Cluster Duality & Mirror Symmetry for Grassmannians

Joint work w/ Konni Rietsch

Mirror symmetry:

- vast program, initiated by physicists ~1980's, providing dictionary between certain varieties & their "mirror dual" varieties
- 2 sides of the dictionary referred to as A-model & B-model

For smooth projective Fano varieties over $\mathbb{C}$ (including G/P):

<u>A-model</u> variety X | <u>B-model</u> variety $X^\vee$ w/ superpotential

- includes Gromov Witten invariants, which can be encoded by the small quantum cohom ring $\longleftrightarrow$ mirror symmetry encodes data on LHS in terms of oscillating integrals of regular function called <u>superpotential</u>

Mirror symmetry for flag varieties:

- Initiated by Givental for type A Fl_n, who gave a formula for the superpotential W
- Rietsch: generalized Givental's superpotential (& some of his work) to G/P
- Marsh-Rietsch: examined Grassmannian case in detail, & in particular gave explicit formula for superpotential W in terms of Plucker coordinates (cluster variables)

- Today — exhibit one aspect of mirror symmetry for Grassmannians.

Overview

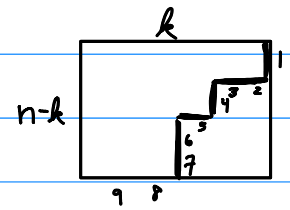
A-model | **B-model**

$X = Gr_{n-k}(\mathbb{C}^n)$

$\check{X}$ = complement of anti-canon divisor in $Gr_k(\mathbb{C}^n)$
- Remove locus where any of cyclically consec Plucker coords vanish

Let $N := k(n-k)$

partition $\leftrightarrow$ vertical steps



partition $\leftrightarrow$ horizontal steps

Plucker coordinates indexed by Young diagrams $\leq (n-k) \times n$

P_J for $J \in \binom{[n]}{n-k}$ | P_K for $K \in \binom{[n]}{k}$

Reduced plabic graph G w/ perm $\pi_{k,n}$ | Same plabic graph G

"plabic" chart $\Phi_G : (\mathbb{C}^*)^N \to X$
(chart for cluster $\mathcal{X}$-variety)

cluster chart $\Phi_G^\vee : (\mathbb{C}^*)^N \to \check{X}$
(chart for cluster $\mathcal{A}$-variety)

Newton-Okounkov body NO_G

$$\left(\frac{\text{NO-body}}{\text{arbitrary variety}} = \frac{\text{moment polytope}}{\text{toric variety}} \right)$$

Write superpotential $W : \check{X} \to \mathbb{C}(q)$ in terms of $\Phi_G^\vee$, & "tropicalize" it

NO_G is polytope, defined as convex hull of integer points

polytope Q_G, defined by inequalities

$$\boxed{\underline{\text{Theorem}} \text{ (Rietsch–W): } NO_G = Q_G}$$

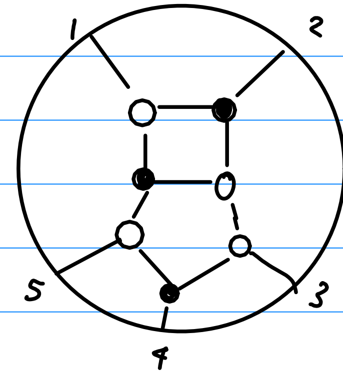
- Related to work of Berenstein Kazhdan (geom crystals), & should be related to Goncharov-Shen, & Gross–Hacking–Keel–Kontsevich

Def: The Grassmannian $Gr_k(\mathbb{C}^n)$ is the set of all k-planes in $\mathbb{C}^n$.
Represent element of $Gr_k(\mathbb{C}^n)$ by full rank $k \times n$ matrix M.
For $I \in \binom{[n]}{k}$, $\Delta_I(M) =$ det of $k \times k$ minor of M located in columns I.
Plucker coordinates.

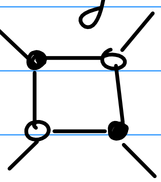
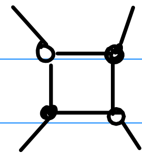
Need combinatorics of plabic graphs (Postnikov)

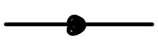
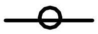
Def: A plabic graph is a planar graph G drawn inside a disk w/ n boundary vertices on the disk, labeled $1, 2, \ldots, n$ in clockwise order.
Each boundary vertex is incident to single edge.
The remaining internal vertices are colored black + white.

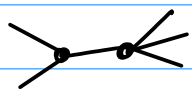
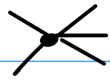
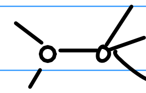
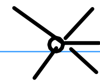
Ex:

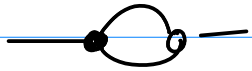


Plabic graphs are considered up to an equivalence relation generated by local moves M1, M2, M3:

(M1)  $\longleftrightarrow$ 

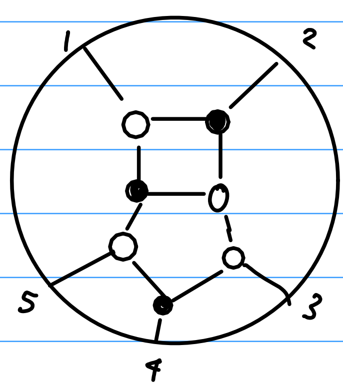
(M2)  $\longleftrightarrow$ and  $\Longleftrightarrow$

(M3)  $\Longleftrightarrow$  and  $\Longleftrightarrow$ 

Def: Plabic graph G called <u>reduced</u> if $\nexists$ $G' \sim G$ s.t. G' contains local configuration 

Def/Lemma: "Rules of the road": turn right at $\bullet$, turn left at $\circ$. Given reduced G, associate <u>trip permutation</u> π_G by starting at each bdy vertex i & following the rules of the road until we reach another bdy vertex $\pi_G(i)$. This will define a permutation π_G.

Ex:



$$\pi_G = \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 4 & 5 & 1 & 2 \end{array}$$

Def: Let

$$\pi_{k,n} = \begin{array}{cccccccc} 1 & 2 & 3 & & k & k+1 & k+2 \ldots & n \\ \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow \\ n-k+1 & n-k+2 & n-k+3 \ldots & & n & 1 & 2 \ldots & n-k \end{array}$$

So in example, $\pi_G = \pi_{3,5}$ $(k=3, n=5)$

Recall: ① Grassmannian has natural totally positive subset, defined as subset where Plucker coords > 0
② $\mathbb{C}[Gr_{k,n}]$ has natural cluster algebra structure (Scott); Plucker coordinates are among the cluster variables.

Plabic graphs provide tool for understanding both the totally pos Grassmannian & the cluster structure on the Grassmannian

Plabic graph G of type $\pi_{k,n}$

"A-model" Postnikov Scott "B-model"

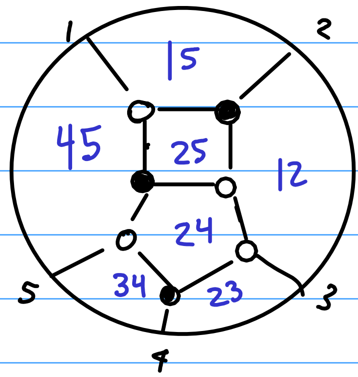
"Plabic chart"
$\Phi_G : (\mathbb{C}^*)^N \to X = Gr_{n-k}(\mathbb{C}^n)$

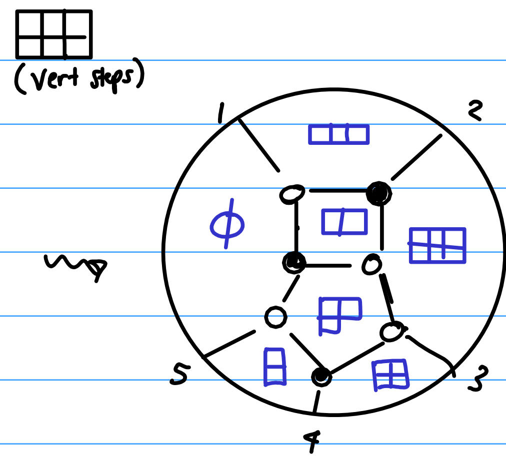
Cluster chart
$\Phi_G^\vee : (\mathbb{C}^*)^N \to \check{X} \subset Gr_k(\mathbb{C}^n)$

Rk: When we restrict domain of Φ_G to $(\mathbb{R}_{>0})^N$, get param. of tot. pos. Grassmannian

Given plabic graph of type $\pi_{k,n}$, use rules of road to label faces by Young diagrams $\leq (n-k) \times k$ rectangle: The trip starting at i partitions disk into 2 parts (left & right): put i in each region on the left. Then map the $(n-k)$-element subset in each region to a Young diagram.

Ex:

$G =$ 

$\rightsquigarrow$ 

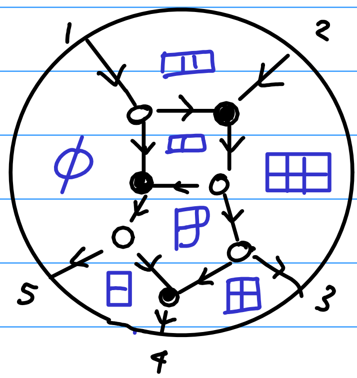
$\pi_G = \pi_{k,n}$ for $k=3$, $n=5$

Rk: The labels in regions of $G \rightsquigarrow$ cluster $\{P_\mu\}$

A-model: Plabic chart $\Phi_G : (\mathbb{C}^*)^N \to Gr_{n-k}(\mathbb{C}^n)$
obtained by choosing perfect orientation $\mathcal{O}$ for G.
(every $\bullet$ has unique outgoing edge
every $\circ$ " " ingoing)

Let I = set of sources of $\mathcal{O}$.

Put variable a_μ in region labeled by μ. Then



$$\Phi_G : (\mathbb{C}^*)^N \to Gr_{n-k}(\mathbb{C}^n)$$
$$\{a_\mu\} \mapsto \Phi_G(\{a_\mu\})$$

defined by:

Plucker coordinate $P_J(\Phi_G(\{a_\mu\}))$
= gen fun for flows
(sets of non-intersecting paths)
from I to J.

Plucker coordinates $\xrightarrow{\text{val}}$ Integer lattice point in $\mathbb{R}^N$
obtained by choosing leading (lowest degree) term in $\frac{P_J}{P_{12}} = p_J$.

	$(2,2)$	$(3,3)$	$(1,1)$	(2)	(3)	$(2,1)$	ϕ
$P_{12} = 1$	0	0	0	0	0	0	0
$P_{13} = a_{(3,3)}$	0	1	0	0	0	0	0
$P_{14} = a_{(2,2)} a_{(3,3)} (1 + a_{(2,1)})$	1	1	0	0	0	0	0
$P_{15} = a_{(2,2)} a_{(3,3)} a_{(1,1)} \; a_{(2,1)}$	1	1	1	0	0	1	0
$P_{23} = a_{(3,3)} a_{(3)}$	0	1	0	0	1	0	0
$P_{24} = a_{(2,2)} a_{(3,3)} a_{(3)} (1 + a_{(2,1)} + a_{(2)} a_{(2,1)})$	1	1	0	0	1	0	0
$P_{25} = a_{(2,2)} a_{(3,3)} a_{(1,1)} \; a_{(3)} \; a_{(2,1)} (1 + a_{(2)})$	1	1	1	0	1	1	0
$P_{34} = a_{(3,3)}^2 \; a_{(2,2)} \; a_{(2)} \; a_{(3)} \; a_{(2,1)}$	1	2	0	1	1	1	0
$P_{35} = a_{(3,3)}^2 \; a_{(2,2)} \; a_{(2)} \; a_{(3)} \; a_{(2,1)} \; a_{(1,1)}$	1	2	1	1	1	1	0
$P_{45} = a_{(3,3)}^2 \; a_{(2,2)}^2 a_{(2)} \; a_{(3)} \; a_{(2,1)} \; a_{(1,1)}$	2	2	1	1	1	1	0

RHS: These are the integer points of a polytope, the NObody NO_G.

$\boxed{\text{B-model}}$ The superpotential $W : \check{X} \to \mathbb{C}(q)$, when $\check{X} \subset Gr_3(\mathbb{C}^5)$. is given by

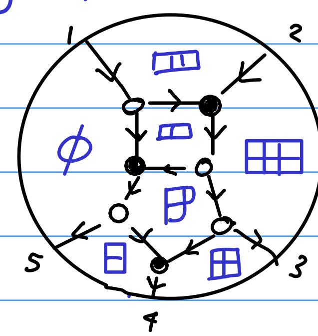
$$P_{(1)} + \frac{P_{(2)}}{P_{(1)}} + \frac{P_{(3)}}{P_{(2)}} + \frac{P_{(2,2)}}{P_{(1)} P_{(1,1)}} + \frac{P_{(3,3)} P_{(1)}}{P_{(2)} P_{(2,2)}} + \frac{P_{(1,1)}}{P_{(1)}} + \frac{P_{(2,2)}}{P_{(1)} P_{(2)}} + \frac{P_{(3,3)}}{P_{(2)} P_{(3)}} + q \frac{P_{(2)}}{P_{(3,3)}}$$

Mnemonic:

$$\begin{tikzcd} 1 \arrow[d] & & & \\ (1) \arrow[r] \arrow[d] & (2) \arrow[r] \arrow[d] & (3) \arrow[d] & \\ (1,1) \arrow[r] & \frac{(2,2)}{(1)} \arrow[r] & \frac{(3,3)}{(2)} \arrow[r] & q \end{tikzcd}$$

we're leaving out the ϕ, or equivalently, choosing $P_\phi = 1$

Rewrite W in terms of the clusters given by



Note: ① W lies in cluster algebra $\mathbb{C}[q]$ so is Laurent in any cluster.

② W is regular on $\check{X} - \bigcup_{\substack{I \text{ cyclically} \\ \text{consec subset of } [n]}} \{p_I = 0\}$

To rewrite W we use

$$P_{(1)} = \frac{P_{(2,2)} P_{\phi} + P_{(1,1)} P_{(2)}}{P_{(2,1)}}$$

+ substitute into W.

$P_{\phi} = 1$. We get:

Note: each term in numer is adj to each term in denom

$$W = \frac{P_{(2,2)}}{P_{(2,1)}} + \frac{P_{(1,1)} P_{(2)}}{P_{(2,1)} P_{\phi}} + \frac{P_{(3)}}{P_{(2)}} + \frac{P_{(2,1,1)}}{P_{(1,1)}} + \frac{P_{(3,3)} P_{\phi}}{P_{(2)} P_{(2,1,1)}} + \frac{P_{(3,3)} P_{(1,1)}}{P_{(2,1)} P_{(2,2,1)}} + \frac{P_{(2,2,1)}}{P_{(2,1)}} + \frac{P_{(3,3)} P_{\phi}}{P_{(1)} P_{(3)}} + q \frac{P_{(2)}}{P_{(3,3)}}$$

"Tropicalize W":

Define polytope Q_G by inequalities, one for each term of W:

$b_{(2,2)} - b_{(2,1)} \geq 0 \qquad b_{(1,1)} + b_{(2)} - b_{(2,1)} \geq 0 \qquad b_{(3)} - b_{(2)} \geq 0 \qquad b_{(2,1,1)} - b_{(1,1)} \geq 0$

$b_{(3,3)} - b_{(2)} - b_{(2,1,1)} \geq 0 \qquad b_{(3,3)} + b_{(1,1)} - b_{(2,1)} - b_{(2,2,1)} \geq 0 \qquad b_{(2,2,1)} - b_{(2,1)} \geq 0$

$b_{(3,3)} - b_{(1)} - b_{(3)} \geq 0 \qquad 1 + b_{(2)} - b_{(3,3)} \geq 0$

Theorem: $Q_G = NO_G$!

Rk: Q_G isomorphic to Gelfand-Tsetlin polytope...
integer pts $\leftrightarrow$ basis for related representation of GL_n.

Idea: Prove $Q_G = NO_G$ for particular nice choice of G.
Then show that if one changes G by square move, both polytopes change by tropical cluster relation...

(End)