Y-meshes and generalized pentagram maps

Max Glick and Pavlo Pylyavskyy
University of Minnesota

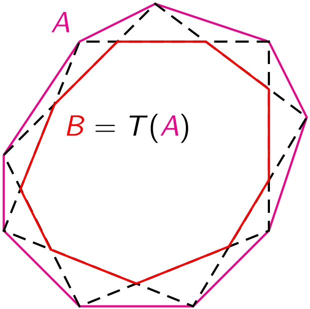
July 27, 2015

[arxiv:1503.02057]

Plan

1. Introduce Y-meshes.
2. Explain the connection between Y-meshes and cluster algebras.
3. Explore the dynamics of Y-meshes.

The pentagram map [Schwartz 1992]



$T =$ the pentagram map

The pentagram family

Name	Introduction
Pentagram map	[Schwartz 1992]
Higher diagonal	[Schwartz 2001]
Higher pentagram	[Gekhtman, Shapiro, Tabachnikov, Vainshtein 2012]
Lower pentagram	[GSTV 2012]
Short diagonal hyperplane	[Khesin, Soloviev 2013]
Dented map	[KS 2015]
(I, J)-map	[KS 2015]
Pentagram spirals	[Schwartz 2013]

The pentagram family

Name	Integrability
Pentagram map	[Ovsienko, Schwartz, Tabachnikov 2010]
Higher diagonal	[GSTV 2012]
Higher pentagram	[Gekhtman, Shapiro, Tabachnikov, Vainshtein 2012]
Lower pentagram	[GSTV 2012]
Short diagonal hyperplane	[Khesin, Soloviev 2013]
Dented map	[KS 2015]
(I, J)-map	
Pentagram spirals	k=1: [Mari Beffa 2014]

The pentagram family

Name	Cluster structure
Pentagram map	[G. 2011]
Higher diagonal	[GSTV 2012]
Higher pentagram	[Gekhtman, Shapiro, Tabachnikov, Vainshtein 2012]
Lower pentagram	[GSTV 2012]
Short diagonal hyperplane	
Dented map	
(I, J)-map	
Pentagram spirals	

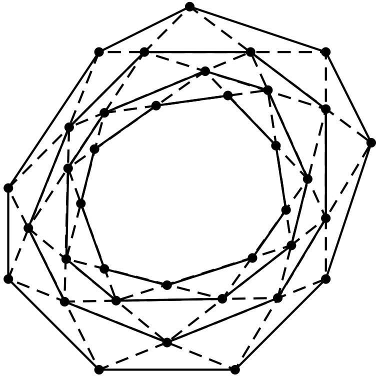
The pentagram family

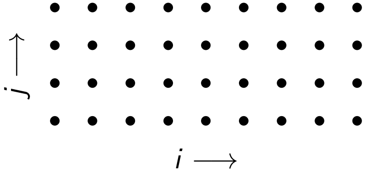
Name	Cluster structure
Pentagram map	[G. 2011]
Higher diagonal	[GSTV 2012]
Higher pentagram	[Gekhtman, Shapiro, Tabachnikov, Vainshtein 2012]
Lower pentagram	[GSTV 2012]
Short diagonal hyperplane	D odd: [GP 2015]
Dented map	$\gcd(D, k) = 1$: [GP 2015]
(I, J)-map	
Pentagram spirals	k=1: [GP 2015]

Collinear quadruples for the pentagram map

$P_{i,j} = \text{ vertex } i \text{ of } T^j(A)$

$\Longrightarrow P_{i,j}, P_{i,j+1}, P_{i+1,j+1}, P_{i+2,j}$ collinear for all i, j

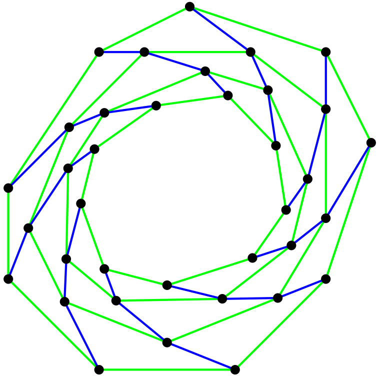


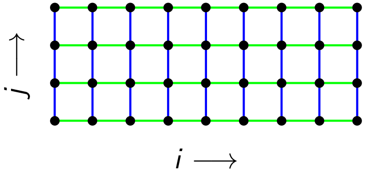


Collinear quadruples for the pentagram map

$P_{i,j} =$ vertex i of $T^j(A)$

$\Longrightarrow P_{i,j}, P_{i,j+1}, P_{i+1,j+1}, P_{i+2,j}$ collinear for all i, j

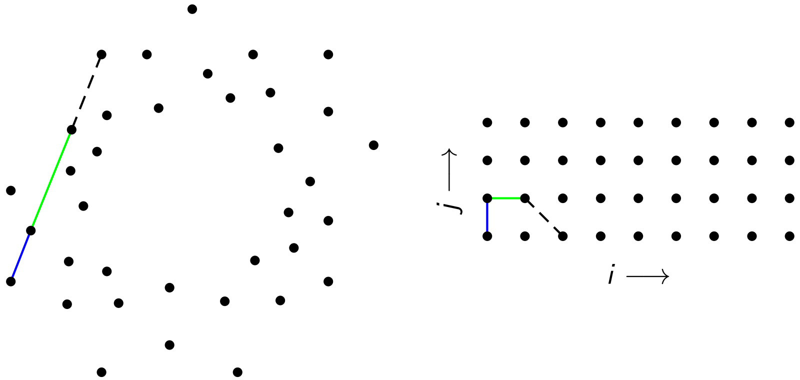




Collinear quadruples for the pentagram map

$$P_{i,j} = \text{ vertex } i \text{ of } T^j(A)$$

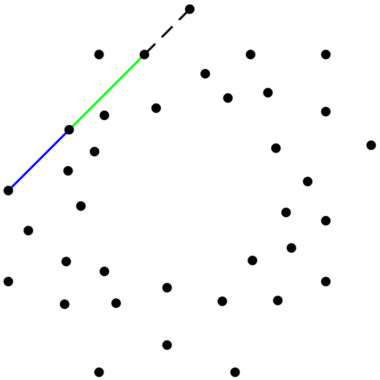
$$\implies P_{i,j}, P_{i,j+1}, P_{i+1,j+1}, P_{i+2,j} \text{ collinear for all } i, j$$

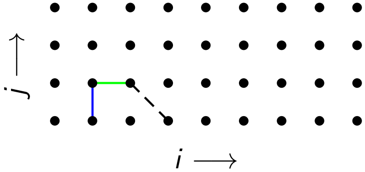


Collinear quadruples for the pentagram map

$$P_{i,j} = \text{ vertex } i \text{ of } T^j(A)$$

$$\implies P_{i,j}, P_{i,j+1}, P_{i+1,j+1}, P_{i+2,j} \text{ collinear for all } i, j$$

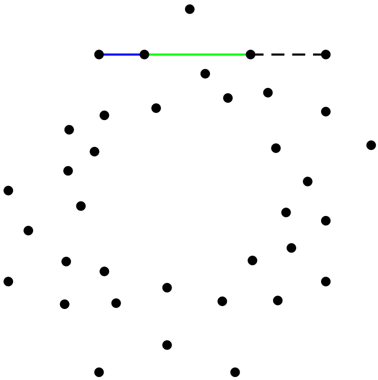


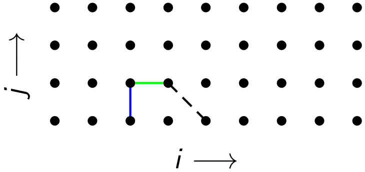


Collinear quadruples for the pentagram map

$$P_{i,j} = \text{ vertex } i \text{ of } T^j(A)$$

$$\Longrightarrow P_{i,j}, P_{i,j+1}, P_{i+1,j+1}, P_{i+2,j} \text{ collinear for all } i, j$$

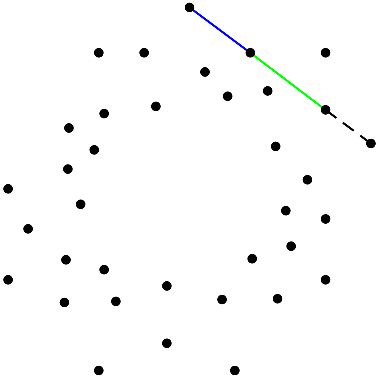


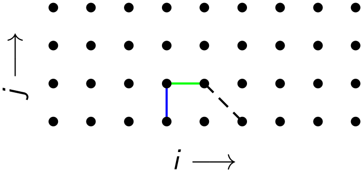


Collinear quadruples for the pentagram map

$$P_{i,j} = \text{ vertex } i \text{ of } T^j(A)$$

$$\Longrightarrow P_{i,j}, P_{i,j+1}, P_{i+1,j+1}, P_{i+2,j} \text{ collinear for all } i, j$$

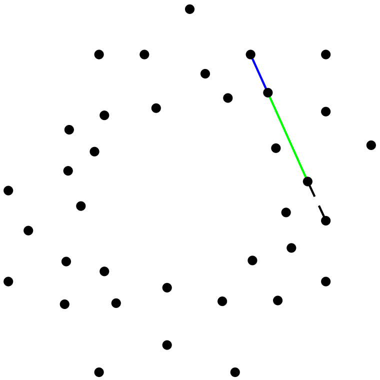


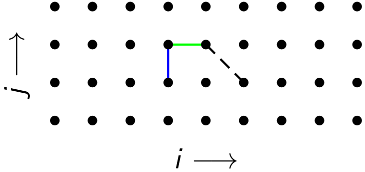


Collinear quadruples for the pentagram map

$$P_{i,j} = \text{ vertex } i \text{ of } T^j(A)$$

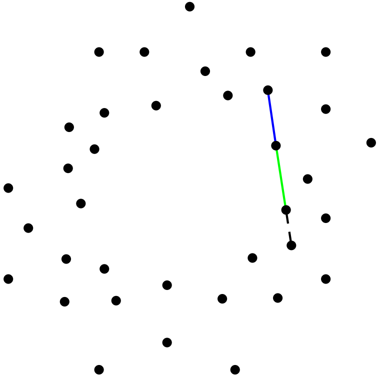
$$\Longrightarrow P_{i,j}, P_{i,j+1}, P_{i+1,j+1}, P_{i+2,j} \text{ collinear for all } i, j$$

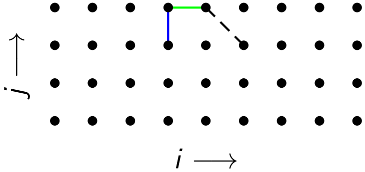




Collinear quadruples for the pentagram map

$$P_{i,j} = \text{ vertex } i \text{ of } T^j(A)$$
$$\implies P_{i,j}, P_{i,j+1}, P_{i+1,j+1}, P_{i+2,j} \text{ collinear for all } i, j$$





Definition of Y-meshes

Let $S = \{a, b, c, d\} \subseteq \mathbb{Z}^2$, $D \geq 2$ an integer. A Y-mesh of type S and dimension D is a family of points P_r and lines L_r in $\mathbb{RP}^D$ indexed by $r \in \mathbb{Z}^2$ such that:

- $P_{r+a}, P_{r+b}, P_{r+c}, P_{r+d}$ lie on L_r for all $r \in \mathbb{Z}^2$.

Definition of Y-meshes

Let $S = \{a, b, c, d\} \subseteq \mathbb{Z}^2$, $D \geq 2$ an integer. A Y-mesh of type S and dimension D is a family of points P_r and lines L_r in $\mathbb{RP}^D$ indexed by $r \in \mathbb{Z}^2$ such that:

- $P_{r+a}, P_{r+b}, P_{r+c}, P_{r+d}$ lie on L_r for all $r \in \mathbb{Z}^2$.
- (Genericity) $P_{r+a}, P_{r+b}, P_{r+c}, P_{r+d}$ distinct and $L_{r-a}, L_{r-b}, L_{r-c}, L_{r-d}$ distinct for all $r \in \mathbb{Z}^2$.

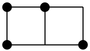
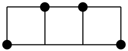
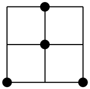
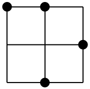
Definition of Y-meshes

Let $S = \{a, b, c, d\} \subseteq \mathbb{Z}^2$, $D \geq 2$ an integer. A Y-mesh of type S and dimension D is a family of points P_r and lines L_r in $\mathbb{RP}^D$ indexed by $r \in \mathbb{Z}^2$ such that:

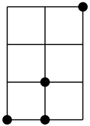
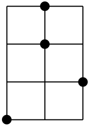
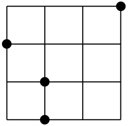
- $P_{r+a}, P_{r+b}, P_{r+c}, P_{r+d}$ lie on L_r for all $r \in \mathbb{Z}^2$.
- (Genericity) $P_{r+a}, P_{r+b}, P_{r+c}, P_{r+d}$ distinct and $L_{r-a}, L_{r-b}, L_{r-c}, L_{r-d}$ distinct for all $r \in \mathbb{Z}^2$.
- (Nondegeneracy) The $P_{i,j}$ do not lie in a proper subspace of $\mathbb{RP}^D$.

A zoo of examples

Many dynamical systems give rise to Y-meshes of some type S, including:

Name	S	D
pentagram		2
higher pentagram		3
short diagonal hyperplane		3
dented pentagram		3

A zoo of examples (continued)

Name	S	D
		2
		4
		6

Coordinates on Y-meshes

If P is a Y-mesh of type $S = \{a, b, c, d\}$, then let

$$y_r(P) = -[P_{r+a}, P_{r+c}, P_{r+b}, P_{r+d}]$$

for all $r \in \mathbb{Z}^2$, where

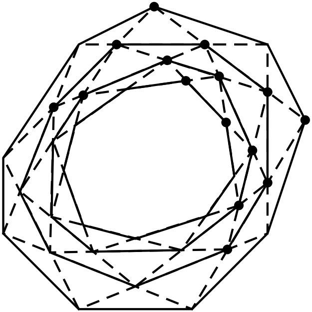
$$\begin{aligned}[x_1, x_2, x_3, x_4] &= \text{cross ratio of } x_1, x_2, x_3, x_4 \\ &= \frac{(x_1 - x_2)(x_3 - x_4)}{(x_2 - x_3)(x_4 - x_1)}.\end{aligned}$$

Transition equation for the y-variables

Theorem

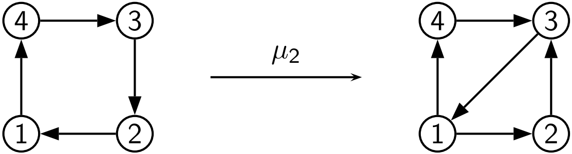
Let $y_r = y_r(P) = -[P_{r+a}, P_{r+c}, P_{r+b}, P_{r+d}]$. *Then*

$$y_{r+a+b} y_{r+c+d} = \frac{(1 + y_{r+a+c})(1 + y_{r+b+d})}{(1 + y_{r+a+d}^{-1})(1 + y_{r+b+c}^{-1})}.$$



Y-patterns [Fomin, Zelevinsky 2007]

$$(y_1, y_2, y_3, y_4) \qquad (y_1 \frac{1}{1+y_2^{-1}}, \frac{1}{y_2}, y_3(1+y_2), y_4)$$



Y-patterns are a family of discrete dynamical systems encoded by directed graphs (quivers) satisfying many nice properties:

- subtraction free rational expressions
- a natural Poisson structure [Gekhtman, Shapiro, Vainshtein]
- frequent cancellation of factors under iteration
- (sometimes) combinatorial formulas for iterates

Y-seeds and mutations

A Y-seed is a pair $(\mathbf{y}, Q)$ where $\mathbf{y} = (y_1, \ldots, y_n)$ is a collection of rational functions and Q is a quiver, i.e. a directed graph on vertex set $\{1, 2, \ldots, n\}$ without oriented 2-cycles.

Given a Y-seed $(\mathbf{y}, Q)$ and some $k \in \{1, \ldots, n\}$, the mutation $\mu_k(\mathbf{y}, Q) = (\mathbf{y}', Q')$, where

Y-seeds and mutations

A Y-seed is a pair $(\mathbf{y}, Q)$ where $\mathbf{y} = (y_1, \ldots, y_n)$ is a collection of rational functions and Q is a quiver, i.e. a directed graph on vertex set $\{1, 2, \ldots, n\}$ without oriented 2-cycles.

Given a Y-seed $(\mathbf{y}, Q)$ and some $k \in \{1, \ldots, n\}$, the mutation $\mu_k(\mathbf{y}, Q) = (\mathbf{y}', Q')$, where

- The vector $\mathbf{y}'$ is obtained from $\mathbf{y}$ via the following steps:
  1. For each $j \to k$ in Q, multiply y_j by $1 + y_k$.
  2. For each $k \to j$ in Q, multiply y_j by $\frac{1}{1+y_k^{-1}}$.
  3. Invert y_k.

Y-seeds and mutations

A Y-seed is a pair $(\mathbf{y}, Q)$ where $\mathbf{y} = (y_1, \ldots, y_n)$ is a collection of rational functions and Q is a quiver, i.e. a directed graph on vertex set $\{1, 2, \ldots, n\}$ without oriented 2-cycles.

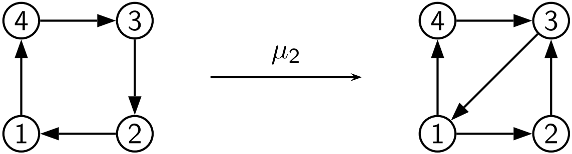
Given a Y-seed $(\mathbf{y}, Q)$ and some $k \in \{1, \ldots, n\}$, the mutation $\mu_k(\mathbf{y}, Q) = (\mathbf{y}', Q')$, where

- The vector $\mathbf{y}'$ is obtained from $\mathbf{y}$ via the following steps:
  1. For each $j \to k$ in Q, multiply y_j by $1 + y_k$.
  2. For each $k \to j$ in Q, multiply y_j by $\frac{1}{1+y_k^{-1}}$.
  3. Invert y_k.
- The quiver Q' is obtained from Q via the following steps:
  1. For every length 2 path $i \to k \to j$, add an arc from i to j.
  2. Reverse the orientation of all arcs incident to k.
  3. Remove all oriented 2-cycles.

An example of a Y-seed mutation

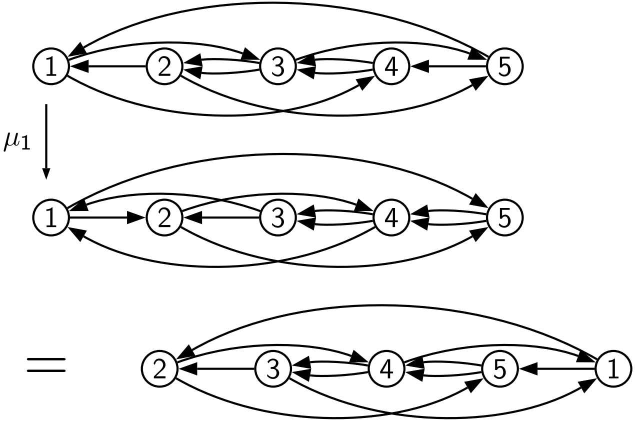
$$(y_1, y_2, y_3, y_4)$$

$$\left(y_1 \frac{1}{1+y_2^{-1}}, \frac{1}{y_2}, y_3(1+y_2), y_4\right)$$



Periodic quiver [Fordy, Marsh 2011]

A quiver Q on $V = \{1, 2, \ldots, n\}$ is periodic if $\mu_1(Q) \cong Q$ with the isomorphism induced by a cyclic permutation.



Generalization of Fordy, Marsh

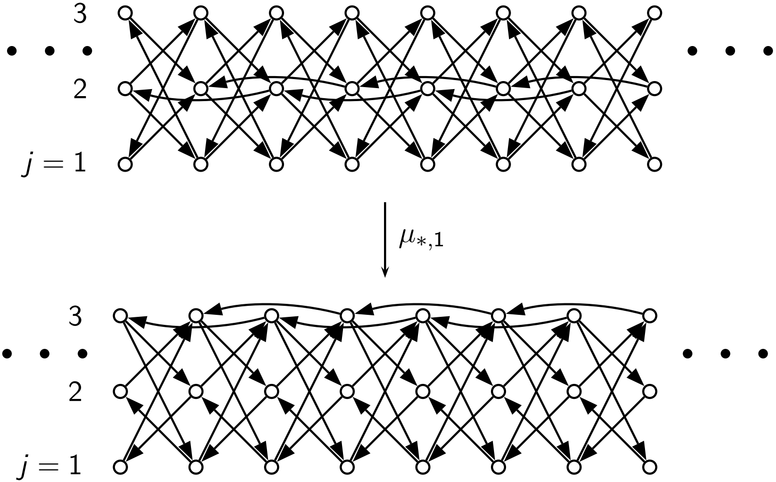
Say a quiver Q on $V = \mathbb{Z} \times \{1, 2, \ldots, l\}$ is periodic if

- there are no arrows $(i, 1) \to (i', 1)$
- Q is invariant under

$$(i, j) \in V \mapsto (i + 1, j) \in V$$

- the composition of all the $\mu_{i,1}$ produces $Q' \cong Q$ via a cyclic permutation of the rows.

Example of two dimensional, periodic quiver



From Y-meshes to cluster algebras

Theorem

Fix $S = \{a, b, c, d\} \subseteq \mathbb{Z}^2$. There is a periodic quiver Q_S on $V = \mathbb{Z} \times \{1, 2, \ldots, l\}$ with $l = c_2 + d_2 - a_2 - b_2$ such that (certain of) the y-variables transform under mutation as

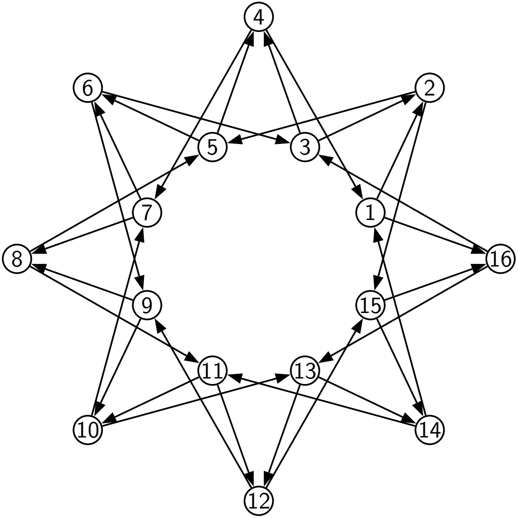
$$y_{r+a+b} y_{r+c+d} = \frac{(1 + y_{r+a+c})(1 + y_{r+b+d})}{(1 + y_{r+a+d}^{-1})(1 + y_{r+b+c}^{-1})}.$$

just like the cross ratios of a Y-mesh of type S.

Example: the pentagram map

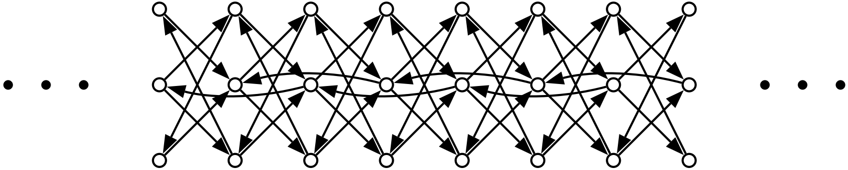
$S = \{(0,0), (2,0), (0,1), (1,1)\}$

$Q_S =$



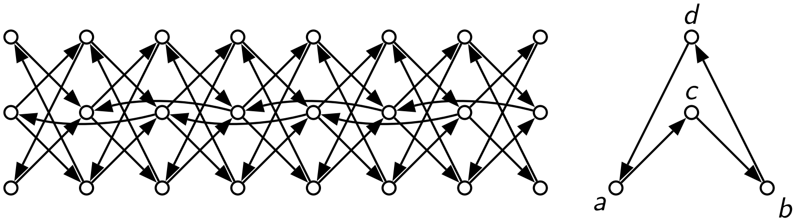
Example: the short diagonal hyperplane map

$S = \{(-1,0),(1,0),(0,1),(0,2)\}$



Construction of Q_S

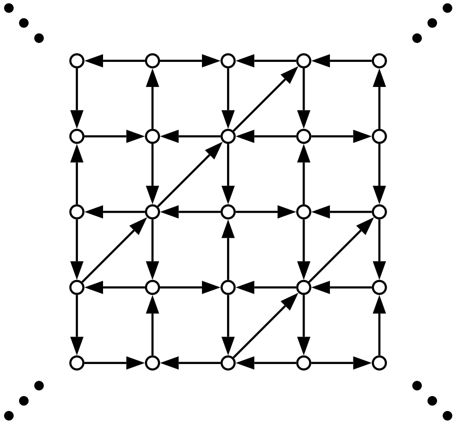
- Vertex set $V = \mathbb{Z} \times \{1, 2, \ldots, l\}$, where $l = c_2 + d_2 - a_2 - b_2$.
- Each $(i, 1)$ has two outgoing arrows with displacement $c - a$, $d - b$ and two incoming arrows with displacement $a - d$, $b - c$.
- The remaining arrows are forced (using a similar construction as [Fordy, Marsh 2011]) by the periodicity condition.



Quivers on tori

Theorem

For each S, the quiver Q_S can be embedded on a torus with alternating orientations around each vertex.



Questions about Y-meshes

Fix a dimension D and type S.

1. Do Y-meshes exist?
2. Is a Y-mesh determined uniquely by the points of m consecutive rows for some m?
3. How can we construct the $P_{i,m+1}$ from the $P_{i,j}$ with $1 \leq j \leq m$?
4. What is the minimal m as above?

The maximal dimension

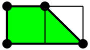
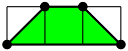
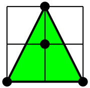
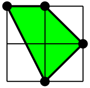
Let $S = \{a, b, c, d\}$ with $a_2 \leq b_2 < c_2 \leq d_2$. Suppose $b - a, c - a, d - a$ generate all of $\mathbb{Z}^2$. Let

$$D(S) = 2(\text{area of convex hull of } S) - 1.$$

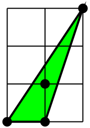
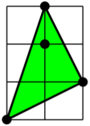
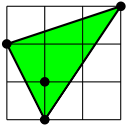
Theorem

A Y-mesh of type S and dimension D exists if and only if $2 \leq D \leq D(S)$.

The maximal dimension: examples

Name	conv(S)	Area	$D(S)$
pentagram		1.5	2
higher pentagram		2	3
short diagonal hyperplane		2	3
dented pentagram		2	3

The maximal dimension: examples

Name	conv(S)	area	$D(S)$
		1.5	2
		2.5	4
		3.5	6

The dynamics of Y-meshes

Assume $S = \{a, b, c, d\} \subseteq \mathbb{Z}^2$ and $a_2 \leq b_2 < c_2 \leq d_2$.

Proposition

A Y-mesh of type S satisfies

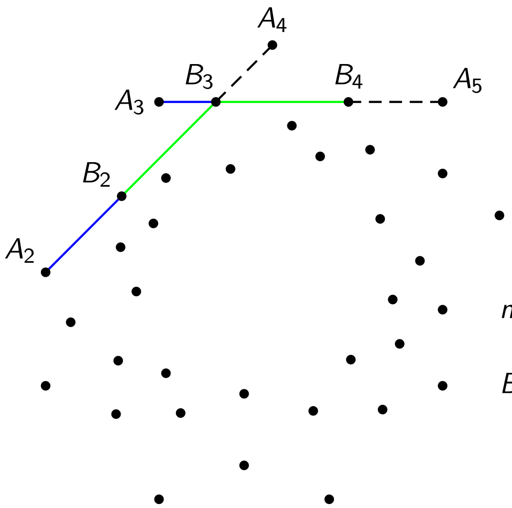
$$P_r = \langle P_{r-(d-a)}, P_{r-(d-b)} \rangle \cap \langle P_{r-(c-a)}, P_{r-(c-b)} \rangle$$
$$P_r = \langle P_{r+(c-b)}, P_{r+(d-b)} \rangle \cap \langle P_{r+(c-a)}, P_{r+(d-a)} \rangle$$

for all $r \in \mathbb{Z}^2$.

Corollary

A Y-mesh is uniquely determined by its points on $m = d_2 - a_2$ consecutive rows.

Example: $S = \{(0,0),(2,0),(0,1),(1,1)\}, D = 2$



$m = d_2 - a_2 = 1$

$B_i = \langle A_{i-1}, A_{i+1} \rangle \cap \langle A_i, A_{i+2} \rangle$

Notation: $A_i = P_{i,1}, B_i = P_{i,2}$

The dynamics of Y-meshes (continued)

- The $P_{i,j}$ with $1 \leq j \leq m = d_2 - a_2$ satisfy certain collinearity and coplanarity conditions.
- These conditions propagate under the map

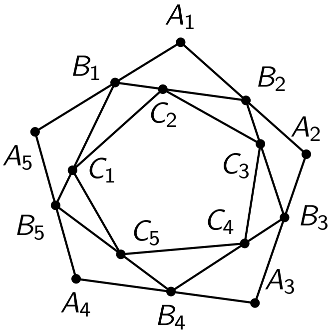
$$(P_{i,j})_{j=1\ldots m} \longmapsto (P_{i,j})_{j=2\ldots m+1}$$

given by

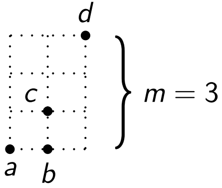
$$P_r = \langle P_{r-(d-a)}, P_{r-(d-b)} \rangle \cap \langle P_{r-(c-a)}, P_{r-(c-b)} \rangle$$

for $r = (i, m+1)$.

Example: The gopher map



$S =$



The relations

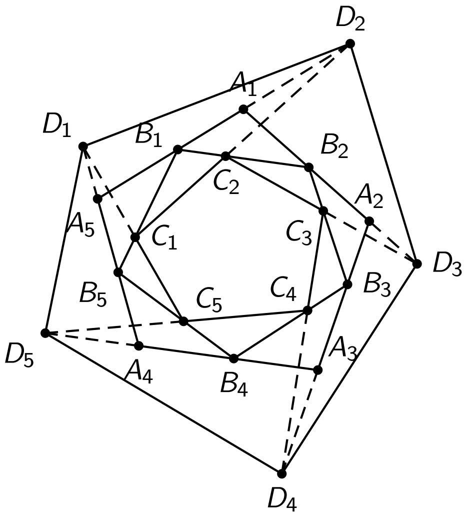
A_{i-1}, A_i, B_i collinear

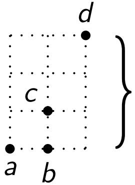
B_{i-1}, B_i, C_i collinear

The map

$D_i = \langle A_{i-2}, A_{i-1} \rangle \cap \langle C_{i-1}, C_i \rangle$

Example: The gopher map



$S =$ 

$m = 3$

The relations

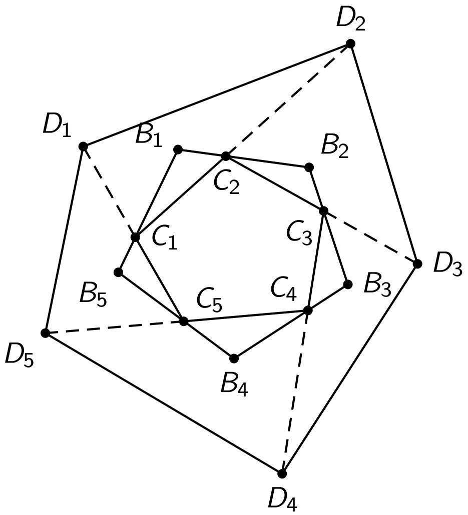
A_{i-1}, A_i, B_i collinear

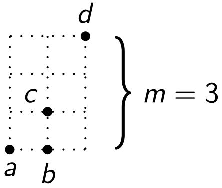
B_{i-1}, B_i, C_i collinear

The map

$D_i = \langle A_{i-2}, A_{i-1} \rangle \cap \langle C_{i-1}, C_i \rangle$

Example: The gopher map



$S =$ 

The relations

A_{i-1}, A_i, B_i collinear

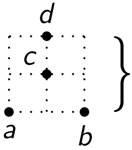
B_{i-1}, B_i, C_i collinear

The map

$D_i = \langle A_{i-2}, A_{i-1} \rangle \cap \langle C_{i-1}, C_i \rangle$

Example: short diagonal hyperplane

$S =$



$m = 2$

$D = 3$

The relations

A_{i-1}, B_i, A_{i+1} collinear

$A_{i-1}, A_{i+1}, B_{i-1}, B_{i+1}$ coplanar

The map

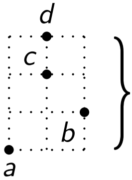
$C_i = \langle A_{i-1}, A_{i+1} \rangle \cap \langle B_{i-1}, B_{i+1} \rangle$

Proposition

$C_i = \langle A_{i-1}, A_{i+1} \rangle \cap \langle A_{i-2}, A_i, A_{i+2} \rangle$, i.e. $A \mapsto C$ is the short diagonal hyperplane map of Khesin and Soloviev.

Example: the rabbit map

$S =$



$m = 3$

$D = 2$

<u>The relations</u>

A_{i-1}, B_{i+1}, C_i collinear

A_{i+1}, B_i, C_i collinear

<u>The map</u>

$D_i = \langle A_{i-1}, B_{i+1} \rangle \cap \langle B_{i-1}, C_{i+1} \rangle$

Proposition

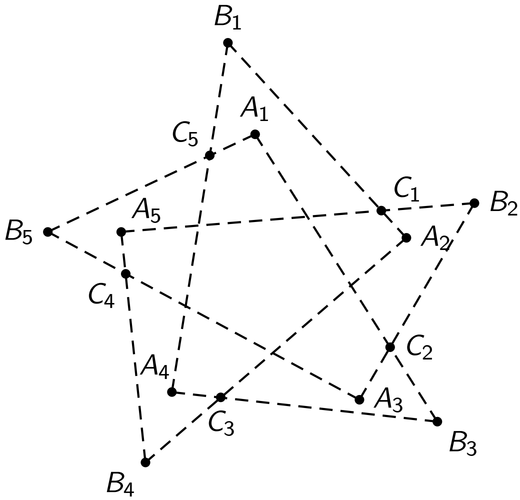
The Y-mesh is determined by the points on two consecutive rows.

Proof.

$C_i = \langle A_{i-1}, B_{i+1} \rangle \cap \langle A_{i+1}, B_i \rangle$ □

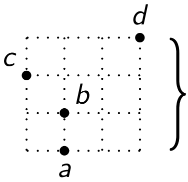
Example: the rabbit map (continued)

$$C_i = \langle A_{i-1}, B_{i+1} \rangle \cap \langle A_{i+1}, B_i \rangle$$



Example: the elephant map

$S =$



$m = 3$

$D = 6$

<u>The relations</u>

A_i, B_i, C_{i-1} collinear

A_i, B_{i-1}, C_{i+2} collinear

$A_{i-2}, B_{i-2}, B_{i+1}, C_{i+1}$ coplanar

<u>The map</u>

$D_i = \langle A_{i-2}, B_{i-2} \rangle \cap \langle B_{i+1}, C_{i+1} \rangle$

Proposition

$$B_i = V_{i-7} \cap V_{i-4} \cap V_{i-3} \cap V_i$$

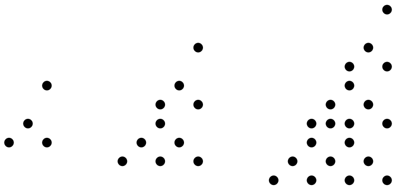
where

$$V_i = \langle A_i, A_{i+1}, A_{i+4}, A_{i+5}, A_{i+7} \rangle$$

Bonus results

- Certain "multiratios" of points are also y-variables in the cluster algebra.
- For $D = 2$, the minimum number of rows needed to determine the Y-mesh is $m = \max(d_2 - b_2, c_2 - a_2)$.
- Fix $k < D$. We have a conjectural fractal-like description of the maximal subsets $X \subseteq \mathbb{Z}^2$ satisfying

$$\dim\langle\{P_r : r \in X\}\rangle = k.$$



$k = 1$ $\quad$ $k = 2$ $\quad$ $k = 3$