Holographic QCD at finite (imaginary) chemical potential

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Contents:

- The Roberge-Weiss phase diagram.
- Holographic QCD & chemical potential: context.
- Holographic QCD & chemical potential: results.
- Conclusions and perspectives.

The Roberge-Weiss phase diagram

QCD with imaginary baryon chemical potential μ_B

- Motivation: sign problem in lattice QCD, real μ_B challenging.
- Analytic continuation from imaginary μ_B.
- Rich phase diagram: Roberge-Weiss [Roberge-Weiss 1986].
- $\theta_B \equiv \mu_B / T_c(0)$ is an angle: $Z = Tr(e^{-\beta H + i\theta_B N_B})$.
- Free energy density f at $T > T_c$:
  - Periodic: $f(\theta_B) = f(\theta_B + 2\pi k)$.
  - Depends on θ_B / N_c.

The Roberge-Weiss phase diagram

QCD with imaginary baryon chemical potential μ_B

- For N_f massless Weyl (anti)fundamental fermions:

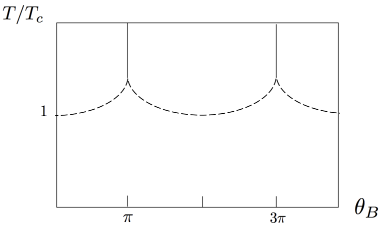
$$f(\theta_B) = \frac{N_c N_f}{12} T^4 \mathrm{min}_k \left(\frac{\theta_B - 2\pi k}{N_c} \right)^2$$

- Large T: f has first order discontinuities at $\theta_B = (2k+1)\pi$.
- Critical temperature for deconfinement:

$$\frac{T_c(\theta_B)}{T_c(0)} \sim 1 + a\,\theta_B^2 \qquad a > 0$$

The Roberge-Weiss phase diagram

Phase diagram:



Phase diagram in holography?

- Top-down model closest to (planar) QCD ("Holographic QCD"): Witten-Sakai-Sugimoto [Witten 1998, Sakai-Sugimoto 2004].
- $D4 - D8/\bar{D}8$: non supersymmetric, confining theory with only quarks in fundamental.

Contents:

Context: Holographic YM

$D4$-branes wrapped on $S^1_{x_4}$, at low energy:
4d YM theory + (adjoint) KK modes [Witten 1998]

Dual gravity solution:

$$ds^2 = \left(\frac{u}{R}\right)^{3/2}\left[dx_\mu dx^\mu + f(u)dx_4^2\right] + \left(\frac{u}{R}\right)^{-3/2}\left[\frac{du^2}{f(u)} + u^2 d\Omega_4^2\right]$$

$$f(u) = 1 - \frac{u_0^3}{u^3} \qquad e^\phi = g_s\left(\frac{u}{R}\right)^{3/4} \qquad F_4 = \dots$$

- In IR: $\mathbb{R}^{1,3} \times$ cigar$_{(u,x_4)} \times S^4$.
- $g_{00}(u_0) \neq 0$ (regular): confinement.
- Period of $x_4 \sim 1/M_{KK}$.
- In gravity regime KK modes are NOT decoupled.
- If x_0 compact: theory at finite $T < T_c$ (period of $x_0 \sim 1/T$).

Context: Holographic YM

$D4$-branes wrapped on $S^1_{x_4}$, at low energy:
4d YM theory + (adjoint) KK modes [Witten 1998]

Dual gravity solution at $T > T_c$:

$$ds^2 = \left(\frac{u}{R}\right)^{3/2}\left[\tilde{f}(u)dx_0^2 + dx_a dx^a + dx_4^2\right] + \left(\frac{u}{R}\right)^{-3/2}\left[\frac{du^2}{\tilde{f}(u)} + u^2 d\Omega_4^2\right]$$

$$\tilde{f}(u) = 1 - \frac{u_T^3}{u^3} \qquad e^{\phi} = g_s\left(\frac{u}{R}\right)^{3/4} \qquad F_4 = \dots$$

- In IR: $\mathbb{R}^3 \times$ cigar$_{(u,x_0)}$ $\times S^1_{x_4} \times S^4$.
- $g_{00}(u_T) = 0$: deconfinement.
- Period of $x_0 \sim 1/T$.

Symmetry between solutions: $x_0 \leftrightarrow x_4, \;\; u_0 \leftrightarrow u_T, \;\; M_{KK} \leftrightarrow T$.

Context: Holographic YM

D4-branes wrapped on $S^1_{x_4}$, at low energy:
4d YM theory + (adjoint) KK modes [Witten 1998]

Difference of free energy densities (from on-shell actions):

$$\Delta f = -\left(\frac{2N_c^2\lambda_4}{2187\pi^2 M_{KK}^2}\right)\left[(2\pi T)^6 - M_{KK}^6\right]$$

Notation: $\lambda_4 = g_{YM}^2 N_c = 2\pi g_s l_s M_{KK} N_c$.

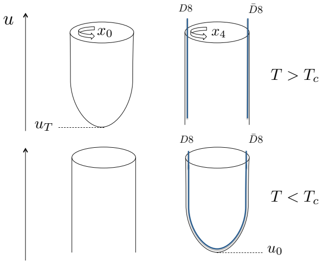
Critical temperature for deconfinement: $T_c = \frac{M_{KK}}{2\pi}$.

Context: Holographic QCD

Probe $D8/\bar{D}8$-branes at antipodal points on $S^1_{x_4}$:
massless (anti)quarks in the fundamental [Sakai-Sugimoto 2004]

- $T > T_c$: branes fall separately into horizon (χSR)
- $T < T_c$: branes connect at tip of cigar (χSB)

  $U(N_f) \times U(N_f) \to SU(N_f) \times U(1)_B$



- Critical temperature for chiral symmetry restoration coincides with critical temperature for deconfinement T_c.

[Aharony-Sonnenschein-Yankielowicz 2007]

Dynamical quarks:
backreaction of $D8/\bar{D}8$-branes on Witten's background

- Dynamical flavors in holography: "flavor brane" backreaction (probe branes $\sim$ quenched approximation).
- Localized $D8/\bar{D}8$ configuration: too difficult.

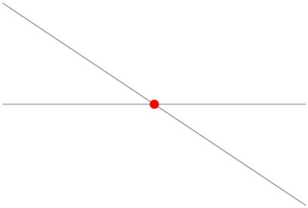
  (PDEs: a limiting case in [Burrington-Kaplunovsky-Sonnenschein 2007]).
- Difficult technical problem (PDEs) made simpler by "smearing technique" (ODEs).

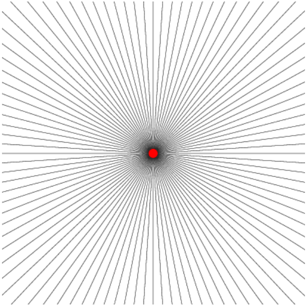
  [Bigazzi-Casero-Cotrone-Kiritsis-Paredes 2005, Casero-Nunez-Paredes 2006]

Context: Smearing technique

Dynamical quarks:
backreaction of $D8/\bar{D}8$-branes on Witten's background

- Smearing: homogeneous distribution of flavor branes along transverse dimensions $\Rightarrow$ recover symmetry.
- In dual field theory: $U(N_f) \to U(1)^{N_f}$.





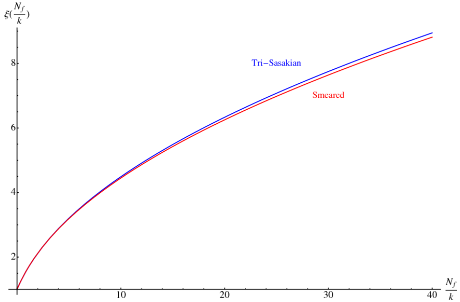
Picture from [Nunez-Paredes-Ramallo 2010]

Is it a good approximation?

E.g. Free energy F in flavored ABJM [Conde-Ramallo 2011]:

$$F(S^3) = \frac{\pi\sqrt{2}}{3} k^{1/2} N^{3/2} \xi\left(\frac{N_f}{k}\right)$$

Comparison of smeared gravity with localized field theory result:



Dynamical quarks:
backreaction of $D8/\bar{D}8$-branes on Witten's background

- Complete solution ($D8/\bar{D}8$-branes uniformly distributed on $S^1_{x_4}$ s.t. $U(1)$ symmetry recovered): not yet.

  (Still complicated second order non linear coupled equations).

- Smeared configuration at first order in $\frac{N_f}{N_c}$: analytic solutions!

Contents:

Confined phase ($T < T_c$)

- Metric:

$$ds^2 = e^{2\lambda}(-dt^2 + dx_a dx^a) + e^{2\tilde{\lambda}} dx_4^2 + l_s^2 e^{-4\phi+8\lambda+2\tilde{\lambda}+8\nu} d\rho^2 + l_s^2 e^{2\nu} d\Omega_4^2$$

- Action:

$$2k_0^2 S = \int d^{10}x \sqrt{-g}\left[e^{-2\phi}\left(R + 4(\partial\phi)^2\right) - \frac{1}{2}|F_4|^2\right]$$
$$-\frac{2k_0^2 N_f T_8 M_{KK}}{\pi}\int d^{10}x \frac{\sqrt{-(g + 2\pi\alpha' F)}}{\sqrt{g_{44}}} e^{-\phi} \quad \leftarrow \quad \text{smeared DBI}$$

- Solution for brane gauge field: $A_t = \mu \quad \Rightarrow \quad F = 0$: zero-density, finite-chemical potential (validity: small μ).

Confined phase ($T < T_c$)

Field expansion in N_f/N_c:

$$\Psi(\rho) = \Psi_{Witten}(\rho) + \epsilon_f \Psi_1(\rho) + \mathcal{O}(\epsilon_f^2)$$

where:

$$\epsilon_f \equiv \frac{1}{12\pi^3} \lambda_4^2 \frac{N_f}{N_c} \ll 1$$

(Horrible but) analytic solution (in $r \equiv \frac{u_0^3}{l_s^3 g_s^2}\rho$):

$$\lambda_1 = \frac{3}{8}f + y - \frac{1}{4}(A_2 - A_1) - \frac{1}{4}(B_2 - B_1)r$$

$$\tilde{\lambda}_1 = -\frac{1}{8}f + y - \frac{1}{4}(A_2 + B_2 r) - \frac{3}{4}(A_1 + B_1 r)$$

$$\phi_1 = \frac{11}{8}f + y - \frac{1}{4}(A_1 + B_1 r) - \frac{3}{4}(A_2 + B_2 r)$$

$$\nu_1 = \frac{11}{24}f + q$$

with:

$$f = \frac{4}{9}e^{-3r/2}\,{}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{13}{6};\frac{3}{2},\frac{3}{2};e^{-3r}\right)$$

$$y = C_2 - \coth\left(\frac{3r}{2}\right)\left(C_1 + C_2\left(\frac{3r}{2}+1\right)\right) + z$$

$$q = \frac{1}{12}(A_1 - 5A_2 + r(B_1 - 5B_2)) + \frac{5}{3}z - \coth\left(\frac{3r}{2}\right)(M_1 + M_2(3r+2)) + 2M_2$$

$$z = -\frac{e^{-9r/2}\left(e^{-3r}+1\right)\left(9e^{3r}\,{}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{19}{6};\frac{3}{2},\frac{3}{2};e^{-3r}\right) + {}_3F_2\left(\frac{3}{2},\frac{3}{2},\frac{19}{6};\frac{5}{2},\frac{5}{2};e^{-3r}\right)\right)}{162\left(1-e^{-3r}\right)}$$

$$-\frac{8e^{-3r/2}\left(10e^{-3r}+3\right)\,{}_2F_1\left(\frac{1}{6},\frac{1}{2};\frac{3}{2};e^{-3r}\right)}{819\left(1-e^{-3r}\right)} + \frac{e^{-15r/2}\left(38e^{3r}+8e^{6r}-40\right)}{273\left(1-e^{-3r}\right)^{13/6}}$$

Holographic QCD & chemical potential: results

Features

- IR regular (Ricci and Kretschmann scalars, with $B_1 = 6C_2$, $B_2 = 0$, $M_2 = \frac{C_2}{6}$).
- UV divergent (for whatever choice of integration constants): Landau pole.
  - Compact radial variable: $x = e^{-3r/2} \to$ IR: $x \sim 0$, UV: $x \sim 1$.
  - In UV:

$$\frac{1}{g^2_{YM,x}} \approx \frac{1}{g^2_{YM}}\left[1 - \frac{3}{7}\epsilon_f \frac{2^{5/6}}{(1-x)^{1/6}}\right] \qquad \Rightarrow \qquad x_{LP} = 1 - 2^5 (3/7)^6 \epsilon_f^6$$

  - Physics reliable at $x \ll x_{LP} \sim 1$.

Holographic QCD & chemical potential: results

Examples of observables

- From fundamental string: string tension

$$T_s = \sqrt{g_{00}g_{11}}_{|x=0} = \frac{2}{27\pi}\lambda_4 M_{KK}^2 \left[1 + 1.13\epsilon_f\right]$$

- From wrapped $D4$: baryon (vertex) mass

$$m_B = \frac{1}{27\pi}\lambda_4 N_c M_{KK} \left[1 + 0.95\epsilon_f\right]$$

- From fluctuation of $D8$ gauge fields: vector meson masses
($\epsilon_f = 0.02$, "naive comparison")

$$\frac{m^2_{\rho(1450)}}{m^2_\rho} \sim 3.7 \qquad \text{Without flavors}: \frac{m^2_{\rho(1450)}}{m^2_\rho} \sim 4.3 \qquad \text{Experiment}: \frac{m^2_{\rho(1450)}}{m^2_\rho} \sim 3.5$$

$$\frac{m^2_{a_1(1260)}}{m^2_\rho} \sim 2.37 \qquad \text{Without flavors}: \frac{m^2_{a_1(1260)}}{m^2_\rho} \sim 2.39 \qquad \text{Experiment}: \frac{m^2_{a_1(1260)}}{m^2_\rho} \sim 2.51$$

Holographic renormalization

Divergent on-shell action $\Rightarrow$ $S_E^{ren} = (S_E + S_{GH}) + S_{c.t.}^{D4} + S_{c.t.}^{D8}$

- Standard counter-term for $D4$-branes [Mateos-Myers-Thomson 2007]:

$$S_{c.t.}^{D4} = \frac{g_s^{1/3}}{k_0^2 R} \int d^9 x \sqrt{h} \frac{5}{2} e^{-\frac{7}{3}\phi}$$

- For $D8$-branes no existing covariant counter-terms!

Holographic renormalization

Strategy:

- Go to dual frame: $d\tilde{s}^2 \sim e^{-\frac{2}{3}\phi} ds^2$ [Kanitscheider-Skenderis-Taylor 2008].
- Reduce on $S^4 \;\Rightarrow\;$ metric is asymptotically AdS.
- Use standard AdS counter-terms: volume + GH (written in 8d)

$$\tilde{S}^{D8}_{c.t.} \sim \int d^8x \sqrt{\tilde{h}_8}\, e^{2\phi} \left[1 - 2\alpha \tilde{K}_9\right]$$

- Bring back to original frame (probe case):

$$S^{D8}_{c.t.} = -2N_f T_8 \int d^8x \sqrt{h_8} \left[\frac{16}{7} \frac{R}{g_s^{1/3}} e^{-2\phi/3} - \frac{R^2}{g_s^{2/3}} e^{-\phi/3} \left(K_9 - \frac{8}{3} n \cdot \nabla\phi \right) \right]$$

(In backreacted case numeric coeffs. are different).

Features: covariant (good!), non local (bad?).

Free energy density

From renormalized on-shell action: free energy density

$$f = -p = -\frac{2N_c^2\lambda_4}{3^7\pi^2} M_{KK}^4 \left[1 - \frac{4}{7}\frac{\lambda_4^2}{\pi^3}\frac{N_f}{N_c}\frac{\pi^{3/2}}{\Gamma\left(-\frac{2}{3}\right)\Gamma\left(\frac{1}{6}\right)}\right]$$

T-independent $\Rightarrow$ zero entropy ($\mathcal{O}(1)$: confining phase).

Holographic QCD & chemical potential: results

Deconfined phase ($T > T_c$)

- Metric:

$$ds^2 = -e^{2\tilde{\lambda}}dt^2 + e^{2\lambda}dx_a dx^a + e^{2\lambda_s}dx_4^2 + l_s^2 e^{-4\phi+6\lambda+2\tilde{\lambda}+2\lambda_s+8\nu}d\rho^2 + \dots$$

- Solution for brane gauge field "complicated"
  $\Rightarrow$ small charge ("q") expansion
  $\Rightarrow$ "simple" brane gauge field $A_t \sim q(1-\sqrt{1-e^{-3r}})$.
- Fields expanded in $\epsilon_{f\ T} = \frac{\lambda_4^2}{12\pi^3}\frac{2\pi T}{M_{KK}}\frac{N_f}{N_c} \ll 1$ and $q^2 \ll 1$.
- Another (horrible but) analytic solution (let's skip it).
- Features:
  - Horizon in IR.
  - Landau pole in UV.

Holographic renormalization

Divergent on-shell action $\Rightarrow$ $S_E^{ren} = (S_E + S_{GH}) + S_{c.t.}^{D4} + S_{c.t.}^{D8}$

- Sub-leading divergence from $D8$-branes does not match with $T < T_c$ one $\Rightarrow$ cannot use background subtraction!
- Have to add the counter-term (probe case):

$$S_{c.t.}^{D8} = 2N_f T_8 \int d^8x \sqrt{h_8} \left[\frac{2}{7} \frac{R}{g_s^{1/3}} e^{-2\phi/3} - \frac{2}{7} \frac{R^2}{g_s^{2/3}} e^{-\phi/3} \left(K_9 - \frac{8}{3} n \cdot \nabla\phi \right) \right]$$

(In backreacted case numeric coeffs. are different).

Thermodynamics

- From asymptotics of A_t $\Rightarrow$ chemical potential

$$\mu = \frac{8\pi}{27} q \lambda_4 \frac{T^2}{M_{KK}} \qquad \text{Note}: q \sim T^{-2} \ (\text{fixed } \mu)$$

- From electric displacement $\frac{\delta \mathcal{L}}{\delta F_{t\hat{\rho}}}$ $\Rightarrow$ quark number density

$$n_q = \frac{32\pi}{729} q N_c N_f \lambda_4^2 \frac{T^5}{M_{KK}^2} \qquad \text{Note}: q \sim T^{-5} \ (\text{fixed } n_q)$$

- From area of horizon $\Rightarrow$ entropy density

$$s = \frac{256 N_c^2 \pi^4 \lambda_4}{729 M_{KK}^2} T^5 \left[1 + \frac{2}{3} \epsilon_{f\ T} \left(1 + \frac{q^2}{2}\right)\right]$$

- From ADM energy $\Rightarrow$ energy density

$$\varepsilon = \frac{5}{6} \left(\frac{256 N_c^2 \pi^4 \lambda_4}{729 M_{KK}^2}\right) T^6 \left[1 + \frac{24}{35} \epsilon_{f\ T} \left(1 + \frac{7}{9} q^2\right)\right]$$

Thermodynamics

- From on-shell action $\Rightarrow$ Gibbs free energy density (grand canonical ensemble)

$$\omega = -p = -\frac{1}{6}\left(\frac{256 N_c^2 \pi^4 \lambda_4}{729 M_{KK}^2}\right) T^6 \left[1 + \frac{4}{7}\epsilon_{f\ T}\left(1 + \frac{7}{6}q^2\right)\right]$$

- From $\varepsilon = Ts + f$ $\Rightarrow$ Helmholtz free energy density (canonical ensemble)

$$f = -\frac{1}{6}\left(\frac{256 N_c^2 \pi^4 \lambda_4}{729 M_{KK}^2}\right) T^6 \left[1 + \frac{4}{7}\epsilon_{f\ T}\left(1 - \frac{7}{6}q^2\right)\right]$$

- From $\left(\frac{\partial \varepsilon}{\partial T}\right)_{V,n}$ $\Rightarrow$ heat capacity at fixed quark density

$$c_{V,n} = 5\left(\frac{256 N_c^2 \pi^4 \lambda_4}{729 M_{KK}^2}\right) T^5 \left[1 + \frac{4}{5}\epsilon_{f\ T}\left(1 - \frac{1}{3}q^2\right)\right]$$

Holographic QCD & chemical potential: results

Thermodynamics

- From $\left(\frac{\partial \varepsilon}{\partial T}\right)_{V,\mu}$ $\Rightarrow$ heat capacity at fixed chemical potential

$$c_{V,\mu} = 5\left(\frac{256 N_c^2 \pi^4 \lambda_4}{729 M_{KK}^2}\right) T^5 \left[1 + \frac{4}{5}\epsilon_{f\ T}\left(1 + \frac{1}{3}q^2\right)\right]$$

- From $\frac{s}{c_{V,\mu}}$ $\Rightarrow$ speed of sound (squared)

$$c_s^2 = \frac{1}{5}\left[1 - \frac{2}{15}\epsilon_{f\ T}\left(1 - \frac{1}{2}q^2\right)\right]$$

- From $\varepsilon - 3p$ $\Rightarrow$ interaction energy

$$IE = \frac{1}{3}\left(\frac{256 N_c^2 \pi^4 \lambda_4}{729 M_{KK}^2}\right) T^6 \left[1 + \frac{6}{7}\epsilon_{f\ T}\left(1 + \frac{7}{18}q^2\right)\right]$$

Thermodynamics

Consistency checks:

- $s = -(\partial f / \partial T)$ (considering T-dependence of $\epsilon_{f\ T}$ and q in canonical ensemble)
- $s = -(\partial \omega / \partial T)$ (considering T-dependence of $\epsilon_{f\ T}$ and q in grand canonical ensemble)
- $\omega = f - \mu n_q$
- Ok with probe approximation.

Holographic QCD & chemical potential: results

The critical temperature

- From $p_{conf}(T_c) = p_{deconf}(T_c)$:

$$\frac{2\pi T_c}{M_{KK}} = 1 - \frac{1}{126\pi^3}\lambda_4^2 \frac{N_f}{N_c}\left(1 + \frac{12\pi^{3/2}}{\Gamma\left(-\frac{2}{3}\right)\Gamma\left(\frac{1}{6}\right)}\right) - \frac{27}{16\pi}\frac{N_f}{N_c}\frac{\mu^2}{M_{KK}^2}$$

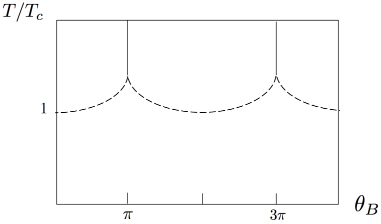
Note: N_f-behavior is comparison scheme dependent.

- At imaginary baryon chemical potential μ_B ($\theta_B = \mu_B / T_c(0)$):

$$\frac{T_c}{T_c(\theta_B = 0)} = 1 + \frac{27}{64\pi^3} N_f N_c \frac{\theta_B^2}{N_c^2}$$

Also in [Rafferty 2011], where it is shown that free energy has first order discontinuities at $\theta_B = (2k+1)\pi$.

Phase diagram:



Qualitatively the same as lattice QCD

Contents:

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- Conclusions and perspectives.

Conclusions and perspectives

Done:

- Derived gravity solutions dual to Witten-Sakai-Sugimoto model with dynamical flavors, at first order in N_f/N_c and q^2, in confined and deconfined phases.
- Studied a few observables (e.g. string tension, hadron masses, etc.).
- Analyzed phase diagram at finite imaginary chemical potential ([Rafferty 2011]).
- Phase diagram of holographic model observed to be in qualitative agreement with lattice QCD.
- Introduced covariant (non-local) counterterms for $D8$-branes.

Conclusions and perspectives

To do (among others):

- Fields/operators dictionary.
- Holographic renormalization.
- Study of other observables (e.g. probe energy loss, entanglement entropy).
- Go beyond leading orders in ϵ_f, q^2.