Primes and almost primes in short intervals: Statistics over $\mathbb{Z}$ and $\mathbb{F}_q[T]$

Brad Rodgers

Department of Mathematics
University of Zürich

Statistics and Number Theory
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Primes at mean spacing

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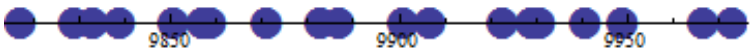
Conjecture (Distribution of primes at mean spacing)

For $I_\lambda := [n, n + \lambda \log n]$,

$$\mathbb{P}_{n \approx X}(\#\{p \in I_\lambda\} = k) := \frac{1}{X} \sum_{n \leq X} \mathbf{1}[\#\{p \in I_\lambda\} = k\}] \sim \frac{e^{-\lambda} \lambda^k}{k!}.$$

Poisson processes

Meaning of the conjecture: in a short interval $[n, n + \log n]$, the primes look like a Poisson process:



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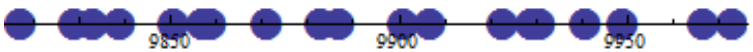

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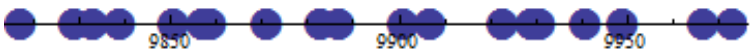

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$$\frac{\pi_k(X)}{X} = \mathbb{P}_{n\approx X}(\Omega(n) = k) \sim e^{-\log\log X}\frac{(\log\log X)^{k-1}}{(k-1)!}.$$

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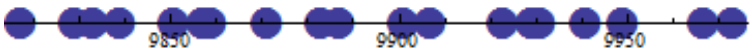

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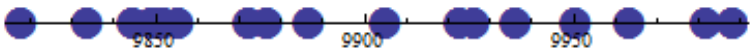

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Moreover (Erdős-Kac):

$$\mathbb{P}_{n\approx X}\left(\frac{\Omega(n) - \log\log X}{\sqrt{\log\log X}} \in [\alpha, \beta]\right) \sim \int_\alpha^\beta \frac{e^{-t^2/2}}{\sqrt{2\pi}}\, dt.$$

(Very) Short intervals

Conjecture (Distribution of primes in very short intervals)

$$\mathbb{P}_{n\approx X}\left(\frac{\pi(n+H)-\pi(n)-H/\log X}{\sqrt{H/\log X}}\in[\alpha,\beta]\right)\sim\int_{\alpha}^{\beta}\frac{e^{-t^2/2}}{\sqrt{2\pi}}\,dt$$

for H growing such that

$$H/\log X\to\infty,\quad \text{yet}\quad \log H/\log X\to 0.$$

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*For a random integer n, think of the event "n is prime" as the flip of a biased coin, that turns up '***prime***' $\frac{1}{\log n}$ of the time, and '***not prime***' otherwise*

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- The Cramer model is consistent with the conjecture above.
- Why $\log H/\log X\to 0$? **The Cramer model would not require this.**

Exceptions to the Cramer model

- The Cramer model does not suggest any $\log H/\log X \to 0$ cutoff for the distribution of primes in very short intervals. But based in part on Hardy-Littlewood heuristics:

Conjecture (Goldston-Montgomery)

For $\epsilon > 0$ and $X^\epsilon < H < X^{1-\epsilon}$

$$\frac{1}{X}\int_0^X (\psi(n+H)-\psi(n)-H)^2\,dn \sim H(\log X - \log H).$$

Conjecture (Montgomery-Soundararajan)

For random $n \approx X$,

$$\frac{\psi(n+H)-\psi(n)-H}{\sqrt{H(\log X - \log H)}} \Rightarrow N(0,1).$$

for $X^\epsilon < H < X^{1-\epsilon}$.

The Riemann zeta function

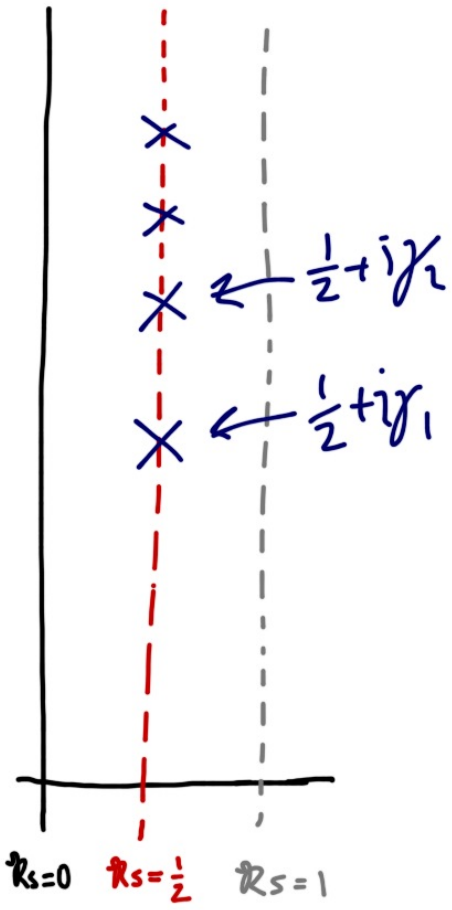

Figure: Zeros of $\zeta(s)$.

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1.$$

- Label non-trivial zeros by $1/2 + i\gamma$.
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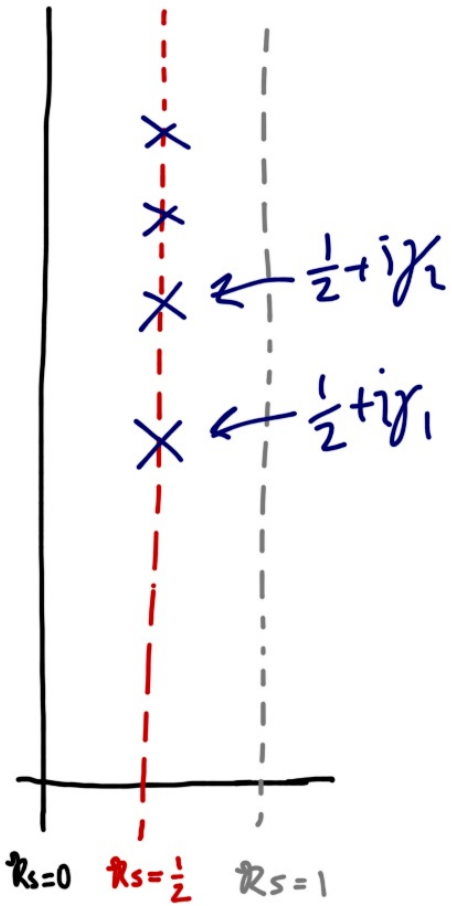

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- Label non-trivial zeros by $1/2 + i\gamma$.
- Riemann hypothesis: γ is always real.
- Riemann hypothesis is equivalent to the statement:

$$\psi(x) := \sum_{n \leq x} \Lambda(n) = x + O(x^{1/2+\epsilon}).$$

Spacing of zeros

Theorem (Riemann - von Mangoldt)

$$N(T) = \#\{\gamma \in (0, T),\ \zeta(\tfrac{1}{2} + i\gamma) = 0\}$$
$$= \tfrac{T}{2\pi} \log(\tfrac{T}{2\pi}) - \tfrac{T}{2\pi} + O(\log T)$$

- The likelihood of finding a zero $\gamma \in [T, T + dx]$ is roughly $\log T/2\pi\, dx$.

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Conjecture (Montgomery's pair correlation conjecture)

For any nice test function η,

$$\frac{1}{N(T)} \sum_{\substack{0 \le \gamma, \gamma' \le T \\ \gamma \neq \gamma'}} \eta\Big(\frac{\log T}{2\pi}(\gamma - \gamma')\Big) \sim \int \eta(x)\Big[1 - \Big(\frac{\sin \pi x}{\pi x}\Big)^2\Big]\, dx$$

- Given a zero γ around height T, the likelihood of finding another zero

$$\gamma' \in [\gamma + \tfrac{2\pi x}{\log T}, \gamma + \tfrac{2\pi x}{\log T} + dx]$$

is roughly

$$\Big[1 - \Big(\frac{\sin \pi x}{\pi x}\Big)^2\Big]\, dx.$$

A histogram of the pair correlation conjecture

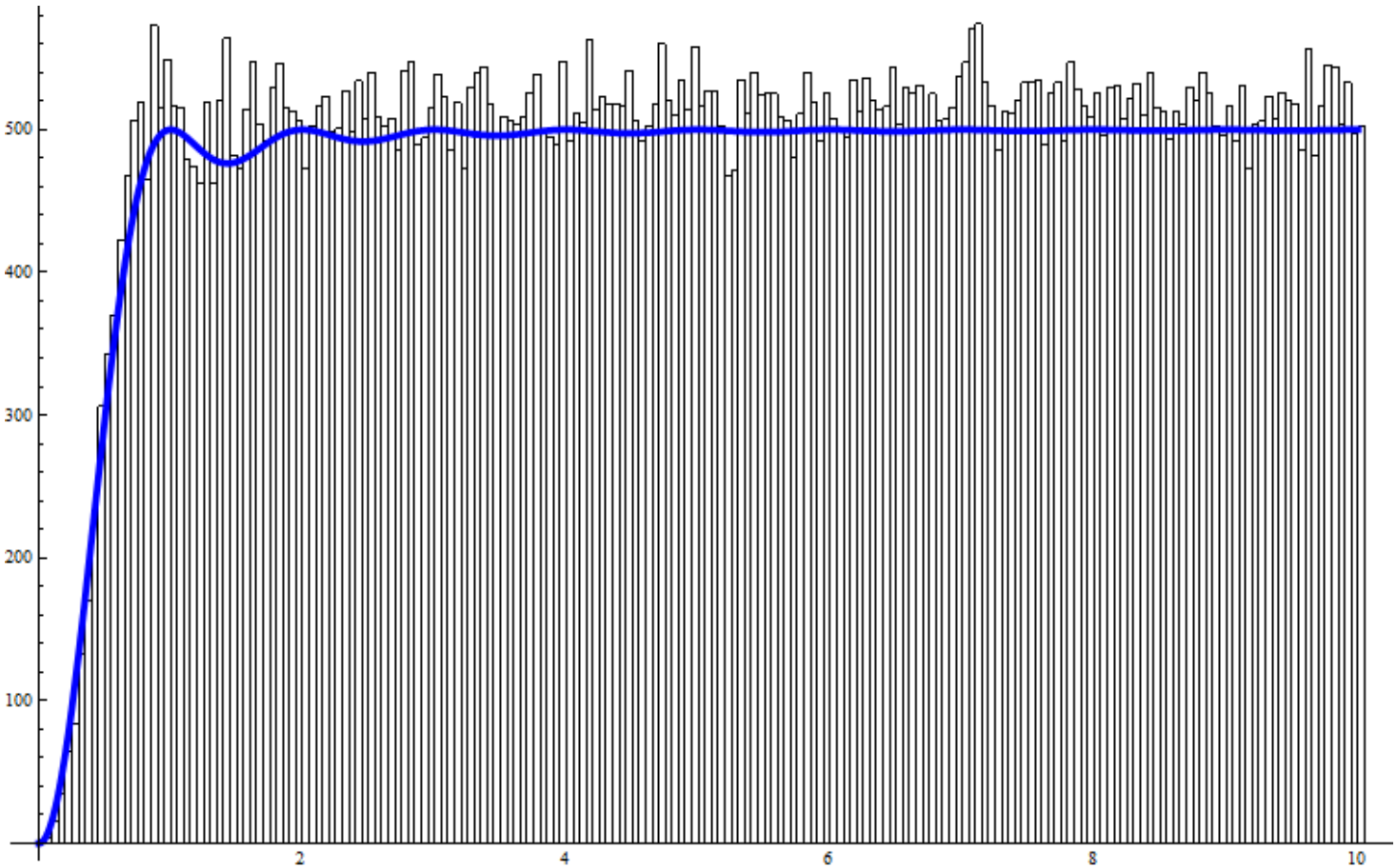


Figure: A histogram of $\frac{\log T}{2\pi}(\gamma - \gamma')$ for the first 10000 zeros, in intervals of size .05, compared to the appropriately scaled prediction $1 - (\frac{\sin \pi x}{\pi x})^2$.

Random matrix statistics

The Institute for Advanced Study

April 7 1972

Dear Atle

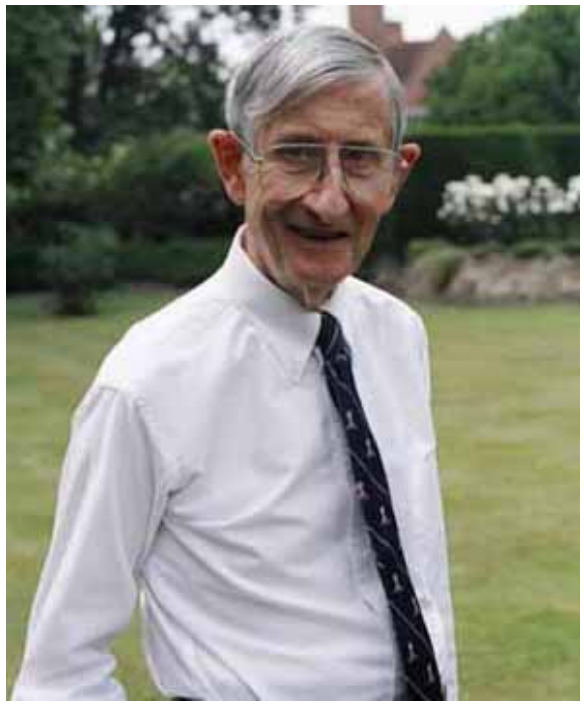
The reference which Dr Montgomery wants is

M. L. Mehta, "Random Matrices"
Academic Press, N.Y. 1967.

Page 76 Equation 6.13
Page 113 Equation 7.61

Showing that the pair-correlation function of zeros of the ζ-function is identical ~~with~~ with that of eigenvalues of a random complex (Hermitian or unitary) matrix of large order.

Freeman Dyson.



This same spacing is seen between the eigenvalues of random matrices!

The GUE Conjecture

- $U(N) := \{g \in \mathcal{M}_{N\times N}(\mathbb{C}) : gg^* = I\}$ with Haar measure.
- Label the eigenvalues of $U(N)$ by

$$\{e^{i2\pi\theta_1}, ..., e^{i2\pi\theta_N}\}, \quad \theta_i \in [-1/2, 1/2)$$

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- N eigenangles in $[-1/2, 1/2]$, so they have density N.
- Then the collection of randomly translated and stretched out zeta zeros:

$$\{\frac{\log T}{2\pi}(\gamma - t)\}$$

look like the collection of stretched out eigenangles

$$\{N\theta_i\}$$

of a random unitary matrix from $U(N)$.

Primes in short intervals

Recall, defining,

$$\widetilde{\psi}(x;H) = \psi(x+H) - \psi(x) - H,$$

Conjecture (Goldston-Montgomery)

For $1 \leq H \leq X^{1-\epsilon}$,

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Rigorously (from prime number theorem):

$$\frac{1}{X}\sum_{n\leq X}\Lambda(n)^2 \sim \log X.$$

Ideas of the Gallagher-Mueller/Goldston-Montgomery proof:

- **The assumption of the pair correlation conjecture.**
- **The explicit formula:**

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With these tools:

$$\frac{1}{X}\int_0^X \tilde{\psi}(x;\, H)^2\, dx \approx \Big(\frac{H}{X}\Big)^2 \int_0^{X/H} \Big|\sum_{n\le X}(\Lambda(n)-1)n^{-it}\Big|^2 dt$$

$$\approx H\int_{U(N)} |\mathrm{Tr}(g^r)|^2\, dg \approx H(\log X - \log H),$$

For $N \approx \log(X/H)$ and $r \approx \log X$.

Higher correlations?

- Bogomolny and Keating showed (heuristically) that the GUE Conjecture is equivalent to estimating the likelihood that

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- This may be done more rigorously: Farmer-Gonek-Lee-Lester; R.
- But: the resulting formulas estimating likelihood still are not 'nice' or combinatorially transparent.

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- By the prime number theorem:

$$\psi_k(x) := \sum_{n\leq x} \Lambda_k(n) = xP_{k-1}(\log x) + o(x)$$

$$\sim kx \log^{k-1} x.$$

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Conjecture (R.)

For $1 \le H \le X^{1-\epsilon}$,

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\text{primes} &\longleftrightarrow \text{irreducible polynomials} \\
\Lambda(n) &\longleftrightarrow \Lambda(f) := \begin{cases} \deg P, & \text{if } f = cP^k, \\ 0, & \text{otherwise.} \end{cases} \\
\sum_{d_1 d_2 = n} &\longleftrightarrow \sum_{\substack{g_1 g_2 = f \\ f_1, f_2 \text{ monic}}}
\end{aligned}
$$

Statistics in short intervals

- Prime number theorem:

$$\frac{1}{q^n} \sum_{\substack{\deg(f)=n \\ \text{monic}}} \Lambda(f) = 1$$

- Almost-prime number theorem:

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- Define

$$\widetilde{\Psi}_k(f,h) = \sum_{\substack{g:\, \deg(f-g)\le h \\ \text{monic}}} \Big(\Lambda_k(g) - [n^k - (n-1)^k]\Big).$$

Statistics in short intervals

Theorem (Keating-Rudnick)

For $0 \leq h \leq n-4$, *as* $q \to \infty$,

$$\frac{1}{q^n} \sum_{\substack{\deg(f)=n \\ \text{monic}}} \widetilde{\Psi}(f;h)^2 \sim q^{h+1}(n-h-2).$$

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Note:

$$X \leftrightarrow q^n, \quad H \leftrightarrow q^{h+1} \quad \log X \leftrightarrow n, \quad \log H \leftrightarrow h+1$$

and

$$\sum_{d=1}^{n-h-2} (d^j-(d-1)^j)(d^k-(d-1)^k) \approx \frac{jk}{j+k-1}(n-h)^{j+k-1}.$$

A naive idea: the zeta function of $\mathbb{F}_q[T]$

$$\mathcal{Z}(u) := \sum_{f \text{ monic}} u^{\deg(f)}$$

$$\left(= \sum_{f \text{ monic}} \frac{1}{|f|^s} \quad \text{for} \quad u = q^{-s} \right)$$

$$= \prod_{P \text{ monic}} \frac{1}{1 - u^{\deg P}}$$

$$= \frac{1}{1 - qu}$$

A better idea: Dirichlet characters over $\mathbb{F}_q[T]$

- Defined in exact analogy with $\mathbb{Z}$.
- For $Q \in \mathbb{F}_q[T]$, and $(g, Q) = (f, Q) = 1$,

$$\frac{1}{\Phi(Q)} \sum_{\chi \pmod{Q}} \overline{\chi(f)}\chi(g) = \begin{cases} 1, & \text{if } f \equiv g \pmod{Q} \\ 0, & \text{otherwise.} \end{cases}$$

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- L-functions are defined in analogously:

$$\begin{aligned} \mathcal{L}(u, \chi) :&= \sum_{f \text{ monic}} \chi(f) u^{\deg(f)} \\ &= \sum_{f \text{ monic}} \frac{\chi(f)}{|f|^s} \quad \text{for} \quad u := q^{-s} \end{aligned}$$

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- Here the Riemann Hypothesis is a theorem: All nontrivial zeros of $\mathcal{L}(u, \chi)$ lie on the circle $|u| = q^{-1/2}$ (or $\Re s = 1/2$).

Ideas of the proof

- **A passage to random matrices:** Let
$$\{q^{-1/2}e^{i2\pi\vartheta_1}, ..., q^{-1/2}e^{i2\pi\vartheta_N}\}$$
be the non-trivial zeros of $\mathcal{L}(u, \chi)$ and define a matrix,
$$\Theta_\chi := \mathrm{diag}(e^{i2\pi\vartheta_1}, ..., e^{i2\pi\vartheta_N}).$$
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- **Localization via harmonic analysis:** Replace the characters n^{-it} used for $\mathbb{Z}$ with $\chi(f)$. (An important role is played by an involution of Keating and Rudnick).
- **The explicit formula:**

Theorem (Weil)

$$\sum_{\substack{\deg(f)=n \\ \text{monic}}} \Lambda(f)\chi(f) = -q^{n/2}\mathrm{Tr}\ \Theta_\chi^n + O(1),$$

A variant explicit formula

- An example:

$$\sum_{\substack{\deg(f)=n \\ \text{monic}}} \Lambda \star \Lambda(f)\chi(f) = \sum_{\substack{\deg(g_1)+\deg(g_2)=n \\ \text{monic}}} \Lambda(g_1)\Lambda(g_2)\chi(g_1 g_2)$$

$$= \sum_{\substack{\ell+\lambda=n \\ \ell,\lambda\geq 1}} \left(\sum_{\substack{\deg(g_1)=\ell \\ \deg(g_2)=\lambda \\ \text{monic}}} \Lambda(g_1)\Lambda(g_2)\chi(g_1)\chi(g_2) \right)$$

$$= q^{n/2} \sum_{\substack{\ell+\lambda=n \\ \ell,\lambda\geq 1}} (-\mathrm{Tr}\ \Theta_\chi^\ell)(-\mathrm{Tr}\ \Theta_\chi^\lambda) + O(q^{(n-1)/2}).$$

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- Define inductively:

$$H_1^{(n)} := -\mathrm{Tr}(g^n),$$

$$H_j^{(n)} = \sum_{\substack{\ell+\lambda=n \\ \ell,\lambda\geq 1}} H_1^{(\lambda)} H_{j-1}^{(\ell)} + nH_{j-1}^{(n)}.$$

Lemma (A higher order explicit formula)

$$\sum_{\substack{\deg(f)=n \\ \text{monic}}} \Lambda_j(f)\chi(f) = q^{n/2} H_j^{(n)}(\Theta_\chi) + O(q^{(n-1)/2}).$$

A short interval dictionary

$$\begin{aligned}
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- Variance of Möbius sums in short intervals (Keating – Rudnick)
- Variance of divisor sums (Keating – Roditty-Gershon – Rudnick)

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Variance of traces

Theorem (Dyson)

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$$= \int_{U(N)} \sum_i e^{-i2\pi r\vartheta_i} \sum_j e^{i2\pi r\vartheta_j} \, dg$$

$$= \int_{U(N)} \sum_{i=j} e^{-i2\pi r\vartheta_i} e^{i2\pi\vartheta_j} \, dg + \int_{U(N)} \sum_{i\neq j} e^{-i2\pi r\vartheta_i} e^{i2\pi r\vartheta_j} \, dg$$

$$= N + \int_{\mathbb{T}^2} e^{i2\pi r(x_1-x_2)} \left(1 - \left(\frac{\sin \pi N(x_1-x_2)}{\sin \pi(x_1-x_2)}\right)^2\right) dx_1 dx_2$$

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Covariance of $H_j^{(r)}$

Lemma (R.)

$$\int_{U(N)} \overline{H_j^{(r)}} H_k^{(r)} \, dg = \sum_{\nu=1}^{r \wedge N} (\nu^j - (\nu-1)^j)(\nu^k - (\nu-1)^k) \quad for \quad r \neq 0$$

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$$H_j^{(r)} = \sum_{\nu=1}^{r\wedge N} (-1)^\nu (\nu^j - (\nu-1)^j)\, s_{(r-\nu+1,1^{\nu-1})}$$

Here $s_{(a,1^b)}$ are hook Schur functions for the partition $(a, \underbrace{1, ..., 1}_{b})$.

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Orthogonality relation:

$$\int_{U(N)} \overline{s_{(a,1^b)}} s_{(c,1^d)}\, dg = \begin{cases} 1, & \text{if } (a,1^b) = (c,1^d) \\ 0, & \text{otherwise.} \end{cases}$$

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for $g \in U(N)$, with $r \approx \log n$, $a \approx \log K$, $b \approx \log(m/K)$ and $N \approx \log(m/H)$, for H the size of the short interval.

Thanks:

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