried to map the problem to hyperbolic space
but somehow never got away from flat space.

Another Tale from the Edge

Thermal Corrections?

The initial density matrix is
not that of a pure state!

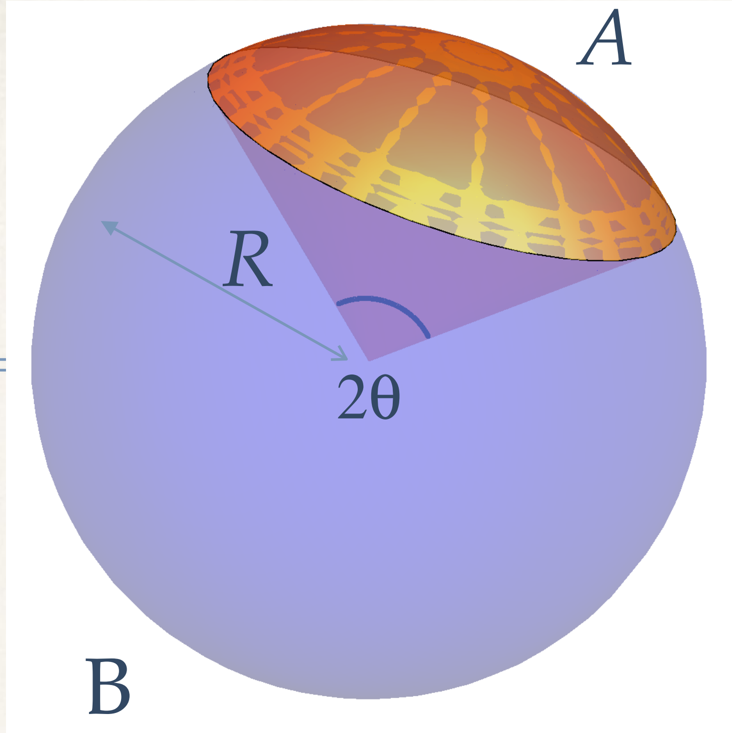
$$\rho(T) = \frac{e^{-H/T}}{\mathrm{tr}(e^{-H/T})}$$

Entanglement entropy measures some combination
of thermal entropy and quantum entanglement.

Why bother with thermal effects?

- Nice to be able to remove them.
- Lessons to be learned from EE in non-traditional contexts.
- Connection to black hole physics.

A Universal Result



In the $RT \ll 1$ limit, for a cap A of opening angle 2θ on the S^3,

$$S_E(A,T) - S_E(B,T) = 2\pi g m R \cot(\theta)\, e^{-m/T} + o(e^{-m/T})$$

(Herzog 2014)

m is the mass gap, ~ $1/R$
g is the degeneracy of the 1st excited state

- Turns out to be true for any CFT in any dimension!
- Subleading in a large N expansion.
- The exp(-m/T) Boltzmann suppression should be true of any gapped QFT (Herzog-Spillane 2012).

Where does it come from?

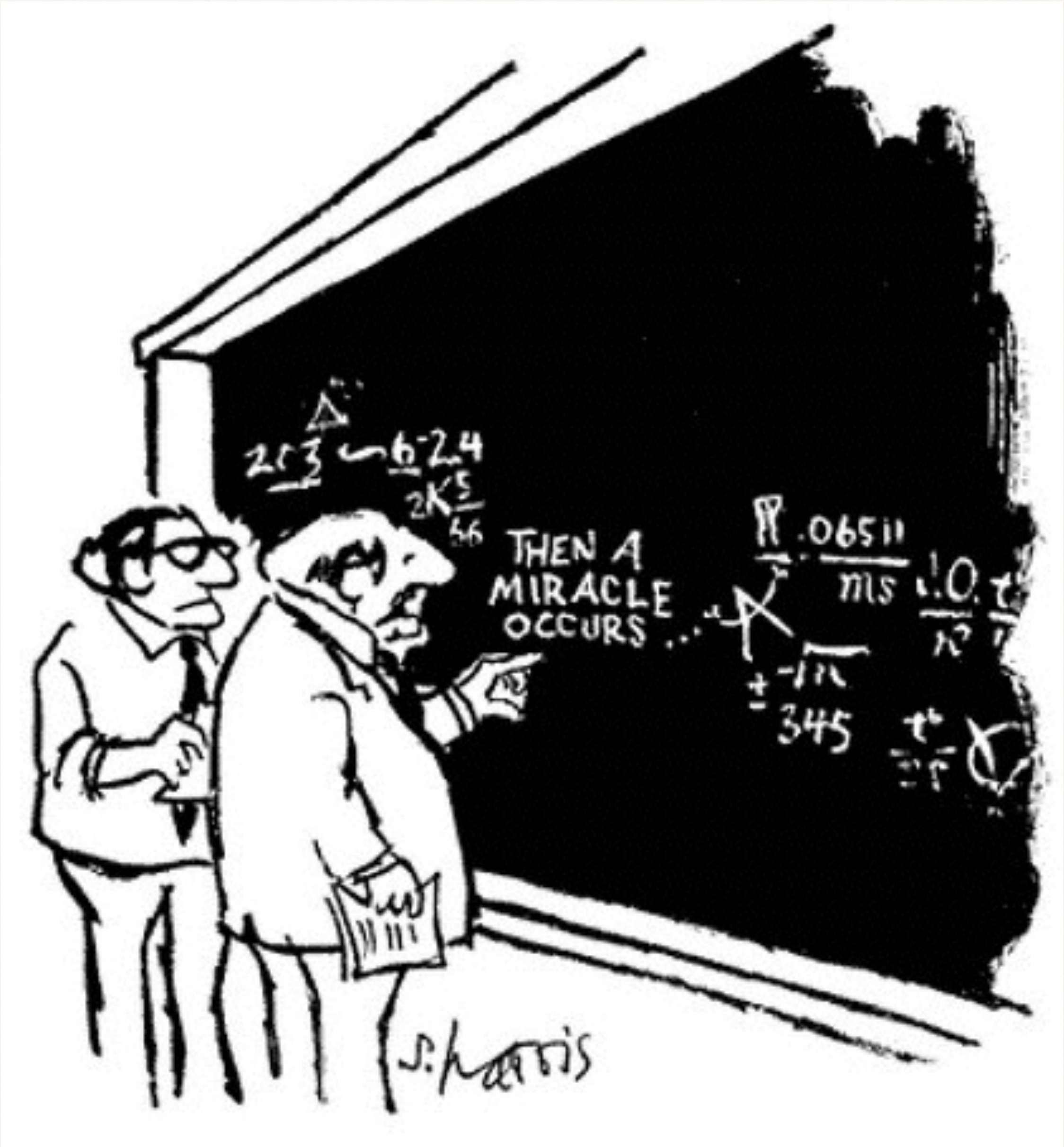
Start with a thermal density matrix $$\rho(T) = \frac{e^{-H/T}}{\mathrm{tr}(e^{-H/T})}$$

(That ρ is mixed means we're not really measuring quantum entanglement.)

Make a small T perturbative expansion

Need to calculate $\langle \psi(x)\psi(y) \log \rho_A(0) \rangle$

where $\psi(x)$ creates the first excited state.


THEN A
MIRACLE
OCCURS
S. Harris

A Special Trick for CFTs

For CFTs and "*A*" a cap on a sphere, $-\log \rho_A(0)$

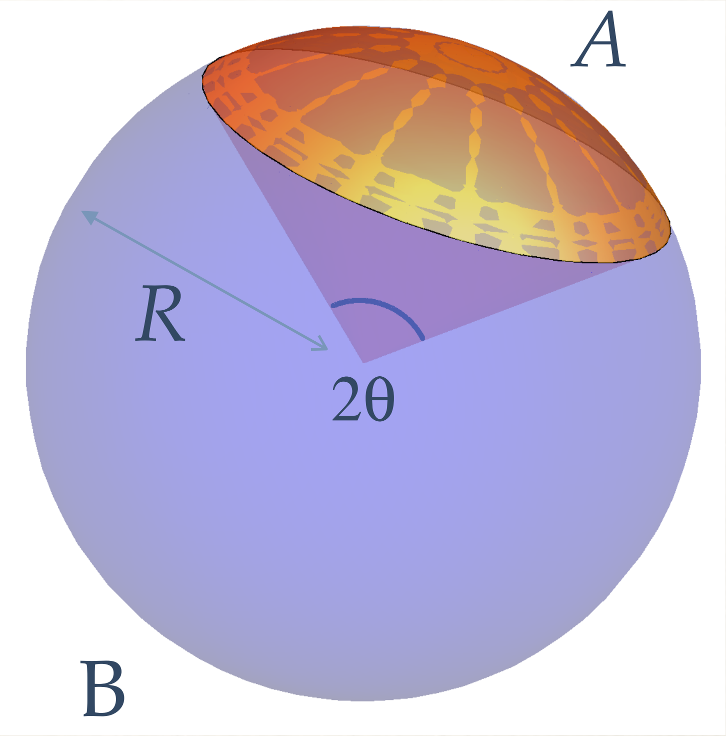
is unitarily related to the Hamiltonian on hyperbolic space.

H is the integral of the *tt* component of the stress-energy tensor $T_{\mu\nu}$.

$$\langle \psi(x)\psi(y) \log \rho_A(0)\rangle \to \langle \psi(x)\psi(y) T_{\mu\nu}(0)\rangle$$

Three point functions involving the stress tensor in CFTs are constrained by symmetry to take relatively simple forms.

Related Result Not Quite Right



From the modular Hamiltonian method

$$S_E(A,T) - S_E(A,0) = gmR\, I_d(\theta)\, e^{-m/T} + \ldots$$

where

$$I_d(\theta) = 2\pi \frac{\mathrm{Vol}(S^{d-2})}{\mathrm{Vol}(S^{d-1})} \int_0^{\theta_0} \frac{\cos\theta - \cos\theta_0}{\sin\theta_0} \sin^{d-2}\theta \, \mathrm{d}\theta$$

But for a scalar field, it turns out other methods match $I_{d-2}(\theta)$.

WHAT'S GOING ON!?!

A Resolution

$$\Delta H = 2\pi\xi \int_{\partial H^{d-1}} \mathrm{d}^{d-2}x\sqrt{\gamma}\,\phi^2$$

- There exists a boundary term that can correct H.
- When $\xi = (d\text{-}2)/4(d\text{-}1)$ (the conformal coupling)

$$I_{d-2}(\theta) \to I_d(\theta)$$

- Suggests whenever CFT has operators of dimension d-2, H may get corrected by boundary terms.

Casini, Mazitelli, Teste (2014)

This particular case

The conformally coupled scalar

$$S = -\frac{1}{2}\int_M [(\partial\phi)^2 + \xi R\phi^2] - \xi \int_{\partial M} K\phi^2$$

trace of extrinsic curvature

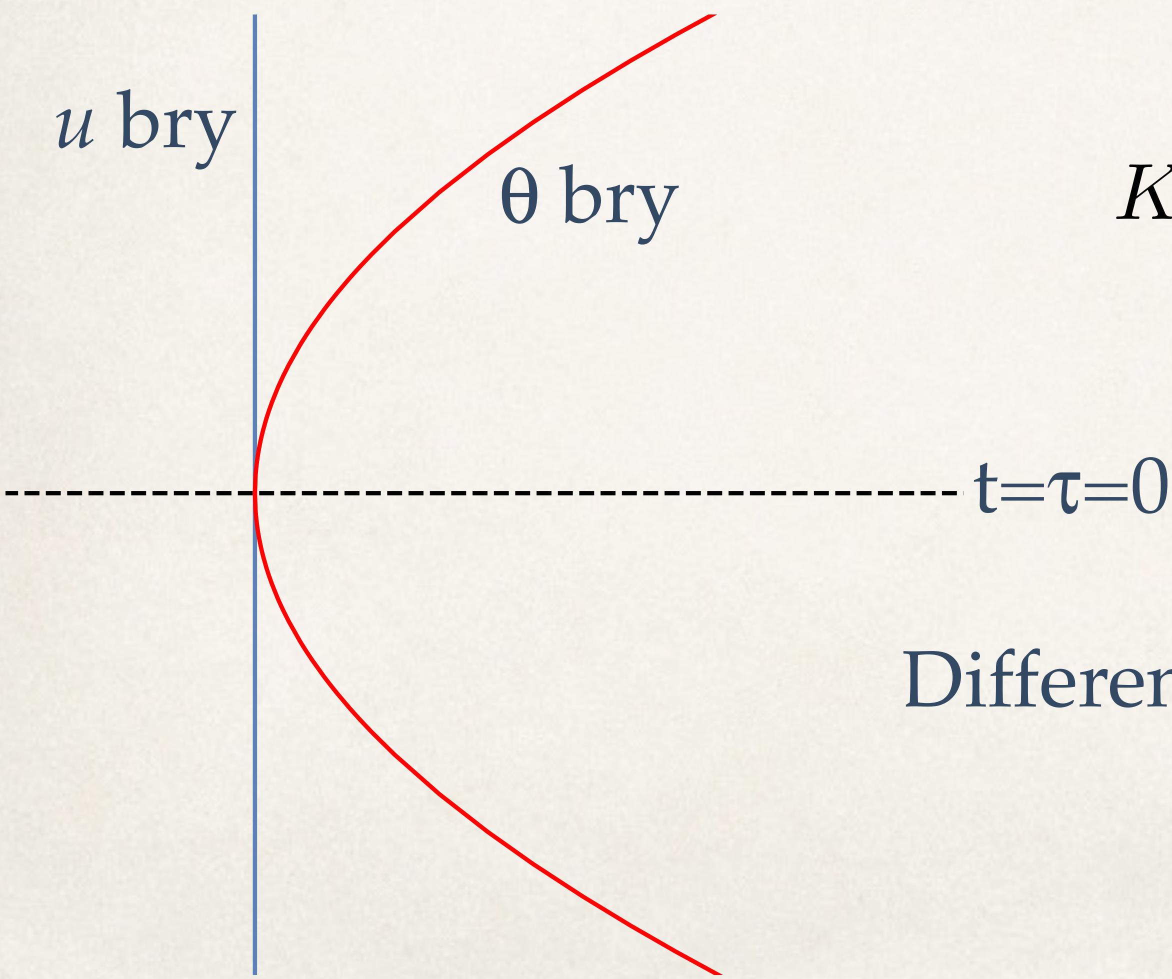
- To define the stress tensor.
- To preserve Weyl scaling symmetry.

Boundary term in action translates into a boundary term in *H*.

In more detail

Let u be the radius of hyperbolic space.
Let θ be the polar angle on the sphere.

Constant u boundary different from pull back of constant θ boundary.



$$K_{(\theta)}|_{\tau=0} = \frac{d-1}{R} \quad , \quad K_{(u)}|_{\tau=0} = \frac{d-2}{R}$$

Difference in K reproduces the shift in H.

Final Remarks

- For certain types of entanglement entropy, mapping to hyperbolic space is a useful tool.
- Hyperbolic space has a boundary, and the boundary has important effects.
    - Thermal corrections.
    - Log contribution to the zero T EE.

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