Lattice Properties of Oriented Exchange Graphs

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(joint with Thomas McConville)

Positive Grassmannians: Applications to integrable systems and super Yang-Mills scattering amplitudes

July 28, 2015

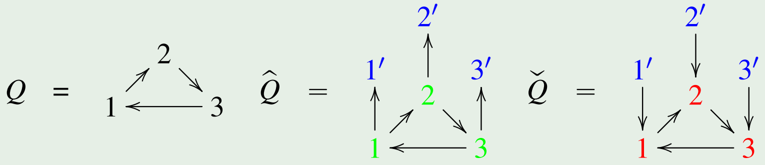
Outline

1. Oriented exchange graphs
2. Torsion classes
3. Biclosed subcategories
4. Application: maximal green sequences

Oriented exchange graphs

Let Q be a finite, connected quiver without loops or 2-cycles whose vertices are $[n] := \{1, 2, \ldots, n\}$.

Example



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Definition

Given a quiver Q, the **framed quiver** (resp. **coframed quiver**) of Q, denoted $\widehat{Q}$ (resp. $\check{Q}$), is formed by

(i) adding a **frozen vertex** i' for each vertex i in Q

(ii) adding an arrow $i \to i'$ (resp. $i \leftarrow i'$) for each vertex i in Q.

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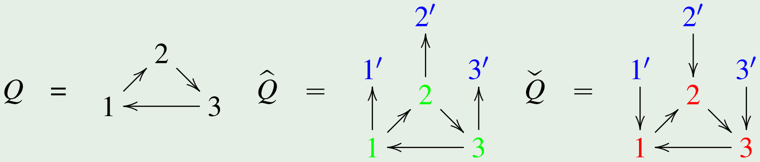

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A nonfrozen vertex of i of $\overline{Q} \in \mathrm{Mut}(\widehat{Q})$ is **green** (resp. **red**) if all arrows between frozen vertices of $\overline{Q}$ and i point away from (resp. toward) i.

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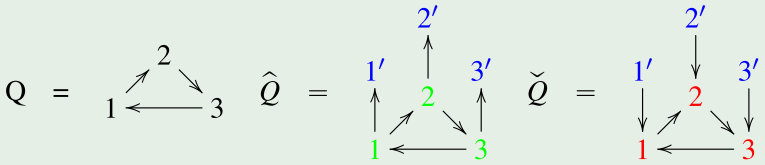

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A **maximal green sequence** of Q is a sequence $\mathbf{i} = (i_1, \ldots, i_k)$ of mutable vertices of $\widehat{Q}$ where

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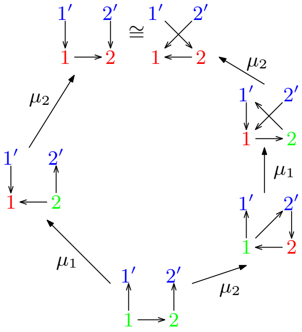
Oriented exchange graphs

Definition (Brüstle-Dupont-Pérotin)

The **oriented exchange graph** of Q, denoted $\overrightarrow{EG}(\widehat{Q})$, is the directed graph with vertices the elements of $\text{Mut}(\widehat{Q})$ and edges $\overline{Q}_1 \longrightarrow \mu_k \overline{Q}_1$ if and only if k is green in $\overline{Q}_1$.



The oriented exchange graph of $Q = 1 \to 2$ has maximal green sequences $(1, 2)$ and $(2, 1, 2)$.

Torsion classes

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Our work is based on ideas developed in [Brüstle-Yang 2014] and [Reading 2006].

Torsion classes

Theorem (Brüstle-Yang)

Let Q be mutation-equivalent to a Dynkin quiver. Then $\overrightarrow{EG}(\widehat{Q}) \cong tors(\Lambda)$ where $\Lambda = \Bbbk Q/I$ is the cluster-tilted (or Jacobian) algebra associated to Q.

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We use $\Gamma(\Lambda\text{-mod})$ to describe the torsion classes of Λ.

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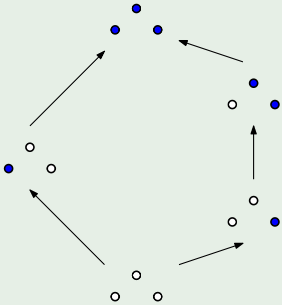

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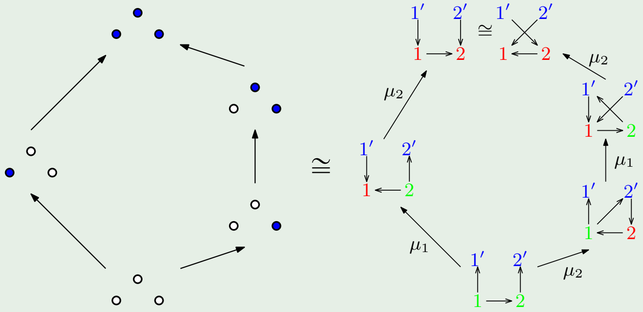

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The partially ordered set $\mathrm{tors}(\Lambda)$ is a **lattice** (i.e. any two torsion classes $\mathcal{T}_1, \mathcal{T}_2 \in \mathrm{tors}(\Lambda)$ have a **join** (resp. **meet**), denoted $\mathcal{T}_1 \vee \mathcal{T}_2$ (resp. $\mathcal{T}_1 \wedge \mathcal{T}_2$)).

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ii) $\mathcal{T}_1 \vee \mathcal{T}_2 = \mathcal{F}ilt(\mathcal{T}_1, \mathcal{T}_2)$ *where* $\mathcal{F}ilt(\mathcal{T}_1, \mathcal{T}_2)$ *consists of objects* X *with a filtration* $0 = X_0 \subset X_1 \subset \cdots \subset X_n = X$ *such that* X_j/X_{j-1} *belongs to* $\mathcal{T}_1$ *or* $\mathcal{T}_2$. *[G.–McConville]*

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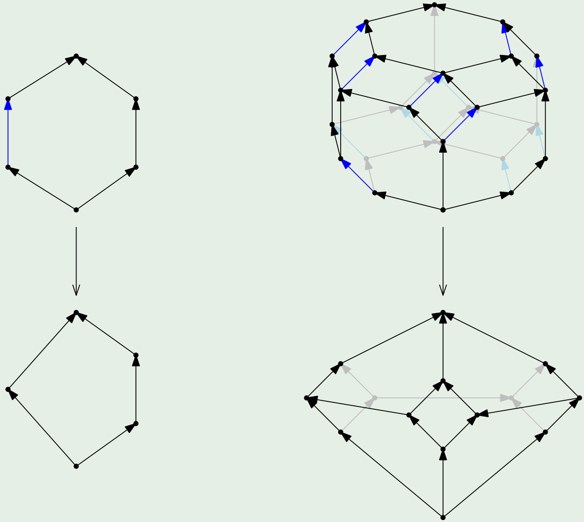
Goal: Realize $\text{tors}(\Lambda)$ as a **quotient** of a lattice with nice properties so that $\text{tors}(\Lambda)$ will inherit these nice properties.

Example

A lattice quotient map $\pi_\downarrow : L \to L/\!\sim$ is a surjective map of lattices.



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$\mathcal{BIC}(Q) :=$ biclosed subcategories of Λ-mod ordered by inclusion

A full, additive subcategory $\mathcal{B}$ of Λ-mod is **biclosed** if

$a)$ $\mathcal{B} = \text{add}(\bigoplus_{i=1}^k X_i)$ for some set of indecomposables $\{X_i\}_{i=1}^k$ (here $\text{add}(\bigoplus_{i=1}^k X_i)$ consists of objects $\bigoplus_{i=1}^k X_i^{m_i}$ where $m_i \geqslant 0$),

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b) $\mathcal{B}$ is **weakly extension closed** (i.e. if $0 \to X_1 \to X \to X_2 \to 0$ is an exact sequence where X_1, X_2, X are indecomposable and $X_1, X_2 \in \mathcal{B}$, then $X \in \mathcal{B}$),

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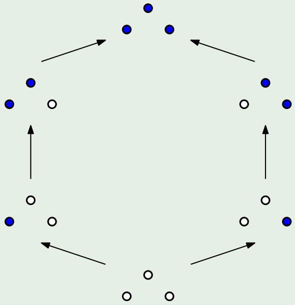

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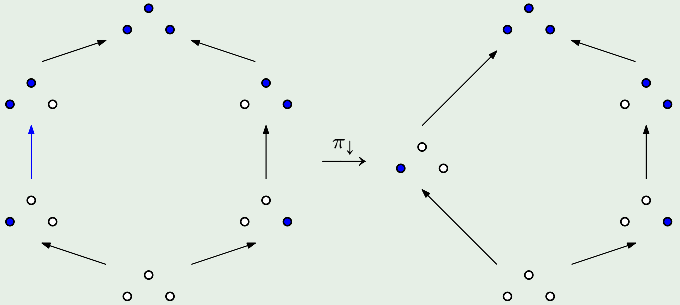

Theorem (G.– McConville)

Let $\mathcal{B} = add(\bigoplus_{i=1}^{k} X_i) \in \mathcal{BIC}(Q)$ *and let*

$$\pi_\downarrow(\mathcal{B}) := add(\bigoplus_{j=1}^{\ell} X_{i_j} : X_{i_j} \twoheadrightarrow Y \implies Y \in \mathcal{B}).$$

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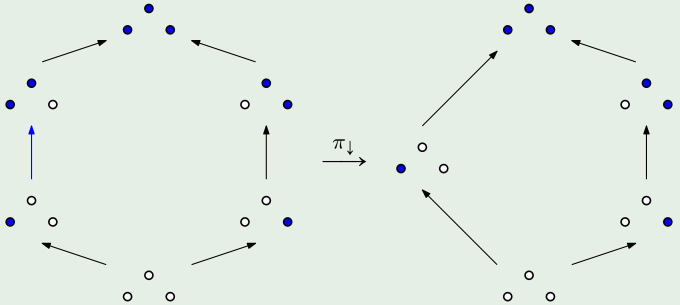

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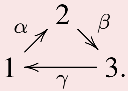
Then $\pi_\downarrow : \mathcal{BIC}(Q) \to tors(\Lambda)$ *is a lattice quotient map.*

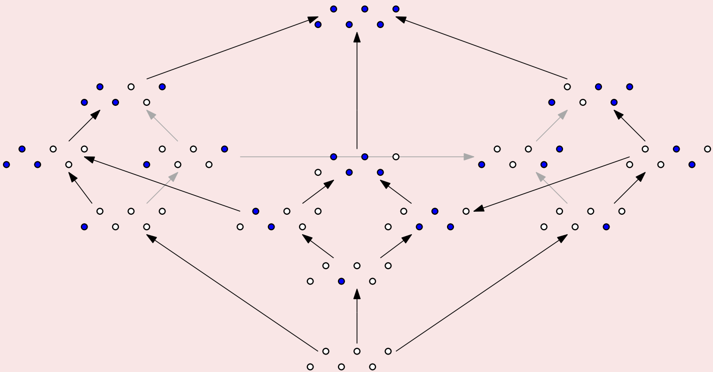
Application: maximal green sequences

Example

Let

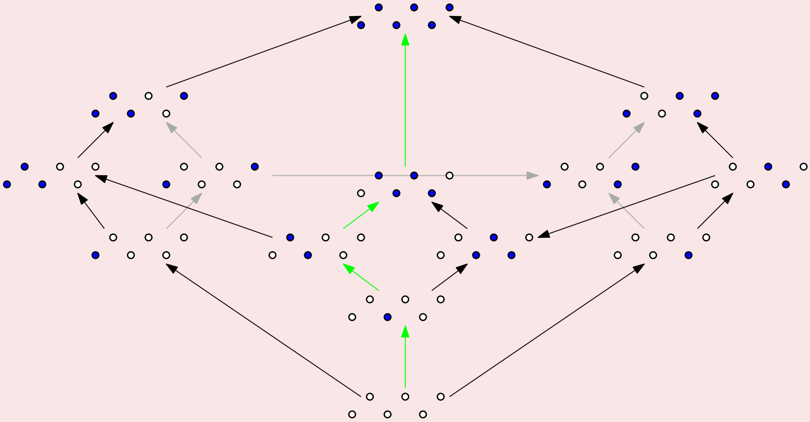
$Q(3)$:= 

Then $\Lambda = \Bbbk Q(3)/\langle \beta\alpha, \gamma\beta, \alpha\gamma \rangle$.



Example

The maximal green sequences of $Q(3)$ are connected by **polygonal flips**.



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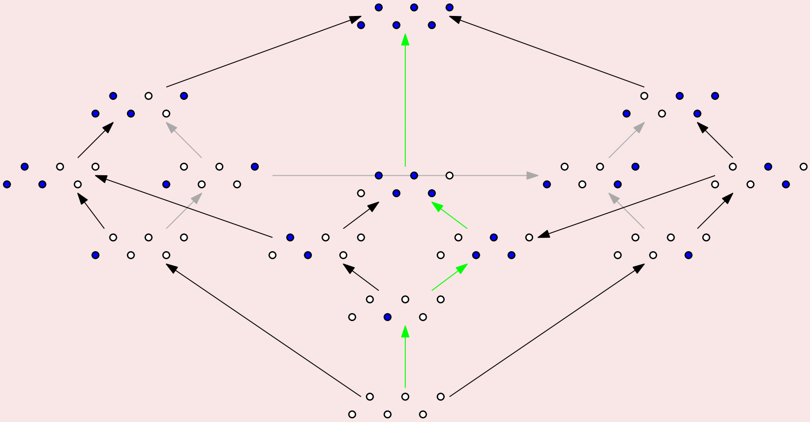

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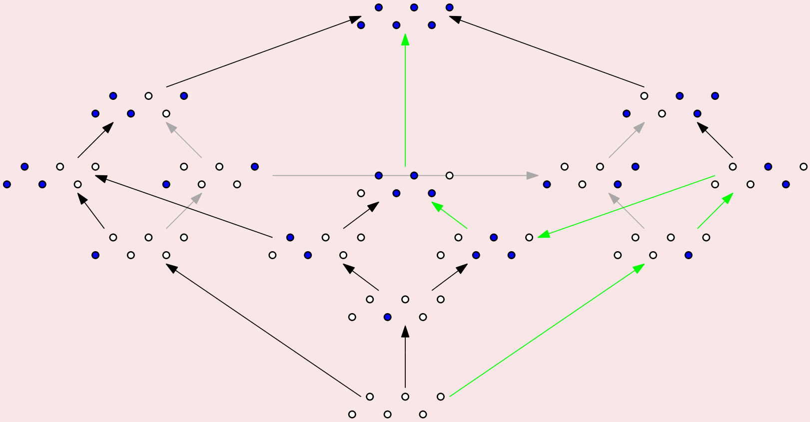

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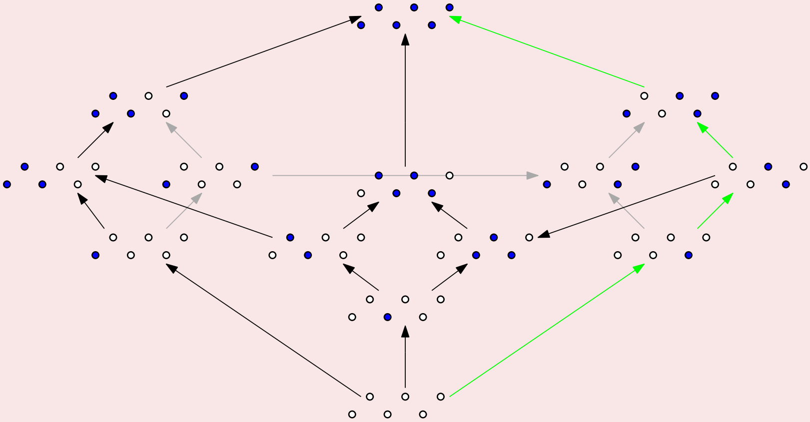

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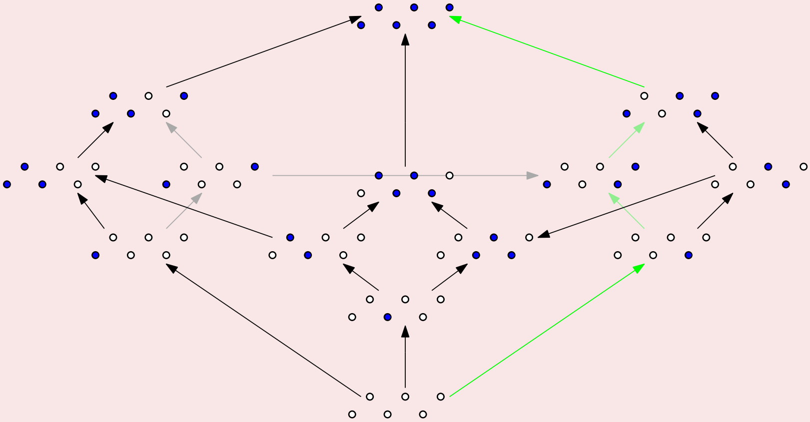

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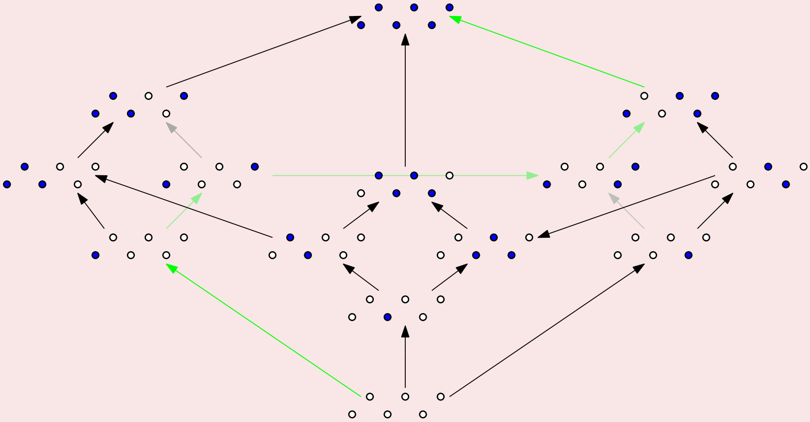

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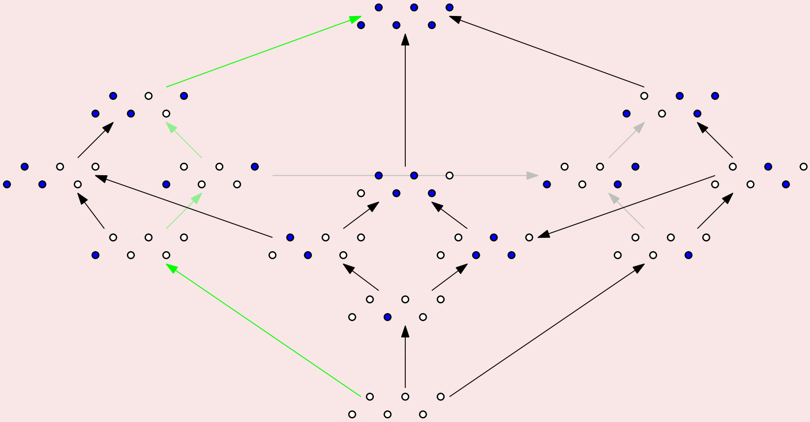

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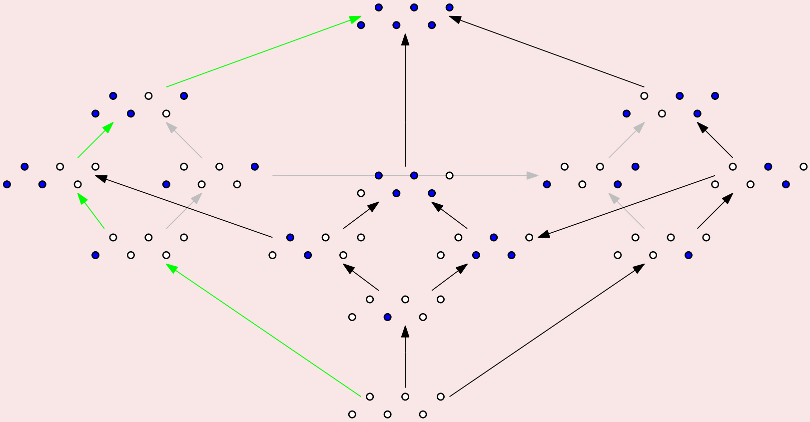

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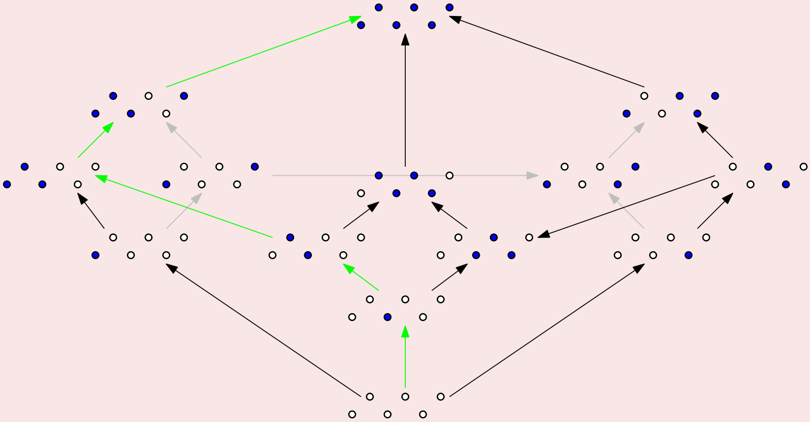

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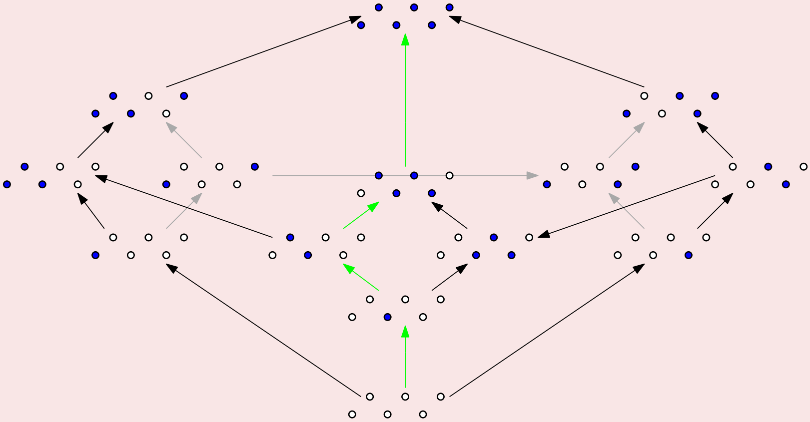

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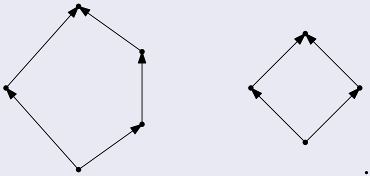

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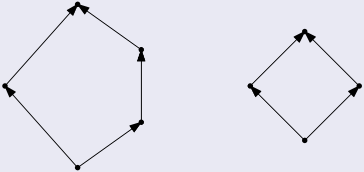

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Corollary (Conjectured by Brüstle-Dupont-Pérotin for any quiver Q)

If Q is mutation-equivalent to $1 \to 2 \to \cdots \to n$ or if Q is an oriented cycle, the set of lengths of the maximal green sequences of Q is of the form $\{\ell_{min}, \ell_{min} + 1, \ldots, \ell_{max} - 1, \ell_{max}\}$ where

$$\ell_{min} := \textit{length of the shortest maximal green sequence of } Q,$$

$$\ell_{max} := \textit{length of the longest maximal green sequence of } Q.$$

Thanks!