Integrable systems in cluster algebra theory

Montreal

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Pentagram map

Introduced by R. Schwarz about 20 years ago.

Studied by
(almost) everything ArXived
Pentagram Map $T$:

Acts on projective equivalence classes of closed and twisted $n$-gons with monodromy $M$. The latter constitute a $2n$-dimensional space, the former is $2n - 8$-dimensional.

A good reference: http://en.wikipedia.org/wiki/Pentagram_map
Corner coordinates: left and right cross-ratios \( X_1, Y_1, \ldots, X_n, Y_n \).

The map is as follows:

\[
X_i^* = X_i \frac{1 - X_{i-1} Y_{i-1}}{1 - X_{i+1} Y_{i+1}}, \quad Y_i^* = Y_{i+1} \frac{1 - X_{i+2} Y_{i+2}}{1 - X_i Y_i}.
\]
Hidden *scaling symmetry*

\[(X_1, Y_1, \ldots, X_n, Y_n) \mapsto (tX_1, t^{-1}Y_1, \ldots, tX_n, t^{-1}Y_n)\]

commutes with the map.

“Easy” invariants:

\[
O_n = \prod_{i=1}^{n} X_i, \quad E_n = \prod_{i=1}^{n} Y_i.
\]

**Monodromy invariants:**

\[
\frac{O_n^{2/3} E_n^{1/3} (\text{Tr } M)}{(\det M)^{1/3}} = \sum_{k=1}^{[n/2]} O_k
\]

are polynomials in \((X_i, Y_i)\), decomposed into homogeneous components; likewise, for \(E_k\) with \(M^{-1}\) replacing \(M\).
Theorem (OST 2010). The Pentagram Map is completely integrable on the space of twisted $n$-gons:
1). The monodromy invariants are independent integrals (there are $2[n/2] + 2$ of them).
2). There is an invariant Poisson structure of corank 2 if $n$ is odd, and corank 4 if $n$ is even, such that these integrals Poisson commute.

Poisson bracket: $\{X_i, X_{i+1}\} = -X_i X_{i+1}$, $\{Y_i, Y_{i+1}\} = Y_i Y_{i+1}$, and the rest $= 0$.

Complete integrability on the space of closed polygons has been proven as well:

He considered the dynamics in the $2n - 1$-dimensional quotient space by the scaling symmetry $(X, Y) \mapsto (tX, t^{-1}Y)$:

$$p_i = -X_{i+1}Y_{i+1}, \quad q_i = -\frac{1}{Y_iX_{i+1}},$$

and proved that it was a $Y$-type cluster algebra dynamics.
Cluster dynamics

Given a *quiver* (an oriented graph with no loops or 2-cycles) whose vertices are labeled by variables $\tau_i$ (rational functions in some free variables), the mutation on vertex $i$ is as follows:

$$
\begin{align*}
\tau_i^* &= \frac{1}{\tau_i}, \\
\tau_j^* &= \frac{\tau_j \tau_i}{1 + \tau_i}, \\
\tau_k^* &= \tau_k(1 + \tau_i);
\end{align*}
$$

the rest of the variables are intact.
The quiver also mutates, in three steps:
(i) for every path $j \to i \to k$, add an edge $j \to k$;
(ii) reverse the orientation of the edges incident to the vertex $i$;
(iii) delete the resulting 2-cycles.

The mutation on a given vertex is an involution.
Example of mutations:

Figure 7. Some quiver mutations applied to the quiver associated with the exchange matrix

\[
\begin{pmatrix}
0 & 1 & 0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0
\end{pmatrix}
\]

Note that in this example the mutated quiver is the same as the initial one except that all the arrows have been reversed. The is an instance of a more general phenomenon described by the following lemma.

Lemma 3.2. Suppose that \((y, B)\) is a \(Y\)-seed of rank \(2n\) such that \(b_{ij} = 0\) whenever \(i, j\) have the same parity (so the associated quiver is bipartite). Assume also that for all \(i\) and \(j\) the number of length 2 paths in the quiver from \(i\) to \(j\) equals the number of length 2 paths from \(j\) to \(i\). Then the \(\mu_i\) for \(i\) odd pairwise commute as do the \(\mu_i\) for \(i\) even. Moreover, \(\mu_{2n-1} \circ \cdots \circ \mu_3 \circ \mu_1(y, B) = (y', -B)\) and \(\mu_{2n} \circ \cdots \circ \mu_4 \circ \mu_2(y, B) = (y'', -B)\) where

\[
y'_j = \begin{cases} y_j & \text{if } j \text{ even} \\
1 - y_j & \text{if } j \text{ odd}
\end{cases}
\]

\[
y''_j = \begin{cases} 1 - y_j & \text{if } j \text{ even} \\
y_j & \text{if } j \text{ odd}
\end{cases}
\]

The proof of this lemma is a simple calculation using the description of quiver mutations above. Note that the term bipartite, as used in the statement of the lemma, simply means that each arc in the quiver connects an odd vertex and an...
Glick’s quiver \((n = 8)\):

![Glick’s quiver](image)

Figure 8. The quiver associated with the exchange matrix \(B_0\) for \(n = 8\) even vertex. No condition on the orientation of the arcs is placed. A stronger notion would require that all arcs begin at an odd vertex and end at an even one. The discussion of bipartite belts in [5] uses the stronger condition. As such, the results proven there do not apply to the current context. We will, however, use the same notation.

Let \(\mu_{even}\) be the compound mutation \(\mu_{even} = \mu_2 \circ ... \circ \mu_4 \circ \mu_2\) and let \(\mu_{odd} = \mu_2 \circ \mu_1 \circ \mu_3 \circ \mu_1\). Equations (2.3)–(2.4) and (3.1)–(3.2) suggest that \(\alpha_1\) and \(\alpha_2\) are instances of \(\mu_{odd}\) and \(\mu_{even}\), respectively. Indeed, let \(B_0\) be the matrix with entries

\[
\begin{align*}
(b_{0_{ij}}) = \\
( -1)^{j-i} & \text{if } i-j \equiv \pm 1 \pmod{2n} \\
( -1)^{j-i+1} & \text{if } i-j \equiv \pm 3 \pmod{2n} \\
0 & \text{otherwise}
\end{align*}
\]

The corresponding quiver in the case \(n = 8\) is shown in Figure 8.

**Proposition 3.3.**

\[\mu_{even}(y, B_0) = (\alpha_2(y), -B_0)\] \[\mu_{odd}(y, -B_0) = (\alpha_1(y), B_0)\]

**Proof.** First of all, \(B_0\) is skew-symmetric and \(b_{0_{ij}} = 0\) for \(i, j\) of equal parity. In the quiver associated to \(B_0\), the number of length 2 paths from \(i\) to \(j\) is 1 if \(|i-j| \in \{2, 4\}\) and 0 otherwise. Therefore, Lemma 3.2 applies to \(B_0\) and \(\mu_{even}\) is given by (3.2).

Both \(\alpha_2\) and \(\mu_{even}\) invert the \(y_j\) for \(j\) even. Now suppose \(j\) is odd. Then \(\alpha_2\) has the effect of multiplying \(y_j\) by \(y_j - 3y_j + 3\)

\[
\frac{(1 + y_j - 1)(1 + y_j + 1)(1 + y_j - 3)(1 + y_j + 3)}{10}
\]
Joint work with Michael Gekhtman, Sergey Tabachnikov, and Alek Vainshtein, ERA 19 (2012), 1–17; and ”Integrable cluster dynamics of directed networks and pentagram maps” Michael Gekhtman, Michael Shapiro, Serge Tabachnikov, Alek Vainshtein with Appendix by A, Izosimov. Generalizing Glick’s quiver (the case of $k = 3$), consider the homogeneous bipartite graph $Q_{k,n}$ where $r = \lfloor k/2 \rfloor - 1$, and $r' = r$ for $k$ even and $r' = r + 1$ for $k$ odd (each vertex is 4-valent):

$$Dynamics: \text{ mutations on all } p\text{-vertices, followed by swapping } p \text{ and } q; \text{ this is the map } \overline{T}_k:$$
\[ q_i^* = \frac{1}{p_i}, \quad p_i^* = q_i \frac{(1 + p_{i-r-1})(1 + p_{i+r+1})p_{i-r}p_{i+r}}{(1 + p_{i-r})(1 + p_{i+r})} , \quad k \text{ even}, \]
\[ q_i^* = \frac{1}{p_{i-1}}, \quad p_i^* = q_i \frac{(1 + p_{i-r-2})(1 + p_{i+r+1})p_{i-r-1}p_{i+r}}{(1 + p_{i-r-1})(1 + p_{i+r})} , \quad k \text{ odd}. \]

The quiver is preserved. The function \( \prod p_i q_i \) is invariant; we restrict to the subspace \( \prod p_i q_i = 1 \).

Invariant Poisson bracket: the variables Poisson commute, unless they are connected by an arrow: \( \{ p_i, q_j \} = \pm p_i q_j \) (depending on the direction).

(This bracket comes from the general theory: GSV, *Cluster algebras and Poisson geometry, AMS, 2010*).
The quivers, for small values of $k$, look like this (for $k = 1$, the arrows cancel out):

The map $\bar{T}_k$ is reversible: $\bar{D}_k \circ \bar{T}_k \circ \bar{D}_k = \bar{T}_k^{-1}$, where

$$\bar{D}_k: p_i \mapsto \frac{1}{q_i}, \quad q_i \mapsto \frac{1}{p_i}, \quad k \text{ even},$$

$$\bar{D}_k: p_i \mapsto \frac{1}{q_{i+1}}, \quad q_i \mapsto \frac{1}{p_i}, \quad k \text{ odd}.$$
Goal: to reconstruct the $x, y$-dynamics and to interpret it geometrically.

**Weighted directed networks on the cylinder and the torus** (A. Postnikov math.CO/0609764, for networks in a disc; GSV book).

Example:

Two kind of vertices, white and black.

Convention: an edge weight is 1, if not specified.

The *cut* is used to introduce a *spectral parameter* $\lambda$. 
Boundary measurements

: the network

\[
\begin{pmatrix}
0 & x & x + y \\
\lambda & 0 & 0 \\
0 & 1 & 1
\end{pmatrix}
\]

Concatenation of networks \(\mapsto\) product of matrices.
Gauge group: at a vertex, multiply the weights of the incoming edges and divide the weights of the outgoing ones by the same function. Leaves the boundary measurements intact.

Face weights: the product of edge weights over the boundary (orientation taken into account). The boundary measurement map to matrix functions factorizes through the space of face weights. (They will be identified with the $p, q$-coordinates).

Poisson bracket (6-parameter): $\{x_i, x_j\} = c_{ij} x_i x_j$, $i \neq j \in \{1, 2, 3\}$
Postnikov moves (do not change the boundary measurements):

Type 1

Type 2

Type 3

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Consider a network whose dual graph is the quiver $Q_{k,n}$.
It is drawn on the torus. Example, $k = 3, n = 5$:

Convention: white vertices of the graph are on the left of oriented edges of
the dual graph.
The network is made of the blocks:

\[
\begin{array}{cccc}
q_{i+r'} & p_i & q_{i+r'+1} \\
q_{i-r-1} & & q_{i-r} \\
p_i & & q_{i+r'}+1
\end{array}
\]

Face weights:

\[
p_i = \frac{y_i}{x_i}, \quad q_i = \frac{x_{i+1}+r}{y_{i+r}}.
\]

This is a projection \( \pi : (x, y) \mapsto (p, q) \) with 1-dimensional fiber.
\((x,y)\)-dynamics: mutation (Postinov type 3 move on each \(p\)-face),

followed by the Postnikov type 1 and 2 moves on the white-white and black-black edge (this interchanges \(p\)- and \(q\)-faces), including moving across the vertical cut, and finally, re-calibration to restore 1s on the appropriate edges. These moves preserve the conjugacy class of the boundary measurement matrix.
Schematically:

1
3
2
2
3
1
3
1
2
2
1
3
3
1
2
2
3
1

mutation

commutation
This results in the map $T_k$:

$$
x_i^* = x_{i-r-1} \frac{x_{i+r} + y_{i+r}}{x_{i-r-1} + y_{i-r-1}}, \quad y_i^* = y_{i-r} \frac{x_{i+r+1} + y_{i+r+1}}{x_{i-r} + y_{i-r}}, \quad k \text{ even},
$$

$$
x_i^* = x_{i-r-2} \frac{x_{i+r} + y_{i+r}}{x_{i-r-2} + y_{i-r-2}}, \quad y_i^* = y_{i-r-1} \frac{x_{i+r+1} + y_{i+r+1}}{x_{i-r-1} + y_{i-r-1}}, \quad k \text{ odd}.
$$

The map $T_k$ is conjugated to the map $\overline{T}_k$: $\pi \circ T_k = \overline{T}_k \circ \pi$.

**Relation with the pentagram map:** the change of variables

$$
x_i \mapsto Y_i, \quad y_i \mapsto -Y_i X_{i+1} Y_{i+1},
$$

identifies $T_3$ with the pentagram map.
Complete integrability of the maps $T_k$

The ingredients are suggested by the combinatorics of the network. Invariant Poisson bracket (in the “stable range” $n \geq 2k - 1$):

$$\{x_i, x_{i+l}\} = -x_ix_{i+l}, 1 \leq l \leq k - 2; \quad \{y_i, y_{i+l}\} = -y_iy_{i+l}, 1 \leq l \leq k - 1;$$

$$\{y_i, x_{i+l}\} = -y_ix_{i+l}, 1 \leq l \leq k - 1; \quad \{y_i, x_{i-l}\} = y_ix_{i-l}, 0 \leq l \leq k - 2;$$

the indices are cyclic.

The functions $\prod x_i$ and $\prod y_i$ are Casimir. If $n$ is even and $k$ is odd, one has four Casimir functions:

$$\prod_{i \text{ even}} x_i, \quad \prod_{i \text{ odd}} x_i, \quad \prod_{i \text{ even}} y_i, \quad \prod_{i \text{ odd}} y_i.$$
Lax matrices, monodromy, integrals: for $k \geq 3$,

$$L_i = \begin{pmatrix}
0 & 0 & 0 & \ldots & x_i & x_i + y_i \\
\lambda & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
& & & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & 1 & 1
\end{pmatrix},$$

and for $k = 2$,

$$L_i = \begin{pmatrix}
\lambda x_i & x_i + y_i \\
\lambda & 1
\end{pmatrix}.$$ 

The boundary measurement matrix is $M(\lambda) = L_1 \cdots L_n$. The characteristic polynomial

$$\det(M(\lambda) - z) = \sum I_{ij}(x, y)z^i \lambda^j.$$ 

is $T_k$-invariant: the integrals $I_{ij}$ are in involution.
Zero curvature (Lax) representation:

\[ L_i^* = P_i L_{i+r-1} P_{i+1}^{-1} \]

where \( L_i \) are the Lax matrices and

\[
P_i = \begin{pmatrix}
0 & \frac{x_i}{\lambda \sigma_i} & \frac{y_{i+1}}{\lambda \sigma_{i+1}} & 0 & \ldots & 0 & 0 \\
0 & 0 & \frac{x_{i+1}}{\lambda \sigma_{i+1}} & \frac{y_{i+2}}{\lambda \sigma_{i+2}} & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & \frac{x_{i+k-4}}{\lambda \sigma_{i+k-4}} & \frac{y_{i+k-3}}{\lambda \sigma_{i+k-3}} & 0 \\
\frac{1}{\sigma_{i+k-2}} & 0 & 0 & \ldots & 0 & \frac{x_{i+k-3}}{\lambda \sigma_{i+k-3}} & 1 \\
\frac{1}{\sigma_{i+k-2}} & 0 & \frac{1}{\lambda \sigma_{i+k-1}} & 0 & \ldots & 0 & 0 \\
0 & 0 & \frac{1}{\lambda \sigma_{i+k-1}} & 0 & \ldots & 0 & 0 \\
\end{pmatrix},
\]

with \( \sigma_i = x_i + y_i \).
Geometric interpretations

Twisted corrugated polygons in $\mathbb{RP}^{k-1}$ and $k-1$-diagonal maps

Let $k \geq 3$. Let $\mathcal{P}_{k,n}$ be the space of projective equivalence classes of generic twisted $n$-gons in $\mathbb{RP}^{k-1}$; one has: $\dim \mathcal{P}_{k,n} = n(k-1)$.

Let $\mathcal{P}_{0, k,n} \subset \mathcal{P}_{k,n}$ consist of the polygons with the following property: for every $i$, the vertices $V_i, V_{i+1}, V_{i+k-1}$ and $V_{i+k}$ span a projective plane. These are corrugated polygons. Projective duality preserves corrugated polygons.

The consecutive $k-1$-diagonals of a corrugated polygon intersect. The resulting polygon is again corrugated. One gets a pentagram-like $k-1$-diagonal map on $\mathcal{P}_{0, k,n}$. For $k = 3$, this is the pentagram map.
Coordinates: lift the vertices $V_i$ of a corrugated polygon to vectors $\tilde{V}_i$ in $\mathbb{R}^k$ so that the linear recurrence holds

$$\tilde{V}_{i+k} = y_{i-1} \tilde{V}_i + x_i \tilde{V}_{i+1} + \tilde{V}_{i+k-1},$$

where $x_i$ and $y_i$ are $n$-periodic sequences. These are coordinates in $P_{k,n}^0$. In these coordinates, the map is identified with $T_k$. The same functions $x_i, y_i$ can be defined on polygons in the projective plane. One obtains integrals of the “deeper” diagonal maps on twisted polygons in $\mathbb{RP}^2$. 
Case $k = 2$

Consider the space $S_n$ of pairs of twisted $n$-gons $(S^-, S)$ in $\mathbb{RP}^1$ with the same monodromy. Consider the projectively invariant projection $\phi$ to the $(x, y)$-space (cross-ratios):

$$x_i = \frac{(S_{i+1}^- - S_{i+2}^-)(S_i^- - S_{i+1}^-)}{(S_i^- - S_{i+1})(S_{i+1}^- - S_{i+2}^-)}$$

$$y_i = \frac{(S_{i+1}^- - S_{i+1})(S_{i+2}^- - S_{i+2})(S_i^- - S_{i+1}^-)}{(S_{i+1}^- - S_{i+2})(S_i^- - S_{i+1})(S_{i+1}^- - S_{i+2}^-)}.$$

Then $x_i, y_i$ are coordinates in $S_n/PGL(2, \mathbb{R})$. 
Define a transformation $F_2(S^-, S) = (S, S^+)$, where $S^+$ is given by the following local \textbf{leapfrog} rule: given points $S_{i-1}, S_i^-, S_i, S_{i+1}$, the point $S_i^+$ is obtained by the reflection of $S_i^-$ in $S_i$ in the projective metric on the segment $[S_{i-1}, S_{i+1}]$:

The projection $\phi$ conjugates $F_2$ and $T_2$. 
In formulas:

\[
\frac{1}{S_i^+ - S_i} + \frac{1}{S_i^- - S_i} = \frac{1}{S_{i+1} - S_i} + \frac{1}{S_{i-1} - S_i},
\]

or, equivalently,

\[
\frac{(S_i^+ - S_{i+1})(S_i - S_i^-)(S_i - S_{i-1})}{(S_i^+ - S_i)(S_{i+1} - S_i)(S_i^- - S_{i-1})} = -1,
\]

(Toda-type equations).
In $\mathbb{CP}^1$, a circle pattern interpretation (generalized Schramm’s pattern):
Thank you!