# Bipartite Graphs and Microlocal Geometry

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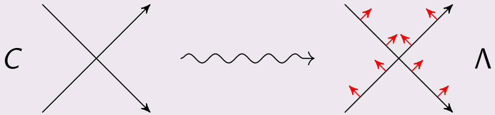
joint work in progress with V. Shende, D. Treumann, and E. Zaslow

# Legendrians and Lagrangians

- Let $S$ be a surface, $T^*S$ its cotangent bundle, and $T^\infty S = T^*S/\mathbb{R}_+$ its cosphere bundle, thought of as the boundary of $T^*S$ "at infinity".

**Definition: (Generic) Legendrian Knots in $T^\infty S$**

A knot $\Lambda \subset T^\infty S$ is Legendrian if it is "lifted" from an oriented immersed curve $C \subset S$ by leftward conormal directions:



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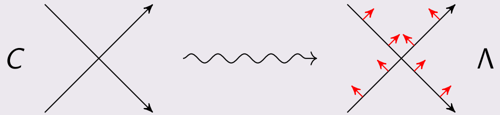


- $T^*S$ has the Liouville form $\lambda = \sum p_i dq_i$ and the symplectic form $\omega = d\lambda$.
- A surface $L \subset S$ is an exact Lagrangian if $\lambda|_L$ is exact.
- $L \subset T^*S$ is Lagrangian $\implies \overline{L} \cap T^\infty S$ is Legendrian

# Bipartite Surface Graphs

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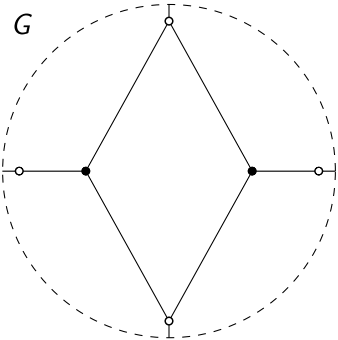


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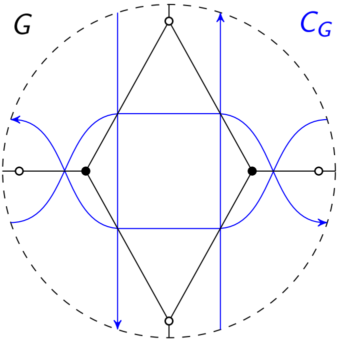


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# The Conjugate Surface, after Goncharov-Kenyon

- Associate to bipartite $G \subset S$ a surface $L_G$, the conjugate surface, which
  1. has a projection $L_G \to S$ with image the white and black regions of $S \smallsetminus C_G$ and which is 1-1 except above crossings.
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- $H_1(L_G;\mathbb{Z})$ comes with a basis: boundaries of faces of $G$.
- Gives canonical coordinates on the algebraic torus $H^1(L_G;\mathbb{C}^*)$:

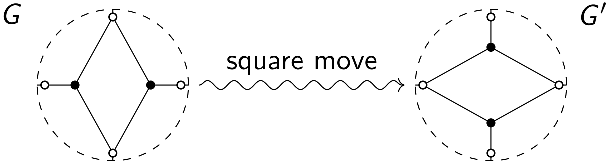
$$\mathbb{C}[H^1(L_G;\mathbb{C}^*)] \cong \mathbb{C}[y_1^{\pm 1}, \dots, y_n^{\pm 1}],$$

  where $y_i$ measures the holonomy around $i$th face of $G$.
- Taking intersection pairings in $H_1(L_G;\mathbb{Z})$ defines a quiver whose vertices are the faces of $G$.

# Examples of Bipartite Surface Graphs

- $S = D^2$, $G$ "reduced": positroid varieties (Postnikov,...), soliton graphs (Kodama-L. Williams,...)
- $S$ arbitrary, $C_G$ "contractible to punctures": cluster algebras from surfaces (Gekhtman-Shapiro-Vainshtein, Fock-Goncharov, Fomin-Shapiro-Thurston,...), (wild) $SL_n$-character varieties (Fock-Goncharov, Goncharov,...)
- $S = T^2$, $G$ "minimal": toric integrable systems, pentagram maps (Goncharov-Kenyon, Fock-Marshakov, Cherkis-Ward, Glick-Pylyavskyy, Musiker-Lai,...)
- All appear in diverse physical contexts: brane tilings, bipartite field theories, on-shell diagrams,... (Hanany-Kennaway, Feng-He-Kennaway-Vafa, Yamazaki, Franco-Eager-Schaeffer, Franco, Gaiotto-Moore-Neitzke, Arkani-Hamed-Bourjaily-Cachazo-Goncharov-Postnikov-Trnka,...)

- Given a quadrilateral face of $G$, we can produce a new graph $G'$ by performing a square move:



# The Square Move and Cluster Transformations

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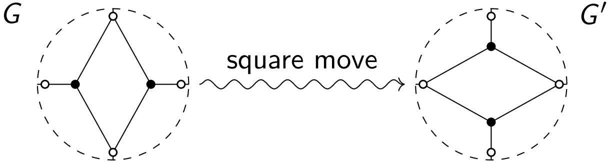


## General Pattern

For "special" equivalence classes of graphs there is a space $X$ (positroid variety, character variety, integrable system,...) which contains all $H^1(L_G;\mathbb{C}^*)$ as open subvarieties:

$$\begin{tikzcd} H^1(L_G;\mathbb{C}^*) \arrow[dd, dashed, "\text{$\mathcal{X}/y$-type cluster transformation}"'] \arrow[dr, hook] & \\ & X \\ H^1(L_{G'};\mathbb{C}^*) \arrow[ur, hook] & \end{tikzcd}$$

# The Conjugate Lagrangian

- Let $\Lambda_G \subset T^\infty S$ be the Legendrian lift of $C_G$.

**Proposition (Shende-Treumann-W.-Zaslow)**

The projection $L_G \to S$ factors through an exact Lagrangian embedding $L_G \hookrightarrow T^*S$, unique up to Hamiltonian isotopy, such that $\overline{L_G} \cap T^\infty S = \Lambda_G$.

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- Not hard to prove, but a ton of information is contained in the embedding $L_G \hookrightarrow T^*S$!
- In particular, it completely determines the "general pattern" above (the space $X$, the maps $H^1(L_G; \mathbb{C}^*) \hookrightarrow X$, and hence the cluster transformations $H^1(L_G; \mathbb{C}^*) \dashrightarrow H^1(L_{G'}; \mathbb{C}^*)$), and extends it to bipartite graphs that aren't "special".

**Definition Sketch**

Given a Legendrian $\Lambda \subset T^\infty S$, the Fukaya category $Fuk_\Lambda(T^*S)$ has objects exact Lagrangians $L$ with $\mathbb{C}$-local systems and such that $\overline{L} \cap T^\infty S \subset \Lambda$. Intersection points between Lagrangians define morphisms, modulo analytic relations.

# The Fukaya Category

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- Say an object $L$ of $Fuk_\Lambda(T^*S)$ has microlocal rank one if its local system is rank one and its boundary is equal to $\Lambda$.
- Let $\mathcal{M}(\Lambda)$ be the moduli space of objects in $Fuk_\Lambda(T^*S)$ of microlocal rank one.

## Proposition (Shende-Treumann-W.-Zaslow)

The canonical inclusion $H^1(L_G; \mathbb{C}) \hookrightarrow \mathcal{M}(\Lambda_G)$ is open.

# Isotopy Invariance

- Shende-Treumann-Zaslow '14: View $Fuk_\Lambda(T^*S)$ and $\mathcal{M}(\Lambda)$ as Legendrian knot invariants (and study them microlocally).
- Let $\Lambda_t$, $t \in [0,1]$ be a Legendrian isotopy. Then there is an associated isomorphism $\mathcal{M}(\Lambda_0) \cong \mathcal{M}(\Lambda_1)$, compatible with composition.

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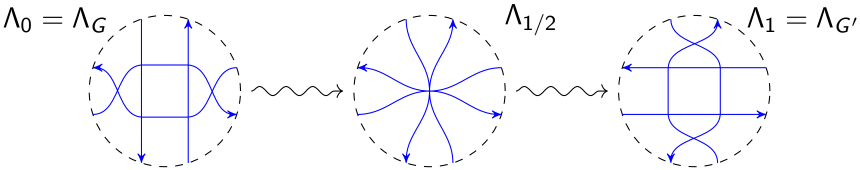
- The square move defines an isotopy from $\Lambda_G$ to $\Lambda_{G'}$:




- The family $L_t$ of conjugate Lagrangians develops a singularity at $t = 1/2$ ($L_1$ obtained from $L_0$ by Lagrangian surgery).

# Cluster Transformations from Conjugate Lagrangians

**Theorem (Shende-Treumann-W.-Zaslow)**

Let $G, G' \subset S$ be related by a square move. The isotopy isomorphism between $\mathcal{M}(\Lambda_G)$ and $\mathcal{M}(\Lambda_{G'})$ birationally identifies the open subsets $H^1(L_G; \mathbb{C}^*)$ and $H^1(L_{G'}; \mathbb{C}^*)$ by an $\mathcal{X}/y$-type cluster transformation in their face coordinates:

$$\begin{tikzcd} H^1(L_G; \mathbb{C}^*) \arrow[r, hook] \arrow[d, dashed, "\substack{\mathcal{X}/y\text{-type cluster}\\ \text{transformation}}"'] & \mathcal{M}(\Lambda_G) \arrow[d, equal, "\substack{\text{isotopy}\\ \text{isomorphism}}", "\wr"'] \\ H^1(L_{G'}; \mathbb{C}^*) \arrow[r, hook] & \mathcal{M}(\Lambda_{G'}) \end{tikzcd}$$

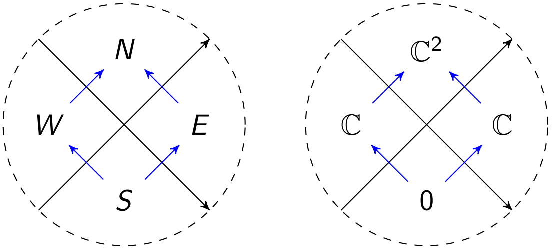
- Consistent with general picture, e.g. from Picard-Lefschetz theory, of how Lagrangian surgery affects local systems (Auroux, Seidel, Kontsevich-Soibelman, ...).

**Definition: Constructible Sheaves**

A constructible sheaf on $S$ with singular support on a Legendrian link $\Lambda \subset T^\infty S$ is:

1. A $\mathbb{C}$-local system on each component of $S \setminus \pi(\Lambda)$
2. Possibly noninvertible leftward maps across strands of $\pi(\Lambda)$

such that $S \to W \oplus E \to N$ is exact at each crossing. It has microlocal rank one if all maps in (2) are 1-1 with 1-dim'l cokernel.




microlocal rk one $\Lambda$-constructible sheaf $=$ pair of linearly independent vectors in $\mathbb{C}^2$

# Microlocal Geometry, after Kashiwara-Schapira

- Let $Sh_\Lambda(S)$ be the category of constructible sheaves on $S$ with singular support on $\Lambda$.

**Theorem (Nadler-Zaslow)**

There is a quasi-equivalence

$$\pi_S : Fuk_\Lambda(T^*S) \cong Sh_\Lambda(S),$$

canonical up to a choice of spin structure on $S$.

- Rough Idea: for an object $L$ of $Fuk_\Lambda(T^*S)$, the stalk of $\pi_S(L)$ at $x \in S$ will be $\mathrm{Hom}_{Fuk}(T^*_x S, L)$.
- In particular, alternate description of $\mathcal{M}(\Lambda)$: the space of microlocal rank one sheaves with singular support on $\Lambda$.
- (Warning: We omit important homological aspects from the present discussion.)

# Isotopy Isormphisms Revisited

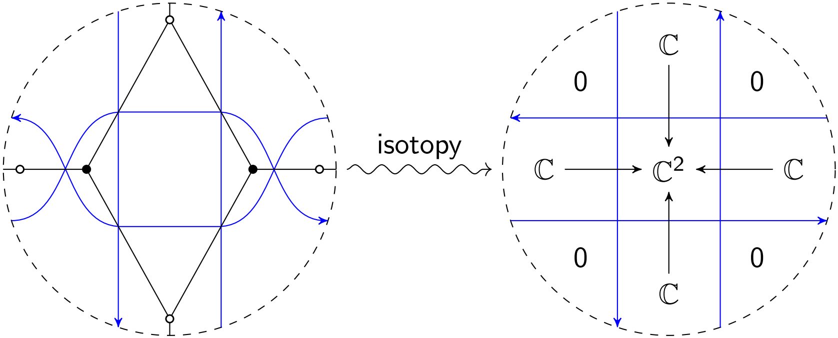
Now we can say what the canonical inclusions $H^1(L_G; \mathbb{C}^*) \hookrightarrow \mathcal{M}(\Lambda_G)$ have to do with the standard examples from earlier:

1. For "special" Legendrian knots $\Lambda$ the space $\mathcal{M}(\Lambda)$, viewed as a space of sheaves, is manifestly isomorphic to some fundamental geometric object $X$ (e.g. positroid variety, character variety, . . . ).
2. "Special" bipartite graphs $G$ are those whose associated Legendrian $\Lambda_G$ is isotopic to a "special" Legendrian $\Lambda$, and using isotopy isomorphisms we then have

$$H^1(L_G; \mathbb{C}^*) \hookrightarrow \mathcal{M}(\Lambda_G) \cong \mathcal{M}(\Lambda) \cong X.$$

# Basic Example: $X$ = Open Positroid Cell of $\mathrm{Gr}_{2,4}$

- If $S$ has boundary, we can consider a framed moduli space $\mathcal{M}^{\mathrm{fr}}(\Lambda)$ of microlocal rank one sheaves with a trivialization along the boundary.



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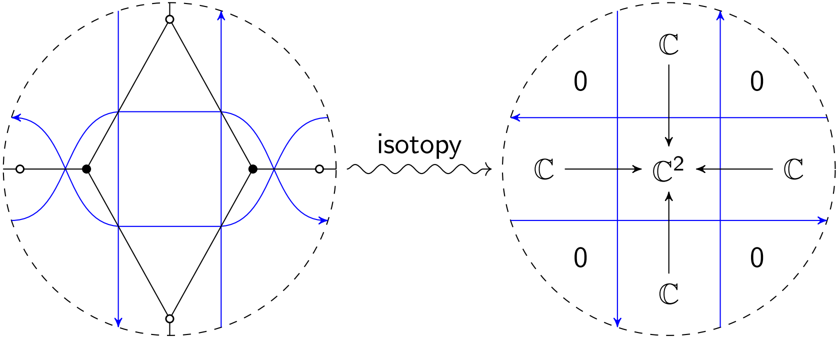


$$\mathcal{M}^{\mathrm{fr}}(\Lambda) = \left\{ \begin{array}{c} \text{4 cyclically-ordered vectors in } \mathbb{C}^2, \\ \text{neighbors linearly independent} \end{array} \right\} / GL_2$$