

CRM, Montreal, October 21, 2015

# Holographic Yang-Mills at finite $\theta$ angle

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- *FB, Aldo L. Cotrone, Roberto Sisca, JHEP 1508 (2015) 090*
- *FB, Aldo L. Cotrone, JHEP 1501 (2015) 104*

# Plan

- Motivations:  $\theta$ -angle in Yang-Mills
- Holographic Yang-Mills at finite  $\theta$ -angle
- Conclusions

# Plan

- Motivations:  $\theta$ -angle in Yang-Mills
- Holographic Yang-Mills at finite  $\theta$ -angle
- Conclusions

Effects of the  $\theta$  parameter interesting but challenging.

# The $\theta$ -angle in Yang-Mills

- Euclidean Lagrangian

$$\mathcal{L}_\theta = \frac{1}{2g_{YM}^2} \text{Tr} F_{\mu\nu} F^{\mu\nu} - \frac{i}{32\pi^2} \theta \epsilon^{\mu\nu\rho\sigma} \text{Tr} F_{\mu\nu} F_{\rho\sigma}$$

- $\theta$  term breaks P,T and hence CP
- $\theta$  multiplies topological charge density  $q(x)$ , whose 4d integral is integer.

$$q(x) = \frac{1}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{Tr} F_{\mu\nu} F_{\rho\sigma}, \quad \int dx^4 q(x) = n \in \mathbb{Z}$$

- Hence  $\theta$  is an angle. Physics invariant under  $\theta \rightarrow \theta + 2\pi$

$$Z[\theta] = \int D[A] e^{-S_\theta}$$

- $\theta$ -dependence due to instantons . Not visible in perturbation theory.

# The $\theta$ -angle in Yang-Mills

- Euclidean Lagrangian

$$\mathcal{L}_\theta = \frac{1}{2g_{YM}^2} \text{Tr} F_{\mu\nu} F^{\mu\nu} - \frac{i}{32\pi^2} \theta \epsilon^{\mu\nu\rho\sigma} \text{Tr} F_{\mu\nu} F_{\rho\sigma}$$

- Effects of  $\theta$  interesting:
  - vacuum structure
  - CP violating effects in QGP [Kharzeev et al]
  - mass and interactions of  $\eta'$  meson in QCD [Witten-Veneziano]
  - cosmology (axions)
- In real world QCD,  $|\theta| < 10^{-10}$  (from neutron EDM). Strong CP problem.

# The $\theta$ -angle in Yang-Mills

- Euclidean Lagrangian

$$\mathcal{L}_\theta = \frac{1}{2g_{YM}^2} \text{Tr} F_{\mu\nu} F^{\mu\nu} - \frac{i}{32\pi^2} \theta \epsilon^{\mu\nu\rho\sigma} \text{Tr} F_{\mu\nu} F_{\rho\sigma}$$

- Real  $\theta$ -angle challenging for Lattice (sign problem: imaginary term)
- Go to imaginary  $\theta$ , then analytically continue to real around  $\theta=0$
- Alternatively, compute n-point correlators of topological charge at  $\theta=0$
- Lattice results obtained in this way: first few terms in  $\theta$  expansion

# The $\theta$ -angle in Lattice Yang-Mills

- Ground state energy density [Vicari, Panagopoulos, 08; Bonati, D'Elia, Vicari, Panagopoulos, 13]

$$f(\theta) - f(0) = \frac{1}{2} \chi_g \theta^2 \left[ 1 + \bar{b}_2 \frac{\theta^2}{N_c^2} + \bar{b}_4 \frac{\theta^4}{N_c^4} + \mathcal{O}(\theta^6) \right] \quad \chi_g = f''(\theta)|_{\theta=0} \neq 0 \text{ topological susceptibility}$$

$$\bar{b}_2 \approx -0.2 \text{ (from } N_c=3, \dots, 6 \text{ data)} \quad |b_4| < 0.001 \quad (N_c=3)$$

- String tension [Del Debbio, Manca, Panagopoulos, Skouropathis, Vicari, 06 ]

$$T_s(\theta) = T_s(0) \left[ 1 + \bar{s}_2 \frac{\theta^2}{N_c^2} + \mathcal{O}(\theta^4) \right], \quad \bar{s}_2 \approx -0.9 \text{ (from } N_c=3, \dots, 6 \text{ data)}$$

- Lowest  $0^{++}$  glueball mass [Del Debbio, Manca, Panagopoulos, Skouropathis, Vicari, 06 ]

$$M_{0^{++}}(\theta) = M_{0^{++}}(0) \left[ 1 + g_2 \theta^2 + \mathcal{O}(\theta^4) \right], \quad g_2 \approx -0.06(2), \quad (N_c = 3)$$

- Deconfinement temperature [D'Elia, Negro, 12, 13]

$$T_c(\theta) = T_c(0) \left[ 1 + R_\theta \theta^2 + \mathcal{O}(\theta^4) \right], \quad R_\theta \approx -0.0175(7), \quad (N_c = 3)$$

## The $\theta$ -angle in large N Yang-Mills

$$\mathcal{L} = \frac{N_c}{2\lambda} \left[ \text{Tr}F^2 - i\frac{\lambda}{8\pi^2} \frac{\theta}{N_c} \text{Tr}F\tilde{F} \right]$$

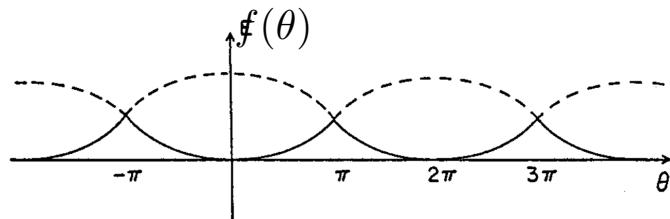
- 't Hooft limit:  $N_c \gg 1$ ,  $\lambda = g_{\text{YM}}^2 N_c$  fixed
- Non trivial  $\theta$  dependence (large N solution of  $U(1)_A$  problem) if  $\theta/N$  fixed.
- Ground state energy density should scale as

$$f(\theta) \equiv \varepsilon(\theta) - \varepsilon(0) = N^2 h \left[ \frac{\theta}{N} \right]$$

- **Puzzle:** should also be periodic in  $\theta$  with  $2\pi$  periodicity.

# The $\theta$ -angle in large N Yang-Mills

- Solution [Witten, 1980]:  $f(\theta)$  multi-branched



$$f(\theta) = N^2 \min_k h \left[ \frac{\theta + 2k\pi}{N} \right]$$

- At  $\theta=(2k+1)\pi$ : expect first order transitions. CP spontaneously broken
- Minimum at  $\theta=0$  (integrand of Euclidean path int. real and positive),  $k=0$ .
- Large N solution of  $U(1)_A$  problem

$$\chi_\infty = \frac{f_{\eta'}^2 m_{\eta'}^2}{4N_f} + O(1/N) \quad \chi_\infty = f''_{YM}(\theta)|_{\theta=0} \neq 0 \text{ topological susceptibility}$$

$$f(\theta) = b N^2 \min_k \left( \frac{\theta + 2k\pi}{N} \right)^2 + O(\theta^4/N^4)$$

# Plan

- Motivations:  $\theta$ -angle in Yang Mills
- Holographic Yang-Mills at finite  $\theta$ -angle
- Conclusions

Exact  $\theta$ -dependence in a non-susy large N Yang-Mills model  
with dual gravity description. Explicit realization of expected YM features.  
Qualitative matching with lattice YM at small  $\theta$ . Benchmark beyond small  $\theta$  ?

## Witten's holographic Yang-Mills

- $N_c$  D4-branes in IIA string theory
- Low energy physics: 5d  $SU(N_c)$  Super-Yang-Mills theory.
- $N_c$  D4-branes on  $S^1_{x4}$  of radius  $R_4 = 1/M_{KK}$  with antiperiodic fermions.
- Low energy: 4d non-susy  $SU(N_c)$  Yang-Mills + adjoint KK modes [Witten 1998]
- Holography: Large  $N_c$ , strong coupling regime mapped into dual gravity description (open/closed string duality)
- Can add  $\theta$  term to the model, no sign problem.
- To leading order in  $\theta/N_c$  done in [Witten 1998].

## Witten's holographic Yang-Mills

- **Gravity action** (closed string description), sourced by the N D4-branes

$$S = \frac{1}{2k_0^2} \int d^{10}x \sqrt{-g} \left[ e^{-2\phi} (\mathcal{R} + 4(\partial\phi)^2) - \frac{1}{2}|F_4|^2 - \frac{1}{2}|F_2|^2 \right]$$

- **Gauge theory action** (open string description), wrapped D4-branes (IR expansion)

$$S = \frac{1}{8\pi g_s l_s M_{KK}} \int d^4x \text{Tr} F_{\mu\nu} F^{\mu\nu} - \frac{i}{32\pi^2 l_s} \int_{S_{x^4}} C_1 \int d^4x \epsilon^{\mu\nu\rho\sigma} \text{Tr} F_{\mu\nu} F_{\rho\sigma}$$

$$F_2 = d C_1 \quad \lambda_4 = g_{YM}^2 N_c = 2\pi g_s l_s M_{KK} N_c \quad \boxed{\theta + 2\pi k = \frac{1}{l_s} \int_{S_{x_4}} C_1}$$

- **Holography**: gravity picture dual to gauge theory at  $\lambda_4 \gg 1$ ,  $N_c \gg 1$
- **Leading order in  $\theta/N_c$**  : treat  $F_2$  as a probe [Witten 1998].

# The $\theta$ -backreacted gravity solution

[Barbon, Pasquinucci 99; Dubovsky, Lawrence, Roberts 2011] ( $x_4 \sim x_4 + 2\pi/M_{KK}$ )

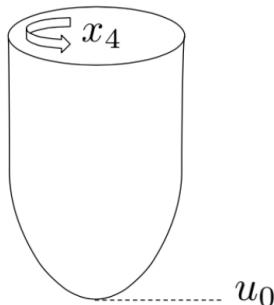
$$ds_{10}^2 = \left(\frac{u}{R}\right)^{3/2} \left[ \sqrt{H_0} dx_\mu dx^\mu + \frac{f}{\sqrt{H_0}} dx_4^2 \right] + \left(\frac{R}{u}\right)^{3/2} \sqrt{H_0} \left[ \frac{du^2}{f} + u^2 d\Omega_4^2 \right]$$

$$f = 1 - \frac{u_0^3}{u^3}, \quad H_0 = 1 - \frac{u_0^3}{u^3} \frac{\Theta^2}{1 + \Theta^2} \quad e^\phi = g_s \left(\frac{u}{R}\right)^{3/4} H_0^{3/4}, \quad C_1 = \frac{\Theta}{g_s} \frac{f}{H_0} dx^4, \quad F_4 = 3R^3 \omega_4$$

$$\int_{S^4} F_4 = 8\pi^3 l_s^3 g_s N_c, \quad R = (\pi g_s N_c)^{1/3} l_s \quad \theta + 2\pi k = \frac{1}{l_s} \int_{S_{x_4}} C_1$$

$$\Theta \equiv \frac{\lambda_4}{4\pi^2} \left( \frac{\theta + 2k\pi}{N_c} \right)$$

$$u_0 = \frac{4R^3}{9} M_{KK}^2 \frac{1}{1 + \Theta^2}$$



- $(u, x_4)$  subspace is a cigar
- $g_{00}(u_0) \neq 0$  (regular) : confinement
- KK modes NOT decoupled
- Small curvature if  $\Theta \ll \lambda_4^{1/4}$

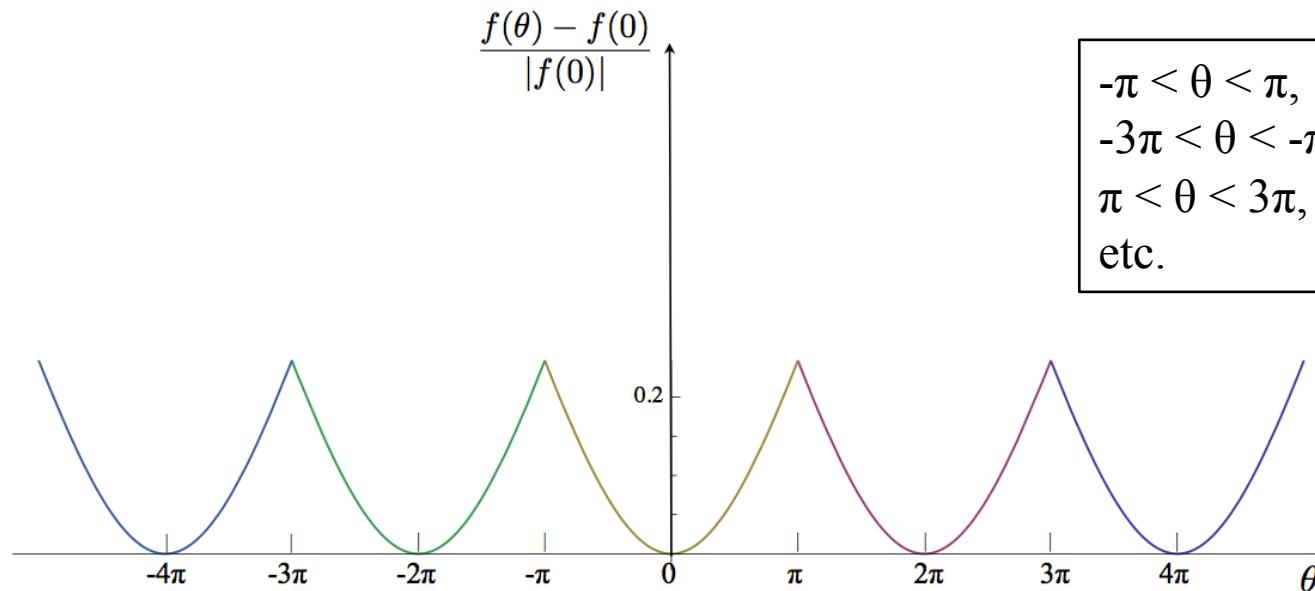
# The ground-state energy density

From holographic relation  $Z = e^{-V_4 f(\theta)} \approx e^{-S_{E \text{ on-shell}}^{\text{ren}}}$

$$f(\Theta) = -\frac{2N_c^2 \lambda_4}{3^7 \pi^2} \frac{M_{KK}^4}{(1 + \Theta^2)^3}$$

$$\Theta \equiv \frac{\lambda_4}{4\pi^2} \left( \frac{\theta + 2k\pi}{N_c} \right)$$

$$f(\theta) = \min_k f(\Theta)$$



Expected structure explicitly realized

# The ground-state energy density

- Expansion around  $\theta=0$

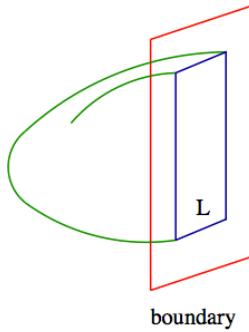
$$f(\theta) - f(0) = \frac{1}{2} \chi_g \theta^2 \left[ 1 + \bar{b}_2 \frac{\theta^2}{N_c^2} + \bar{b}_4 \frac{\theta^4}{N_c^4} + \mathcal{O}(\theta^6) \right]$$

$$\chi_g = \frac{\lambda_4^3 M_{KK}^4}{4(3\pi)^6} \quad \bar{b}_2 = -\frac{\lambda_4^2}{8\pi^4}, \quad \bar{b}_4 = \frac{5\lambda_4^4}{384\pi^8}$$

- Cfr. with Lattice Yang-Mills:  $\bar{b}_2 \approx -0.2$        $|b_4| < 0.001$ 
  - Qualitative agreement with Lattice
  - Prediction:  $b_4 > 0$  (it would be nice to check it on the lattice)

# The string tension

Rectangular Wilson loop: from minimal open string surface [Maldacena; Rey, Yee '98 ]



$$W(\mathcal{C}) = \text{Tr} \left[ P \exp \left( i \oint_{\mathcal{C}} A \right) \right] \sim e^{-S_{NG}^r}$$

$$\langle W[\mathcal{C}] \rangle \approx e^{-TV(L)} \quad V(L) = T_s L$$

$$S_{NG}^r \approx -\frac{1}{2\pi\alpha'} T \sqrt{-g_{00}g_{xx}}|_{u=u_0} L \quad T_s = \frac{1}{2\pi\alpha'} \sqrt{-g_{00}g_{xx}}|_{u=u_0}$$

$$T_s = \frac{2\lambda_4}{27\pi} M_{KK}^2 \frac{1}{(1 + \Theta^2)^2}$$

$$T_s = \frac{2\lambda_4}{27\pi} M_{KK}^2 \left( 1 - \frac{\lambda_4^2}{8\pi^4} \frac{\theta^2}{N_c^2} + \frac{3\lambda_4^4}{256\pi^8} \frac{\theta^4}{N_c^4} + \mathcal{O}(\theta^6) \right)$$

$$T_{s\,lat} = T_{s\,lat}(0) \left[ 1 + \bar{s}_2 \frac{\theta^2}{N_c^2} + \dots \right], \quad \bar{s}_2 \approx -0.9$$

# ‘t Hooft loop and oblique confinement

- Monopole-antimonopole potential from minimal action of D2 wrapping  $S_{x^4}$

$$S_{D2} = -T_2 \int d^3\xi e^{-\hat{\phi}} \sqrt{-\det(g + \mathcal{F})} + T_2 \int \hat{C}_1 \wedge \mathcal{F},$$

- Equivalent to M2-brane wrapping the  $y = \Theta x_4$  cycle in

$$ds_{11}^2 = e^{-\frac{2}{3}\hat{\phi}} ds_{10}^2 + e^{\frac{4}{3}\hat{\phi}} (dy - \hat{C}_{x_4} dx_4)^2.$$

- ‘t Hooft loop has an area law at finite theta (magnetic screening only at theta=0)

$$T_m = \frac{1}{27\pi^2} M_{KK}^2 \lambda_4 \frac{|\theta + 2k\pi|}{(1 + \Theta^2)^2} \equiv T_s \frac{|\theta + 2k\pi|}{2\pi}$$

- Dyons are screened under certain conditions (oblique confinement)

$$T_{dy} = -pT_s + qT_m = \left( -p + \frac{\theta}{2\pi} q \right) T_s \quad \theta = 2\pi(p/q)$$

See also [Gross, Ooguri, 98]

# The scalar glueball mass

- $0^{++}$  glueball spectrum  $\rightarrow \langle \text{Tr}F^2(x)\text{Tr}F^2(y) \rangle = \sum_n c_n e^{-M_n|x-y|}$
- Solve e.o.m of dual gravity scalar field: a metric fluctuation in 11d. Reducing on  $S^4$ :

$$h_{ab} = H_{ab}(u)e^{-ik\cdot x} \quad H_{ab}(u) = \frac{u}{R}H(u) \text{ diag} \left( 0, 1, 1, 1, 0, -\frac{3}{1+\Theta^2}, 0 \right)$$

$$H''(u) + \frac{4u^3 - u_0^3}{u(u^3 - u_0^3)} H'(u) - \frac{M^2 R^3}{u^3 - u_0^3} H(u) = 0$$

- Regularity at  $u=u_0$  and UV normalizability **only if  $M^2>0$  and discrete**

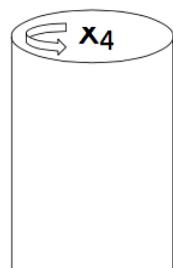
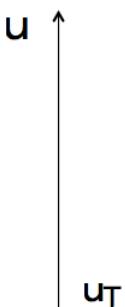
$$M(\Theta) = \frac{M(\Theta=0)}{\sqrt{1+\Theta^2}}$$

$$M(\theta) = M(\theta=0) \left( 1 - \frac{\lambda_4^2}{32\pi^4} \frac{\theta^2}{N_c^2} + \frac{3\lambda_4^4}{2048\pi^8} \frac{\theta^4}{N_c^4} + \mathcal{O}(\theta^6) \right)$$

$$M_{lat}(\theta) = M_{lat}(0) \left( 1 + g_2 \theta^2 + \mathcal{O}(\theta^4) \right), \quad g_2 = -0.06(2)$$

# Finite temperature

Take Euclidean time periodic with period  $1/T$ . **Two possible gravity solutions**



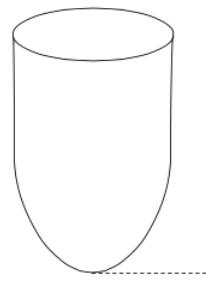
$$ds^2 = \left(\frac{u}{R}\right)^{3/2} \left[ \tilde{f}(u) dx_0^2 + dx_a dx^a + dx_4^2 \right] + \left(\frac{u}{R}\right)^{-3/2} \left[ \frac{du^2}{\tilde{f}(u)} + u^2 d\Omega_4^2 \right],$$

$$\tilde{f}(u) = 1 - \frac{u_T^3}{u^3}.$$

$$T > T_c$$

- black hole solution
- $g_{00}(u_T) = 0$  : deconfinement
- no theta dependence

$$C_1 \sim \theta dx_4, \quad F_2 = 0 \quad [(u, x_4) : \text{cylinder}]$$



$$T < T_c$$

$$u_0$$

- Just Euclidean extension of one discussed before
- Theta-dependence
- confinement

# Deconfinement temperature

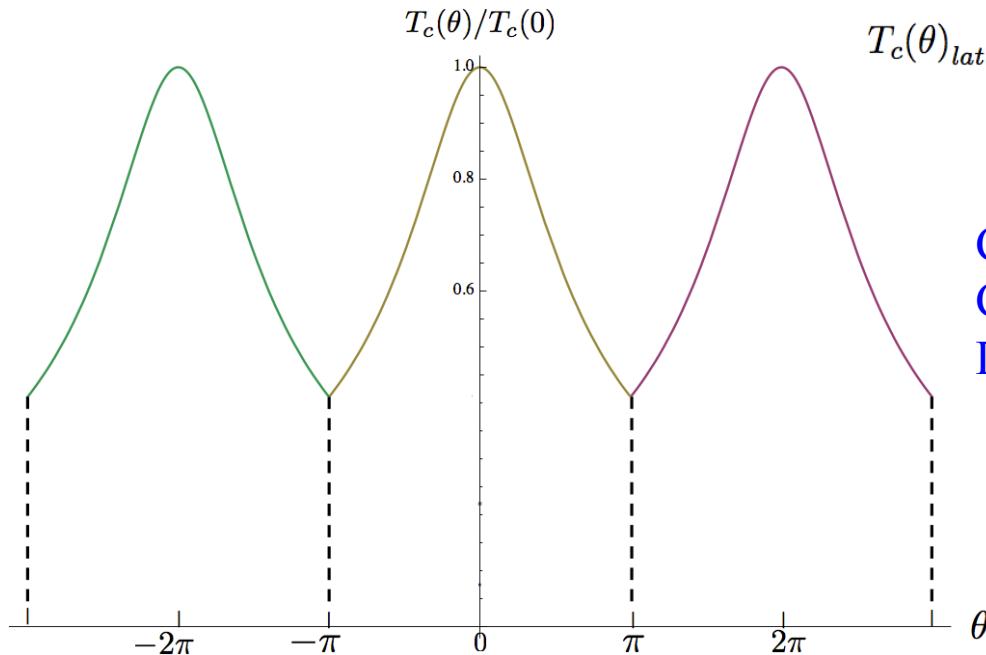
Compare the free energy densities of confined and deconfined phase

$$f = -p = -\frac{2N_c^2\lambda_4}{3^7\pi^2} \frac{M_{KK}^4}{(1+\Theta^2)^3} \equiv \frac{f(0)}{(1+\Theta^2)^3}$$

$$f_{dec} = -p_{dec} = -\frac{1}{6} \frac{256N_c^2\pi^4\lambda_4}{729M_{KK}^2} T^6$$

$$T_c(\Theta) = \frac{M_{KK}}{2\pi} \frac{1}{\sqrt{1+\Theta^2}}$$

$$T_c(\theta) = \frac{M_{KK}}{2\pi} \left[ 1 - \frac{\lambda_4^2}{32\pi^4} \frac{\theta^2}{N_c^2} + \frac{3\lambda_4^4}{2048\pi^8} \frac{\theta^4}{N_c^4} + \mathcal{O}(\theta^6) \right]$$



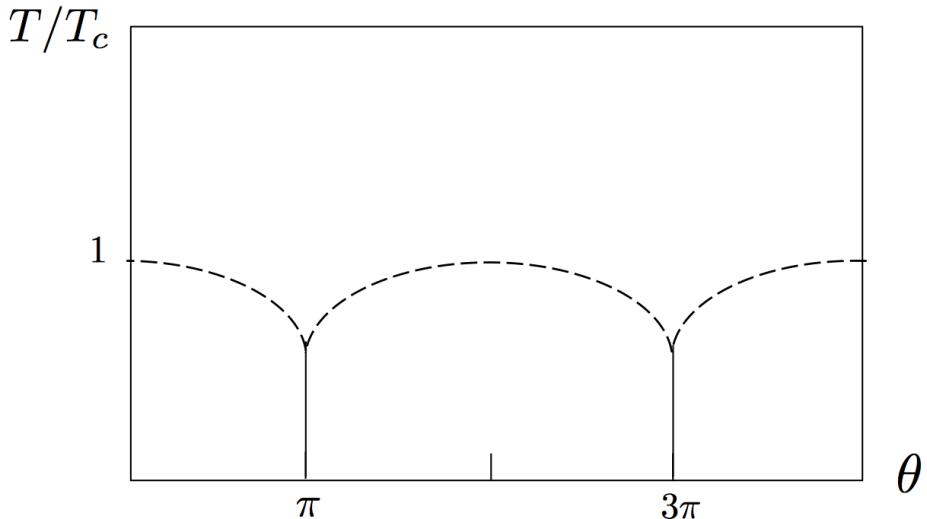
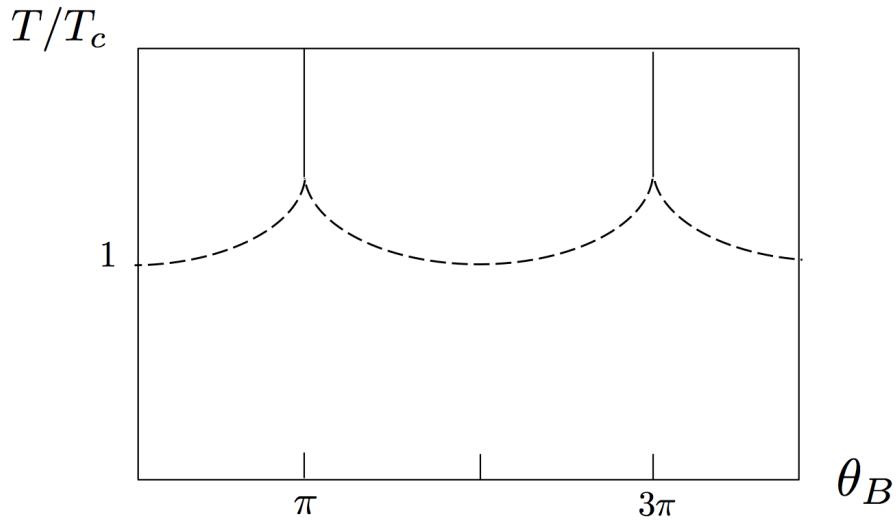
$$T_c(\theta)_{lat} = T_c(0)_{lat} [1 - R_\theta \theta^2 + \mathcal{O}(\theta^4)] , \quad R_\theta = 0.0175(7)$$

Cusps: tri-critical points  
 Colored: deconf. first order transition  
 Dashed: CP-breaking first order transition

- Qualitative comparison with Roberge-Weiss phase diagram of flavored theory (Witten-Sakai-Sugimoto) at finite imaginary chemical potential (see Aldo's talk)

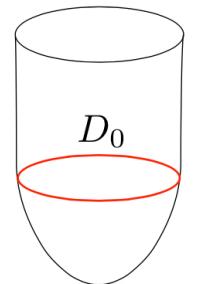
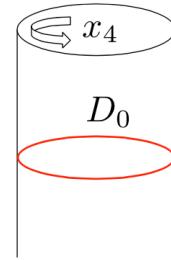
$$Z(T, \theta_B) = \text{Tr} \left( e^{-\frac{H}{T}} e^{i\theta_B B} \right)$$

- $\mu = i \mu_B$ ;  $\theta_B = \mu_B/T$ .  $B$ = baryon number operator.
- Confined phase: d.o.f (hadrons) have integer  $B$ . is  $2\pi$  periodic in  $\theta_B$
- Deconfined phase: quarks,  $B/N_c$  fractional charge.  $Z$  function of  $\theta_B/N_c$

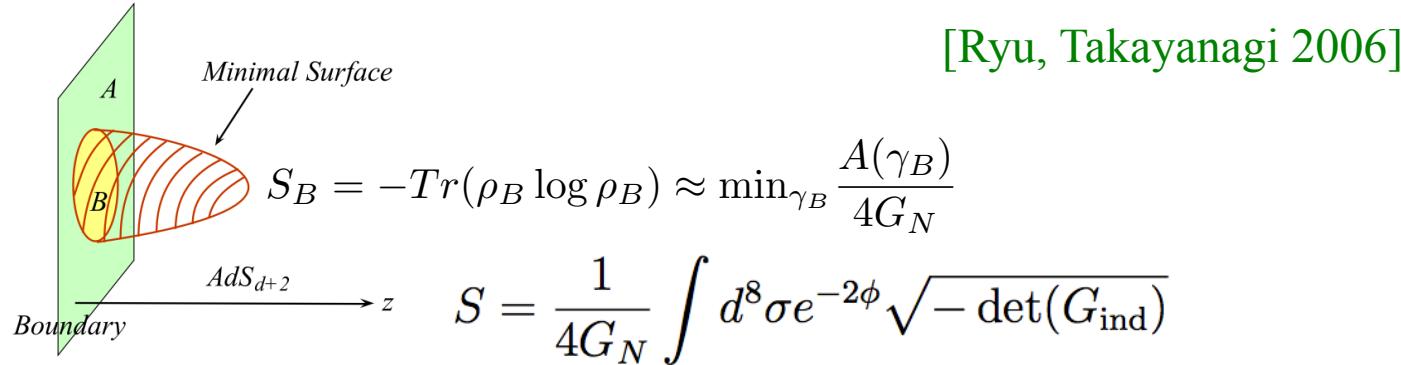


Note: curious qualitative “duality” as noticed for YM and QCD in [D'Elia, Negro, 2013]

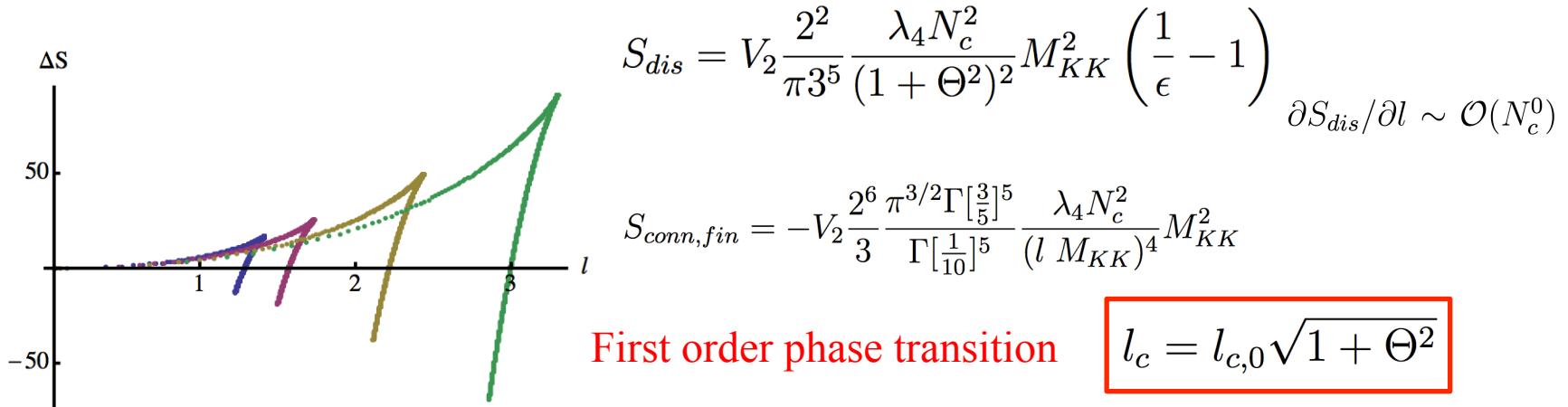
- Comments on the involved degrees of freedom
- $T > T_c$ :
  - $\theta$ : Euclidean D0-branes on  $S_{x4}$  stable: **instanton gas**
  - $\theta_B$ : quarks (fractional baryonic charge)
- $T < T_c$ :
  - $\theta_B$ : baryon gas
  - $\theta$ : Euclidean D0-branes on  $S_{x4}$  vanishing:  
“instanton quarks” (fractional instantonic charge) ?



# Entanglement entropy



Slab geometry:  $B = \{x \in [-l/2, l/2]\}$  [Klebanov, Kutasov, Murugan 07 at  $\theta=0$ ]



# Overview

- Holographic YM results exact in  $\theta$ , large  $N$
  - Observables are those at  $\theta=0$  multiplied by powers of  $(1+\Theta^2)$
  - A factor of  $(1+\Theta^2)^{-1/2}$  for each power of  $M_{KK}$
  - A factor of  $(1+\Theta^2)^{-1}$  for each power of  $\lambda_4$
- 
- Mass scales reduced by  $\theta$  (checked also baryon vertex mass)
  - Structure of  $(T,\theta)$  phase diagram explicitly
  - Agreement with lattice trends at small  $\theta$
  - Benchmarks for subleading coefficients in  $\theta$  expansion

# Work in progress

- $\theta$  dependence in Holographic QCD (Witten-Sakai-Sugimoto)
  - With massless flavors no  $\theta$ -dependence.
  - $\theta$  rotated away by chiral rotations as expected in QCD.
  - Holographically realized by anomaly inflow:  $C_1$  transforms under a gauge transformation of the D8-brane gauge field dual to  $U(1)_A$  [Sakai, Sugimoto 04]
- Take massive quarks: [Aharoni, Kutasov; Hashimoto et al, 2008]
- Energy density at small  $m$ :  $F(\theta) - F(0) = N_f m \Sigma [1 - \cos(\theta/N_f)]$
- Topological susceptibility:
$$\chi = \left. \frac{\partial^2 F(\theta)}{\partial \theta^2} \right|_{\theta=0} = \frac{m \Sigma}{N_f}$$
- Deduce other  $\theta$ -dependent observables

# Thank you