Linearly resummed hydrodynamics from gravity

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based on early work of Lublinsky & Shuryak (2009) and recent works done with Lublinsky and Sharon: 1406.7222, 1409.3095, 1502.08044, 1504.01370

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Outline

Motivation

- Causality of relativistic hydrodynamics: truncation of derivative expansion at any *fixed* order breaks causality; causality is supposed to be recovered by inclusion of all order derivative terms.
- Linearized all order hydro: M. Lublinsky and E. Shuryak: Phys.Rev.D80 (2009) 065026. However, the fluid stress tensor was not fully constructed. (derivative resummation for boost invariant hydro, see works of Heller-Janik-$\cdots$)
- I will use fluid/gravity correspondence to resum all order derivative terms (linearly) for conformal fluid, whose microscopic description is $\mathcal{N} = 4$ super-Yang-Mills theory.

Relativistic hydrodynamics: notations

- Hydro is an effective long-wavelength description for most classical or quantum many-body systems at finite temperature.
- Constitutive relation: $T_{\mu\nu} = (\varepsilon + P)u_\mu u_\nu + P g_{\mu\nu} + \Pi_{\mu\nu}$.
- Conservation law: $\nabla^\mu T_{\mu\nu} = 0$
- $\Pi_{\mu\nu}$ dissipation tensor, derivable from microscopic theory.
- Landau frame: $T_{\mu\nu}u^\nu = -\varepsilon u_\mu \Longrightarrow \Pi_{\mu\nu}u^\nu = 0$.
- CFT fluids in d-dim spacetime: ($T_{\mu\nu}$ is conformal covariant)
  - $\varepsilon = (d-1)P$;
  - $\Pi_{\mu\nu} \longrightarrow \Pi_{\langle\mu\nu\rangle}$: traceless & transverse
    $\Pi_{\langle\mu\nu\rangle} = \frac{1}{2}\Delta_\mu^\alpha \Delta_\nu^\beta \left(\Pi_{\alpha\beta} + \Pi_{\beta\alpha}\right) - \frac{1}{d-1}\Delta_{\mu\nu}\Delta^{\alpha\beta}\Pi_{\alpha\beta}$
  - $\Delta_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$, projection on spatial directions
- Derivative expansion: $\Pi_{\langle\mu\nu\rangle} = -2\eta_0 \nabla_{\langle\mu} u_{\nu\rangle} + 2\tau_{\mathrm{R}}(u\nabla)\nabla_{\langle\mu} u_{\nu\rangle} + \mathcal{O}\left(\partial^2\right)$

Linearized all order hydro (Lublinsky & Shuryak)

Lublinsky & Shuryak Phys.Rev.D80 (2009) 065026

- Linearization: $\nabla \cdots \nabla u (\checkmark)$, but $(\nabla u)^2 (\times)$, similarly for $g_{\mu\nu}$
- $\Pi_{\mu\nu} = -2\eta \nabla_{\langle\mu} u_{\nu\rangle} + \kappa u^\alpha u^\beta C_{\langle\mu\alpha\nu\rangle\beta} + \rho u^\alpha \nabla^\beta C_{\langle\mu\alpha\nu\rangle\beta} + \xi \nabla^\alpha \nabla^\beta C_{\langle\mu\alpha\nu\rangle\beta}$
- $\eta = \eta(\nabla^2, u\nabla),\ \kappa = \kappa(\nabla^2, u\nabla),\ \rho = \rho(\nabla^2, u\nabla),\ \xi = \xi(\nabla^2, u\nabla).$
- In Fourier space, $\eta = \eta(\omega, q^2),\ \kappa = \kappa(\omega, q^2),\ \rho = \rho(\omega, q^2),\ \xi = \xi(\omega, q^2)$
- $\eta,\ \kappa,\ \rho,\ \xi$ from thermal correlators of $T_{\mu\nu} \leftarrow$ AdS/CFT
- Unfortunately, three channels in gravity perturbations are insufficient to determine all four transport coefficient functions.
- We now fixed this problem using fluid/gravity correspondence. (This talk!)

Fluid/gravity correspondence: overview

Policastro-Son-Starinets PRL 87 (2001) 081601, JHEP 0209 (2002) 043

Kovtun-Son-Starinets PRL 94 (2005) 111601

Buchel-Liu PRL93 (2004) 090602

Buchel-Myers-Paulos-Sinha PLB 669 (2008) 364-370

Iqbal-Liu PRD 79 (2009) 025023

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- Hydrodynamic fluctuations can be regarded as gravitational perturbations of black holes in asymptotic AdS spacetime
- $\frac{\eta_0}{s} = \frac{1}{4\pi}$: universal for Einstein gravity (coupled to matters) duals
- 2nd order hydrodynamics:
  Baier-Romatschke-Son-Starinets-Stephanov JHEP 0804 (2008) 100
  Bhattacharyya-Hubey-Minwalla-Rangamani JHEP 0802 (2008) 045
- Towards higher-order hydrodynamics: this talk! (see 1507.02461 for complete classification of 3rd order terms in the nonlinear level)

Novel realization of fluid/gravity correspondence

Bhattacharyya-Hubey-Minwalla-Rangamani JHEP 0802 (2008) 045

- Systematic framework to construct nonlinear fluid dynamics: order by order in boundary derivative expansion; transport coefficients completely determined

- Fluid stress tensor + Conservation laws $\longleftrightarrow$ Solving Einstein equations in asymptotic AdS spacetime

$$S = \frac{1}{16\pi G_N}\int d^5x\sqrt{-G}\,(R+12) + \frac{1}{8\pi G_N}\int d^4x\sqrt{-\gamma}K[\gamma] + S_{\mathrm{c.t.}},$$

$$E_{MN} = R_{MN} - \frac{1}{2}G_{MN}R - 6G_{MN} = 0 : \quad \text{Einstein equations},$$

$$T_{\mu\nu} \equiv -\lim_{r\to\infty} r^2 \frac{2}{\sqrt{-\gamma}}\frac{\delta S}{\delta\gamma^{\mu\nu}} : \quad \text{AdS/CFT dictionary}$$

- Boosted AdS_5 black hole solution

$$ds^2 = -2u_\mu dx^\mu dr - r^2 f(\mathbf{b}r)u_\mu u_\nu dx^\mu dx^\nu + r^2\left(\eta_{\mu\nu} + u_\mu u_\nu\right)dx^\mu dx^\nu,$$

$$f(r) = 1 - 1/r^4, \quad T = 1/(\pi\mathbf{b}), \quad \eta^{\mu\nu}u_\mu u_\nu = -1$$

- Promote constants u_μ and **b** to arbitrary functions of boundary coordinates x_α. Then, do gradient expansion of them around one specific point

$$u_\mu(x) = u_\mu(x_0) + (x - x_0)^\alpha \nabla_\alpha u_\mu(x_0) + \cdots ,$$
$$\mathbf{b}(x) = \mathbf{b}(x_0) + (x - x_0)^\mu \nabla_\mu \mathbf{b}(x_0) + \cdots .$$

- Metric corrections will be solved order by order in derivative expansion. Once done, fluid dynamics can be constructed via AdS/CFT dictionary.
- On-shell hydro: solutions to Einstein equations exit only when **b** and u_μ satisfy certain relations (obtained by solving constraint components of Einstein equations). The resultant stress tensor is conserved.

Derivative resummation in fluid/gravity correspondence

- Drive perturbation: promote boosted AdS_5 black hole solution

$$ds^2 = -2u_\mu(x)dx^\mu dr - r^2 f(\mathbf{b}(x)r)u_\mu(x)u_\nu(x)dx^\mu dx^\nu + r^2\Delta_{\mu\nu}dx^\mu dx^\nu,$$
$$f(r) = 1 - 1/r^4,\ \ T = 1/(\pi\mathbf{b}),\ \Delta_{\mu\nu} = g_{\mu\nu}(x) + u_\mu(x)u_\nu(x),\ x_\mu = (v, x_i)$$

- Boundary metric perturbation during the above promotion: $\eta_{\mu\nu} \to g_{\mu\nu}(x)$
- Instead of gradient expansion, we do linearization:

$$u_\mu(x) = (-1 + \epsilon h_{00}/2, \epsilon u_i(x)) + \mathcal{O}(\epsilon^2),\ \ \mathbf{b}(x) = \mathbf{b}_0 + \epsilon\mathbf{b}_1(x) + \mathcal{O}(\epsilon^2)$$
$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \epsilon h_{\mu\nu}(x) + \mathcal{O}(\epsilon^2)$$

- The seed metric, i.e. the linearized version of ds^2 ($\mathbf{b}_0 = 1$)

$$\begin{aligned} ds^2_{\text{seed}} =& 2drdv - r^2 f(r)dv^2 + r^2\delta_{ij}dx^i dx^j \\ & - \epsilon\Big[2u_i(x)drdx^i + \frac{2}{r^2}u_i(x)dvdx^i + \frac{4}{r^2}\mathbf{b}_1(x)dv^2 \\ & + h_{00}(x)drdv + \frac{1}{r^2}h_{00}(x)dv^2 - r^2 h_{\mu\nu}(x)dx^\mu dx^\nu\Big] + \mathcal{O}(\epsilon^2). \end{aligned}$$

- Bulk metric $ds^2 = G_{MN}dx^M dx^N = ds^2_{\rm seed} + ds^2_{\rm corr}$ where

$$ds^2_{\rm corr} = \epsilon\left(-3h drdv + \frac{k}{r^2}dv^2 + r^2 h\delta_{ij}dx^i dx^j + \frac{2}{r^2}j_i dvdx^i + r^2\alpha_{ij}dx^i dx^j\right)$$

under gauge fixing $G_{rr}=0, \quad G_{r\mu}\propto u_\mu, \quad {\rm Tr}\left[\left(G^{(0)}\right)^{-1}G^{(1)}\right]=0,$

- α_{ij} is traceless & symmetric
- $[h,\ k,\ j_i,\ \alpha_{ij}](r, x^\alpha) \dashrightarrow [h,\ k,\ j_i,\ \alpha_{ij}][u_\mu, h_{\mu\nu}] \longleftarrow$ Einstein equations
- Boundary conditions:
  - regularity of metric correction (at horizon $r=1$)
  - AdS requirement at $r=\infty$ (conformal boundary)
    $h<\mathcal{O}(r^0), \quad k<\mathcal{O}(r^4), \quad j_i<\mathcal{O}(r^4), \quad \alpha_{ij}<\mathcal{O}(r^0)$
  - Landau frame: $\Pi_{\langle\mu\nu\rangle}u^\nu=0$
- We will construct an off-shell hydro: to derive stress tensor, we only need to solve dynamical components of Einstein equations. We finally check the consistency of conservation laws of thus constructed stress tensor and the remaining constraint components of Einstein equations.

Dynamical components of Einstein equations

$$E_{rr}=0:\ \ 0=5\partial_r h+r\partial_r^2 h,\Longrightarrow h=0 \text{ by boundary conditions},$$

$$\begin{aligned}
E_{rv}=0:\ \ \partial_r k=&\ \underline{2r^2\partial u+\frac{1}{3}r\partial_v\partial u-2r^2\partial_k h_{0k}-\frac{1}{6}r\partial^2 h_{00}-\frac{1}{3}r\left(\partial_i\partial_j h_{ij}-\partial^2 h_{kk}\right)}\\
&\underline{+r^2\partial_v h_{kk}}-\frac{2}{3r^2}\partial j-\frac{1}{3}r\partial_i\partial_j\alpha_{ij}-\frac{1}{3r}\partial_r\partial j,
\end{aligned}$$

$$E_{ri}=0:\ \ 0=r\partial_r^2 j_i-3\partial_r j_i+r^3\partial_r\partial_j\alpha_{ij}\underline{+r\left(\partial^2 u_i-\partial_i\partial u\right)+3r^2\partial_v u_i-\frac{3}{2}r^2\partial_i h_{00}},$$

$$\begin{aligned}
E_{ij}=0:\ \ 0=&\ (r^7-r^3)\partial_r^2\alpha_{ij}+(5r^6-r^2)\partial_r\alpha_{ij}+2r^5\partial_v\partial_r\alpha_{ij}+3r^4\partial_v\alpha_{ij}+r^3\partial^2\alpha_{ij}\\
&-r^3\left(\partial_i\partial_k\alpha_{jk}+\partial_j\partial_k\alpha_{ik}-\frac{2}{3}\delta_{ij}\partial_k\partial_l\alpha_{kl}\right)+(1-r\partial_r)\left(\partial_i j_j+\partial_j j_i-\frac{2}{3}\delta_{ij}\partial j\right)\\
&\underline{+(3r^4+r^3\partial_v)\left(\partial_i u_j+\partial_j u_i-\frac{2}{3}\delta_{ij}\partial u\right)+3r^4\partial_v\left(h_{ij}-\frac{1}{3}\delta_{ij}h_{kk}\right)}\\
&\underline{-r^3\left(\partial_i\partial_j h_{00}-\frac{1}{3}\delta_{ij}\partial^2 h_{00}\right)-3r^4\left(\partial_i h_{0j}+\partial_j h_{0i}-\frac{2}{3}\delta_{ij}\partial_k h_{0k}\right)}\\
&\underline{+r^3\left(\partial^2 h_{ij}-\partial_i\partial_k h_{jk}-\partial_j\partial_k h_{ik}+\partial_i\partial_j h_{kk}-\frac{2}{3}\delta_{ij}\partial^2 h_{kk}+\frac{2}{3}\delta_{ij}\partial_k\partial_l h_{kl}\right)}.
\end{aligned}$$

underlined terms are source terms, composed of $u_i(x),\ h_{\mu\nu}(x)$ only. Under $SO(3)$ symmetry, all possible vector and tensor structures (traceless & symmetric) constructed from $u_i(x),\ h_{\mu\nu}(x)$:

Vector	Tensor (traceless & symmetric)
u_i	$t^1_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i - \frac{2}{3}\delta_{ij}\partial u\right)$
$\partial_i \partial u$	$t^2_{ij} = \partial_i\partial_j\partial u - \frac{1}{3}\delta_{ij}\partial^2\partial u$
h_{0i}	$t^3_{ij} = \frac{1}{2}\left(\partial_i h_{0j} + \partial_j h_{0i} - \frac{2}{3}\delta_{ij}\partial_k h_{0k}\right)$
$\partial_i\partial_k h_{0k}$	$t^4_{ij} = \partial_i\partial_j\partial_k h_{0k} - \frac{1}{3}\delta_{ij}\partial^2\partial_k h_{0k}$
$\partial_i h_{00}$	$t^5_{ij} = \partial_i\partial_j h_{00} - \frac{1}{3}\delta_{ij}\partial^2 h_{00}$
$\partial_i h_{kk}$	$t^6_{ij} = \partial_i\partial_j h_{kk} - \frac{1}{3}\delta_{ij}\partial^2 h_{kk}$
$\partial_k h_{ik}$	$t^7_{ij} = \frac{1}{2}\left(\partial_i\partial_k h_{jk} + \partial_j\partial_k h_{ik} - \frac{2}{3}\delta_{ij}\partial_k\partial_l h_{kl}\right)$
$\partial_i\partial_k\partial_l h_{kl}$	$t^8_{ij} = \partial_i\partial_j\partial_k\partial_l h_{kl} - \frac{1}{3}\delta_{ij}\partial^2\partial_k\partial_l h_{kl}$
	$t^9_{ij} = h_{ij} - \frac{1}{3}\delta_{ij}h_{kk}$

Strategy: PDEs$\longrightarrow$ ODEs by basis decomposition and Fourier transform:

$$j_i = V_1 u_i + V_2\partial_i\partial u + V_3 h_{0i} + V_4\partial_i\partial_k h_{0k} + V_5\partial_i h_{00} + V_6\partial_i h_{kk} + V_7\partial_k h_{ik} + V_8\partial_i\partial_k\partial_l h_{kl}$$

$$\alpha_{ij} = 2T_1 t^1_{ij} + T_2 t^2_{ij} + 2T_3 t^3_{ij} + T_4 t^4_{ij} + T_5 t^5_{ij} + T_6 t^6_{ij} + 2T_7 t^7_{ij} + T_8 t^8_{ij} + T_9 t^9_{ij}$$

Fourier transform: $(\partial_v, \partial_i) \longrightarrow (-i\omega, iq_i)$. So, $V_i, T_i \longrightarrow V_i(r, \omega, q^2),\ \ T_i(r, \omega, q^2)$.

We get partially coupled ODEs for $\{V_i,\ T_i\}$.

For viscosities, $\{V_1, V_2, T_1, T_2\}$: (arXiv: 1406.7222, 1409.3095)

$$\begin{cases} 0 = r\partial_r^2 V_1 - 3\partial_r V_1 - q^2 r^3 \partial_r T_1 \underline{-3i\omega r^2 - q^2 r}, \\ 0 = r\partial_r^2 V_2 - 3\partial_r V_2 + \dfrac{1}{3} r^3 \partial_r T_1 - \dfrac{2}{3} q^2 r^3 \partial_r T_2 \underline{-r}, \\ 0 = (r^7 - r^3)\partial_r^2 T_1 + (5r^6 - r^2)\partial_r T_1 - 2i\omega r^5 \partial_r T_1 \\ \qquad - 3i\omega r^4 T_1 + V_1 - r\partial_r V_1 \underline{-i\omega r^3 + 3r^4}, \\ 0 = (r^7 - r^3)\partial_r^2 T_2 + (5r^6 - r^2)\partial_r T_2 - 2i\omega r^5 \partial_r T_2 \\ \qquad - 3i\omega r^4 T_2 + 2V_2 - 2r\partial_r V_2 + \dfrac{1}{3} q^2 r^3 T_2 - \dfrac{2}{3} r^3 T_1. \end{cases}$$

We have similar ODEs for the remaining V_i's and T_i's (arXiv: 1502.08044)

We find some constraints among decomposition coefficients V_i, T_i. The idea is to make suitable linear combinations among V_i's (T_i's), which satisfy very similar ODEs as those of V_i's (T_i's) without source terms. Under the boundary conditions, these suitable combinations have to be zero. (arXiv: 1502.08044)

$$
\begin{aligned}
&T_9 - i\omega T_3 - q^2 T_7 = 0, && i\omega V_3 + q^2 V_7 = 0, \\
&2(T_1 + T_3) - 2i\omega T_5 - q^2(T_2 + T_4) = 0, && (V_1 + V_3) - 2i\omega V_5 - q^2(V_2 + V_4) = 0, \\
&2(T_6 + T_7) - i\omega T_4 - 2q^2 T_8 = 0, && 2V_6 + V_7 - i\omega V_4 - 2q^2 V_8 = 0, \\
&T_5 - T_6 - i\omega T_4 - q^2 T_8 = 0, && V_5 - V_6 - i\omega V_4 - q^2 V_8 + \frac{1}{2} r^3 = 0
\end{aligned}
$$

which are useful in rewriting the fluid stress tensor in a covariant way.

Near-boundary analysis

Near conformal boundary $r = \infty$,

$$V_1 \longrightarrow -i\omega r^3 + \mathcal{O}\left(\frac{1}{r}\right), \quad T_1 \longrightarrow \frac{1}{r} + \frac{t_1(\omega, q^2)}{r^4} + \mathcal{O}\left(\frac{1}{r^5}\right),$$
$$V_2 \longrightarrow -\frac{1}{3}r^2 + \mathcal{O}\left(\frac{1}{r}\right), \quad T_2 \longrightarrow \frac{t_2(\omega, q^2)}{r^4} + \mathcal{O}\left(\frac{1}{r^5}\right).$$

For remaining V_i's and T_i's, we can make similar analysis:

$$V_i \longrightarrow \cdots + \mathcal{O}\left(\frac{\log r}{r}\right), \;\; T_i \longrightarrow \cdots + \frac{t_i(\omega, q^2)}{r^4} + \cdots \times \frac{\log r}{r^4} + \mathcal{O}\left(\frac{\log r}{r^5}\right), \quad i = 3, 4, ..., 9$$

Constraints among t_i's:

$$t_9 - i\omega t_3 - q^2 t_7 = 0, \qquad 2(t_1 + t_3) - 2i\omega t_5 - q^2(t_2 + t_4) = 0,$$
$$2(t_6 + t_7) - i\omega t_4 - 2q^2 t_8 = 0, \quad t_5 - t_6 - i\omega t_4 - q^2 t_8 = 0.$$

$k \longrightarrow \cdots$ by direct integration near $r = \infty$.

Fluid stress tensor via holographic dictionary
Balasubramanian-Kraus Commun.Math.Phys.208 (1999) 413,
Henningson-Skenderis JHEP 9807 (1998) 023 $\cdots$

$$T_{\mu\nu} = \lim_{r\to\infty} \left\{ -2r^2 \left(K_{\mu\nu} - K\gamma_{\mu\nu} + 3\gamma_{\mu\nu} - \frac{1}{2}\mathcal{G}_{\mu\nu}(\gamma) \right) + 2\log\frac{1}{r^2}\, T^a_{\mu\nu} \right\},$$

$$T_{00} = 3 - 12\mathbf{b}_1 - 3h_{00}, \quad T_{0i} = T_{i0} = -4u_i + h_{0i},$$

$$\begin{aligned} T_{ij} = \delta_{ij}\left(1 - 4\mathbf{b}_1\right) + h_{ij} + \Bigg\{ & 8t_1 t^1_{ij} + 4t_2 t^2_{ij} + \left[8t_3 + \frac{1}{24} i\omega \left(3q^2 - 7\omega^2\right) \right] t^3_{ij} \\ & + \left(4t_4 + \frac{5}{36} i\omega \right) t^4_{ij} + \left[4t_5 - \frac{1}{144}\left(q^2 + 21\omega^2\right) \right] t^5_{ij} + \left[4t_6 + \frac{1}{144}\left(13q^2 - \omega^2\right) \right] t^6_{ij} \\ & + \left[8t_7 - \frac{1}{8}\left(\omega^2 + 3q^2\right) \right] t^7_{ij} + \left(4t_8 - \frac{7}{72} \right) t^8_{ij} + \left[4t_9 + \frac{1}{48}\left(7\omega^4 - 7q^4 - 6\omega^2 q^2\right) \right] t^9_{ij} \Bigg\} \end{aligned}$$

Through tensor algebras, $T_{\mu\nu} = (\varepsilon + P)u_\mu u_\nu + Pg_{\mu\nu} + \Pi_{\langle\mu\nu\rangle}$ with $\varepsilon = 3P = 3(\pi T)^3$

$$\Pi_{\langle\mu\nu\rangle} = -2\eta\nabla_{\langle\mu}u_{\nu\rangle} - \zeta\nabla_{\langle\mu}\nabla_{\nu\rangle}\nabla u + \kappa u^\alpha u^\beta C_{\langle\mu\alpha\nu\rangle\beta} + \rho u^\alpha \nabla^\beta C_{\langle\mu\alpha\nu\rangle\beta} + \xi\nabla^\alpha\nabla^\beta C_{\langle\mu\alpha\nu\rangle\beta} + \theta u^\alpha \nabla_\alpha R_{\langle\mu\nu\rangle} \quad \text{where}$$

$$\eta = -4t_1, \quad \zeta = -4t_2, \quad \theta = -6t_2 = 3\zeta/2,$$

$$\kappa = \frac{1}{6}\left[-48\,(t_5 + t_6) + 24i\omega\,(t_2 + 2t_4) + \omega^2\,(48t_8 - 1) - q^2\right],$$

$$\rho = -\frac{4}{3}\left[12\,(t_2 + t_4) + i\omega(1 - 24t_8)\right], \quad \xi = \frac{1}{12}\,(7 - 288t_8)$$

viscosity functions: $\eta(\omega, q^2)$, $\zeta(\omega, q^2)$; gravitational susceptibility of the fluid (GSF): $\kappa(\omega, q^2), \cdots, \rho,\ \xi,\ \theta$.

So far, we establish a map between the dynamical components of Einstein equation and transport coefficients. $T_{\mu\nu}$ is off-shell. We checked that constraint components of Einstein equations $E_{vv} = E_{vi} = 0$ reproduce $\nabla^\mu T_{\mu\nu} = 0$.

Notice: to determine values of transport coefficients, we have to solve these ODEs.

Results I: hydro expansion

- In hydro limit $\omega, q \ll 1$, viscosities and GSFs are expandable

$$\eta = 1 + \frac{1}{2}(2-\ln 2)\, i\omega - \frac{1}{8}q^2 - \frac{1}{48}\left[6\pi - \pi^2 + 12\left(2 - 3\ln 2 + \ln^2 2\right)\right]\omega^2 + \cdots$$

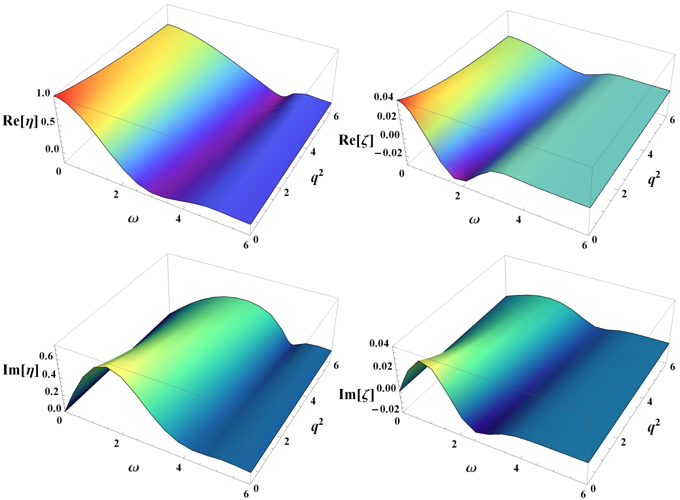
$$\zeta = \frac{1}{12}(5 - \pi - 2\ln 2) + \cdots,$$

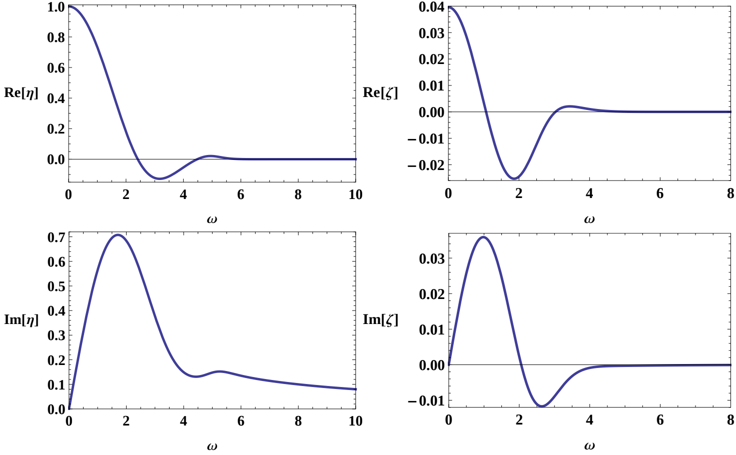
$$\kappa = 2 + \frac{1}{4}(5 + \pi - 6\ln 2)\, i\omega + \cdots, \quad \rho = 2 + \cdots, \quad \xi = \ln 2 - \frac{1}{2}, \quad \theta = \frac{3}{2}\zeta.$$

- $\eta_0/s = 1/(4\pi)$; relaxation time $\tau_{\mathrm{R}} = (2 - \ln 2)/2$
- sound wave dispersion relation gets correction

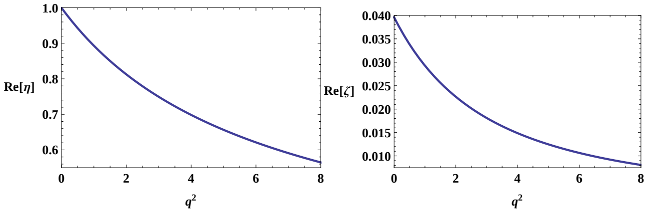
$$\omega = \pm\frac{1}{\sqrt{3}}q - \frac{i}{6}q^2 \pm \frac{1}{24\sqrt{3}}(3 - 2\ln 2)\, q^3 - \frac{1}{864}\left(\pi^2 - 24 + 24\ln 2 - 12\ln^2 2\right) q^4 + \cdots$$

Results II: viscosity functions





Viscosity functions η and ζ vs ω at $q^2 = 0$



Viscosity functions η and ζ vs q^2 at $\omega = 0$

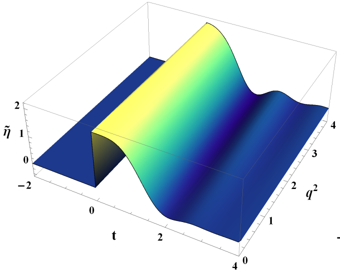
- Viscosities vanish at large momentum, necessary for causality
- Viscosities oscillate when increasing momentum, consistent with expectation that viscosities have infinitely many poles in complex momentum space
- Weak dependence on spatial momentum, so dissipation is quasi-local in space
- ζ is suppressed with respect to η.

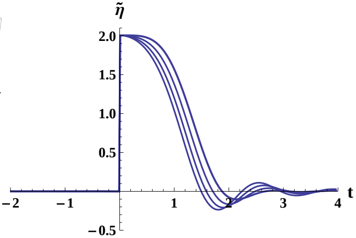
Results III: memory function

$$\Pi_{\langle\mu\nu\rangle}(t) = -\int_{-\infty}^{\infty} dt' 2\tilde{\eta}(t-t', q^2)\nabla_{\langle\mu}u_{\nu\rangle}(t') + \cdots,$$

$$\tilde{\eta}(t, q^2) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \eta(\omega, q^2)e^{-i\omega t}d\omega : \quad \text{memory function;} \quad \cdots$$

Causal theory requires that $\Pi_{\langle\mu\nu\rangle}(t)$ should be affected only by the state of the system in the past. Equivalently, memory function $\tilde{\eta}(t-t') \sim \Theta(t-t')$.





For a causal theory, we have

$$\Pi_{\langle\mu\nu\rangle}(t) = -\int_{-\infty}^{t} dt' 2\tilde{\eta}(t-t', q^2)\nabla_{\langle\mu}u_{\nu\rangle}(t') + \cdots$$

A typical memory function-based formalism would set the low limit of above integral to zero: $-\infty \to 0$. (Kadanoff & Martin Annals Phys. 24 (1963) 419).

$$\text{As a model}: \quad \Pi_{\langle\mu\nu\rangle}(t) = -\int_{0}^{t} dt' 2\tilde{\eta}(t-t', q^2)\nabla_{\langle\mu}u_{\nu\rangle}(t') + \cdots$$

$$\Longrightarrow \text{ initial value problem } \nabla^{\mu}T_{\mu\nu} = 0 \text{ well-defined}$$

We also computed values of GSFs for generic momenta: some of them do not vanish at large momenta, encoding the interference between thermal physics and vacuum pair production, as seen from thermal correlators.

- Two-point correlators from constitutive relation for $T_{\mu\nu}$. To correctly recover the dispersion relations in shear and sound channels (as seen from these two-point correlators), we have to make use of the energy momentum conservation.
- Hydrodynamics dual to Einstein-Gauss-Bonnet gravity: all-order gradient resummation arXiv: 1504.01370
  - For an infinitesimal Gauss-Bonnet coupling, we worked out Gauss-Bonnet corrected viscosities $\eta^{(1)}$, $\zeta^{(1)}$. We also did inverse Fourier transform and found that $\tilde{\eta}^{(1)}$ (also $\tilde{\zeta}^{(1)}$) still has no support in the future.
  - Einstein-Gauss-Bonnet gravity is causal for an infinitesimal Gauss-Bonnet coupling, consistent with previous conclusions.

Summary

- We have consistently determined $T_{\mu\nu}$ of a relativistic conformal fluid ($\mathcal{N} = 4$ SYM at $T \neq 0$). We found that all order derivative terms in $T_{\mu\nu}$ are fully encoded in two momenta-dependent viscosity functions $\eta(\omega, q^2)$ and $\zeta(\omega, q^2)$, and four momenta-dependent GSFs $(\kappa, \rho, \xi, \theta)$.
- At large momenta, the viscosities vanish, as required by causality of relativistic fluid dynamics. Due to infinitely many time derivatives, we turn to memory function approach and make initial value problem well-defined. The infinitely many time derivative was then absorbed into the complicated memory function. The memory function has no support in the future time, implying recovery of causality.

Thanks for your attention!