THE LIE THEORETICAL STRUCTURE OF LIQUID CRYSTAL DYNAMICS

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PLAN OF THE PRESENTATION

Liquid crystal dynamics: two models

Affine Euler-Poincaré reduction

Geometry of non-visocous Ericksen-Leslie nematodynamics

Nonviscous Eringen micropolar model

Eringen implies Ericksen-Leslie

In the variational formulation, the terms modeling dissipation have been eliminated. Reason: want to understand the geometric nature of these equations. The dissipative terms are added later. Think: Euler versus Navier-Stokes.
LIQUID CRYSTALS

Tutorial: http://dept.kent.edu/spie/liquidcrystals/ by B. Senyuk

Liquid crystal: state of matter between crystal and isotropic liquid. Liquid behavior: high fluidity, formation and coalescence of droplets. Crystal behavior: anisotropy in optical, mechanical, electrical, magnetic properties. Long-range orientational order in their molecules and sometimes translational or positional order. Many phases; differ by their structure and physical properties.

History: Friedrich Reinitzer (1857 Prague – 1927 Graz) botanist and chemist. In 1888 he accidentaly discovered a strange behavior of cholesteryl benzoate that would later be called “liquid crystal". Work continued by physicist Otto Lehmann (1855 Konstanz – 1922 Karlsruhe), the “father" of liquid crystal technology. The discovery received plenty of attention at the time but due to no practical use the interest dropped soon. Lehmann was nominated 10 times (1913-22) for the Nobel Prize; never got it.
Georges Friedel (1865 Mulhouse – 1933 Strasbourg) mineralogist and crystallographer; in 1922 first classification of liquid crystals.

Carl Oseen (1879 Lund – 1944 Uppsala) theoretical physicist; first formulation of elasticity theory of liquid crystals.

Since the mid 1960s the entire theoretical and experimental development in liquid crystals has been influenced by the physicist Pierre-Gilles de Gennes (1932 Paris – 2007 Orsay) who got the Nobel Prize in 1991. Unfortunately, his book is not very useful to mathematicians. Better books are by Subrahmanyan Chandrasekhar and especially Epifanio Virga, the most mathematical book I came across.


Main phases of liquid crystals are the nematic, smectic, cholesteric (chiral nematic); molecules behave differently.

nematic $\sim$ “nema” (thread); smectic $\sim$ “smektos”(smeared), or “smechein” (to wash out) + “tikos” (suffix for adjectives of Greek origin); cholesteric $\sim$ “khole”(bile) + “steros”(solid, stiff) + “tikos”; chiral $\sim$ “cheir”(hand), introduced by Kelvin.
From: DoITPoMS, University of Cambridge

From: http://boomeria.org/chemtextbook/cch16.html
Photos by Brian Johnstone. Cooling from liquid crystal state to solid crystal state. The circular discs are the solid crystals.

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Nematic polymer; from: DoITPoMS, University of Cambridge
From B. Senyuk at http://dept.kent.edu/spie/liquidcrystals/
Cholesteric liquid crystal
Iridescent colors of this rose chafer beetle are due to the organization in cholesteric liquid crystal phase of the chitin molecules of the outer part of the shell.
**LIQUID CRYSTAL DYNAMICS**

**Director theory** due to Oseen, Frank, Zöcher, Ericksen and Leslie.

**Micropolar, microstretch**, and **micropomorphic theories**, due to Eringen, which take into account the microinertia of the particles and which is applicable, for example, to **liquid crystal polymers**.

**Ordered micropolar** approach, due to Lhuillier and Rey, which combines the director theory with the micropolar models.

We discuss only nematic liquid crystals (no chirality \( \mathbf{n} \cdot \text{curl} \mathbf{n} \)). We set all dissipation equal to zero; want to understand the conservative case first.

\( \mathcal{D} \subset \mathbb{R}^3 \) bounded domain with smooth boundary. All boundary conditions are ignored: in all integration by parts the boundary terms vanish. We fix a volume form \( \mu \) on \( \mathcal{D} \).
**ERICKSEN-LESLIE DIRECTOR THEORY**  
For nematic and cholesteric liquid crystals

**Key assumption**: only the direction and not the sense of the molecules matter. The preferred orientation of the molecules around a point is described by a unit vector $\mathbf{n} : \mathcal{D} \to S^2$, called the *director*, and $\mathbf{n}$ and $-\mathbf{n}$ are assumed to be equivalent.

**Ericksen-Leslie equations** (*Ericksen [1966], Leslie [1968]*) in a domain $\mathcal{D}$, constraint $\|\mathbf{n}\| = 1$, are:

$$
\begin{cases}
\rho \left( \frac{\partial}{\partial t} \mathbf{u} + \nabla \mathbf{u} \mathbf{u} \right) = \text{grad} \frac{\partial F}{\partial \rho^{-1}} - \partial_j \left( \rho \frac{\partial F}{\partial \mathbf{n},j} \cdot \nabla \mathbf{n} \right), \\
\rho J \frac{D^2}{Dt^2} \mathbf{n} - 2q \mathbf{n} + \mathbf{h} = 0, \quad \mathbf{h} = \rho \frac{\partial F}{\partial \mathbf{n}} - \partial_i \left( \rho \frac{\partial F}{\partial \mathbf{n},i} \right) \\
\frac{\partial}{\partial t} \rho + \text{div}(\rho \mathbf{u}) = 0, \quad \frac{D}{Dt} := \frac{\partial}{\partial t} + \nabla \mathbf{u} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla
\end{cases}
$$

$\mathbf{u}$ *Eulerian velocity*, $\rho$ *mass density*, $\mathbf{n} : \mathcal{D} \to \mathbb{R}^3$ *director* ($\mathbf{n}$ equivalent to $-\mathbf{n}$), $J$ *microinertia constant*, and $F(\mathbf{n}, \mathbf{n}, i)$ is the *free energy*:
A standard choice for $F$ is the **Oseen-Zöcher-Frank free energy**:

$$
\rho F(\rho^{-1}, n, \nabla n) = \frac{1}{2}K_1 (\text{div } n)^2 + \frac{1}{2}K_2 (n \cdot \text{curl } n)^2
$$

\begin{align*}
&+ \frac{1}{2}K_3 \|n \times \text{curl } n\|^2,
\end{align*}

associated to the basic type of director distortions nematics:

(a) splay, (b) bend, (c) twist

**WHAT IS THE VARIATIONAL/HAMILTONIAN STRUCTURE OF THESE EQUATIONS?**

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First key assumption: Replace point particles by small deformable bodies: microfluids. Examples: liquid crystals, blood, polymer melts, bubbly fluids, suspensions with deformable particles, biological fluids. Eringen [1978], [1979], [1981],...

A material particle $P$ in a microfluid is characterized by its position $X$ and by a vector $\Xi$ attached to $P$ that denotes the orientation and intrinsic deformation of $P$. Both $X$ and $\Xi$ have their own motions: $X \mapsto x = \eta(X,t)$ and $\Xi \mapsto \xi = \chi(X,\Xi,t)$ called, respectively, the macromotion and micromotion.

Second key assumption: Material bodies are very small, so a linear approximation in $\Xi$ is permissible for the micromotion:

$$\xi = \chi(X,t)\Xi,$$

where $\chi(X,t) \in GL(3)^+ := \{A \in GL(3) \mid \det(A) > 0\}$. 
The classical Eringen theory considers only three possible groups in the description of the micromotion of the particles:

\[ \text{GL}(3)^+ (\text{micromorphic}) \supset K(3) (\text{microstretch}) \supset SO(3) (\text{micropolar}), \]

\[ K(3) = \left\{ A \in \text{GL}(3)^+ | \text{there exists } \lambda \in \mathbb{R} \text{ such that } AA^T = \lambda I_3 \right\} \]

is a closed subgroup of \( \text{GL}(3)^+ \); associated to rotations and stretch.

The general theory admits other groups describing the micromotion.

We will study only micropolar fluids, i.e., the order parameter group is

\[ \mathbb{O} := SO(3) \]
Eringen’s equations for non-dissipative micropolar liquid crystals:

\[
\begin{align*}
\rho \frac{D}{Dt} u_l &= \partial_l \frac{\partial \psi}{\partial \rho} - \partial_k \left( \rho \frac{\partial \psi}{\partial \gamma^a_l} \gamma^a_k \right), \\
\rho \sigma_l &= \partial_k \left( \rho \frac{\partial \psi}{\partial \gamma^l_k} \right) - \varepsilon_{lmn} \rho \frac{\partial \psi}{\partial \gamma^a_m} \gamma^a_l, \\
\frac{D}{Dt} \rho + \rho \text{div} u &= 0, \\
\frac{D}{Dt} j_{kl} + (\varepsilon_{kpr} j_{lp} + \varepsilon_{lpr} j_{kp}) \nu_r &= 0, \\
\frac{D}{Dt} \gamma^a_l &= \partial_l \nu_a + \nu_{ab} \gamma^b_l - \gamma^a_r \partial_r u_r, \\
\frac{D}{Dt} := \frac{\partial}{\partial t} + u \cdot \nabla & \text{ mat. deriv.}
\end{align*}
\]

sum on repeated indices, \( u \in X(D) \) Eulerian velocity, \( \rho \in F(D) \) mass density, \( \nu \in F(D, \mathbb{R}^3) \) microrotation rate, where we use the standard isomorphism between \( \mathfrak{s}o(3) \) and \( \mathbb{R}^3 \), \( j_{kl} \in F(D, \text{Sym}(3)) \) microinertia tensor (symmetric), \( \sigma_k \) spin inertia is defined by

\[
\sigma_k := j_{kl} \frac{D}{Dt} \nu_l + \varepsilon_{klm} j_{mn} \nu_l \nu_n = \frac{D}{Dt} (j_{kl} \nu_l),
\]

\( \gamma = (\gamma_i^{ab}) \in \Omega^1(D, \mathfrak{s}o(3)) \) wryness tensor, related to \( (\eta, \chi) \) by

\[
\gamma = -\eta^* (\nabla \chi) \chi^{-1} =: \hat{\gamma} = (\hat{\gamma}^a_i),
\]

and \( \psi = \psi(\rho^{-1}, j, \gamma) : \mathbb{R} \times \text{Sym}(3) \times \mathfrak{gl}(3) \to \mathbb{R} \) is the free energy.
VARIATIONAL/HAMILTONIAN STRUCTURE?

Result: Well posedness of the full EL system with viscosity

RELATION BETWEEN ERICKSEN-LESLIE AND ERINGEN?

Eringen’s claim: Eringen theory recovers Ericksen-Leslie theory in the rod-like assumption $j = J(I - n \otimes n)$ with the choice $\gamma = \nabla n \times n$.

Once we have the Euler-Poincaré formulation, it will be clear that $\gamma = \nabla n \times n$ cannot be considered as a definition!

This statement has been controversial due to mistakes:
MORE ABOUT THE RELATIONS BETWEEN THE
ERICSEN-LESLIE-PARODI AND ERINGEN-LEE
THEORIES OF NEMATIC LIQUID CRYSTALS

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Abstract—This paper is a further consideration of the relation between two main phenomenological
theories of nematic liquid crystals: Ericksen–Leslie–Parodi (ELP) and Eringen–Lee (EL). The aim of
the study is to establish the generality of the conclusion which claims that the ELP theory is a
particular case of the EL theory.

According to the analysis presented in the paper this conclusion may be treated as true but only
after modification of the constitutive equation for $f_{m_{ij}}$ in the EL theory by one term arising from
splay deformation. The results of the study are formulated in four conclusions given at the end of the
paper.
This was soon reconsidered in Eringen [1993]

AN ASSESSMENT OF DIRECTOR AND MICROPOLAR THEORIES OF LIQUID CRYSTALS

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Abstract—It is shown that of the two prominent theories of liquid crystals, the Micropolar theory is more general than the Director theory. Under special assumptions, for liquid crystals having rigid rod-like molecular elements, the Micropolar theory reduces to the Director theory. The relationship between the two theories is established fully. An assessment is made of the limitations of both theories and on their domain of applicability.

more precisely
The present discussion is concerned with the relationship of OFEL and E-theories. Already discussions exist in this regard (e.g. [12] and [13]). Rymarz [12] has shown that OFEL-theory (he calls it ELP-theory) is a special case of E-theory, if the stress potential is modified by a splay term $K(\text{div } n)^2$. Below I shall show that this splay term is already present in the E-theory. Consequently, his statement should be modified to: The OFEL-theory is a special case of the E-theory.

The major contributions of this article is not only in this correction, but in the display of relations between the two theories, critical examination of their physical foundations and domain of applicability of each theory in regard to liquid crystals that possess more complicated molecular structures. In particular, this paper should serve the following purposes:

Since then, this remains an open problem.

We solve this problem using techniques of geometric mechanics:

(1) find under what assumptions, Eringen reduces to Ericksen-Leslie

(2) find correct relation between $\gamma$ and $n$ with these assumptions

$\leadsto$ Both Eringen and Rymarz are partially right! Expect to use equivalent formulations of viscous Eringen for well posedness.
EULER-POINCARÉ REDUCTION

Poincaré 1901: Left (right) invariant Lagrangian $L : TG \to \mathbb{R}$, $l := L|_g : g \to \mathbb{R}$. For $g(t) \in G$, let $\xi(t) = g(t)^{-1}\dot{g}(t) \left(\dot{g}(t)g(t)^{-1} \in \mathfrak{g}\right)$. Then the following are equivalent:

(i) $g(t)$ satisfies the Euler-Lagrange equations for $L$ on $G$.

(ii) The variational principle

$$\delta \int_a^b L(g(t), \dot{g}(t))dt = 0$$

holds, for variations with fixed endpoints.

(iii) The Euler-Poincaré equations hold:

$$\frac{d}{dt} \frac{\delta l}{\delta \xi} = \pm \text{ad}^* \xi \frac{\delta l}{\delta \xi}.$$

(iv) The Euler-Poincaré variational principle

$$\delta \int_a^b l(\xi(t))dt = 0$$

holds on $g$, for variations $\delta \xi = \dot{\eta} \pm [\xi, \eta]$, where $\eta(t)$ is an arbitrary path in $g$ that vanishes at the endpoints, i.e $\eta(a) = \eta(b) = 0$. 

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Geometry has led to analytic questions. I am not aware of any serious analysis results for such constrained variational principles.

**Reconstruction**

Solve the Euler-Lagrange equations for a left invariant $L : TG \to \mathbb{R}$

- Form $l := L|_g : g \to \mathbb{R}$
- Solve the Euler-Poincaré equations: $\frac{d}{dt} \frac{\delta l}{\delta \xi} = \text{ad}^*_{\xi} \frac{\delta l}{\delta \xi}$, $\xi(0) = \xi_0$
- Solve linear equation with time dependent coefficients (quadrature): $\dot{g}(t) = g(t)\xi(t)$, $g(0) = e$
- For any $g_0 \in G$ the solution of the Euler-Lagrange equations is $V(t) = g_0 g(t)\xi(t)$ with initial condition $V(0) = g_0 \xi_0$.

EP reduction: free rigid body, ideal fluids, KdV
EP reduction for semidirect products: heavy rigid body, compressible fluids, MHD, GFD. *Holm, Marsden, Ratiu [1998]*
Geometry of complex fluids.*Holm [2002], Gay-Balmaz, Ratiu [2009]*
Right $G$-representation on $V$, $(v, g) \in V \times G \mapsto vg \in V$, induces:

- right $G$-representation on $V^*$: $(a, g) \in V^* \times G \mapsto ag \in V^*$
- right $g$-representation on $V$: $(v, \xi) \in V \times g \mapsto v\xi \in V$
- right $g$-representation on $V^*$: $(a, \xi) \in V^* \times g \mapsto a\xi \in V^*$

Duality pairings: $\langle \cdot, \cdot \rangle_g : g^* \times g \to \mathbb{R}$ and $\langle \cdot, \cdot \rangle_V : V^* \times V \to \mathbb{R}$

Affine right representation: $\theta_g(a) = ag + c(g)$, where $c \in \mathcal{F}(G, V^*)$ is a right group one-cocycle, i.e., $c(fg) = c(f)g + c(g)$, $\forall f, g \in G$. This implies that $c(e) = 0$ and $c(g^{-1}) = -c(g)g^{-1}$. Note that

$$\left. \frac{d}{dt} \right|_{t=0} \theta_{\exp(t\xi)}(a) = a\xi + dc(\xi), \quad \xi \in g, \quad a \in V^*,$$

where $dc : g \to V^*$ is defined by $dc(\xi) := Te_c(\xi)$. Useful to introduce:
• $d_{cT} : V \to \mathfrak{g}^*$ by $\langle d_{cT}(v), \xi \rangle_{\mathfrak{g}} := \langle d_{c}(\xi), v \rangle_V$, for $\xi \in \mathfrak{g}$, $v \in V$

• $\diamond : V \times V^* \to \mathfrak{g}^*$ by $\langle v \diamond a, \xi \rangle_{\mathfrak{g}} := - \langle a\xi, v \rangle_V$ for $\xi \in \mathfrak{g}$, $v \in V$, $a \in V^*$

• then: $\langle a\xi + d_{c}(\xi), v \rangle_V = \langle d_{cT}(v) - v \diamond a, \xi \rangle_{\mathfrak{g}}$

• the semidirect product $S = G \ltimes V$ with group multiplication
  $$(g_1, v_1)(g_2, v_2) := (g_1g_2, v_2 + v_1g_2), \quad g_i \in G, \quad v_i \in V$$

• its Lie algebra $\mathfrak{s} = \mathfrak{g} \ltimes V$ with bracket
  $$\text{ad}_{(\xi_1, v_1)}(\xi_2, v_2) := [(\xi_1, v_1), (\xi_2, v_2)] = ([\xi_1, \xi_2], v_1\xi_2 - v_2\xi_1)$$

• then for $(\xi, v) \in \mathfrak{s}$ and $(\mu, a) \in \mathfrak{s}^* = \mathfrak{g}^* \times V^*$ we have
  $$\text{ad}^*_{(\xi, v)}(\mu, a) = (\text{ad}^*_\xi \mu + v \diamond a, a\xi)$$

In a physical problem (like liquid crystals) we are given:
• \( L : TG \times V^* \rightarrow \mathbb{R} \) right \( G \)-invariant under the action \\
\((v_h, a) \in T_h G \times V^* \mapsto (v_h g, \theta_g(a)) = (v_h g, ag + c(g)) \in T_h g G \times V^* \).

• So, if \( a_0 \in V^* \), define \( L_{a_0} : TG \rightarrow \mathbb{R} \) by \( L_{a_0}(v_g) := L(v_g, a_0) \). Then \( L_{a_0} \) is right invariant under the lift to \( TG \) of right translation of \( G_{a_0}^c \) on \( G \), where \( G_{a_0}^c \) is the \( \theta \)-isotropy group of \( a_0 \).

• Right \( G \)-invariance of \( L \) permits us to define \( l : g \times V^* \rightarrow \mathbb{R} \) by \\
\[ l(v_g g^{-1}, \theta_g^{-1}(a_0)) = L(v_g, a_0). \]

• Curve \( g(t) \in G \), let \( \xi(t) := g(t)g(t)^{-1} \in g \), \( a(t) = \theta_{g(t)^{-1}}(a_0) \in V^* \)

Then \( a(t) \) as the unique solution of the following affine differential equation with time dependent coefficients \\
\[ \dot{a}(t) = -a(t)\xi(t) - dc(\xi(t)), \]

with initial condition \( a(0) = a_0 \in V^*. \)

The following are equivalent:
(i) With $a_0$ held fixed, Hamilton’s variational principle

$$
\delta \int_{t_1}^{t_2} L_{a_0}(g(t), \dot{g}(t)) dt = 0,
$$

holds, for variations $\delta g(t)$ of $g(t)$ vanishing at the endpoints.

(ii) $g(t)$ satisfies the Euler-Lagrange equations for $L_{a_0}$ on $G$.

(iii) The constrained variational principle

$$
\delta \int_{t_1}^{t_2} l(\xi(t), a(t)) dt = 0,
$$

holds on $g \times V^*$, upon using variations of the form

$$
\delta \xi = \frac{\partial \eta}{\partial t} - [\xi, \eta], \quad \delta a = -a \eta - dc(\eta),
$$

where $\eta(t) \in g$ vanishes at the endpoints.

(iv) The affine Euler-Poincaré equations hold on $g \times V^*$:

$$
\frac{\partial}{\partial t} \frac{\delta l}{\delta \xi} = -ad^*_\xi \frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta a} \circ a - dc^T \left( \frac{\delta l}{\delta a} \right).
$$
Lagrangian Approach to Continuum Theories of Perfect Complex Fluids

To apply the previous theorem to complex fluids one makes two key observations:

1. Complex fluids have internal degrees of freedom encoded by the order parameter Lie group $\mathcal{O}$
2. New kind of advection equation: 
\[
\frac{D}{Dt} \gamma^a_l = \partial_l \nu_a + \nu_{ab} \gamma^b_l - \gamma^a_r \partial_l u_r
\]

Geometrically, this means:

1. Enlarge the “particle relabeling group" $\text{Diff}(\mathcal{D})$ to the semidirect product $G = \text{Diff}(\mathcal{D}) \circledast \mathcal{F}(\mathcal{D}, \mathcal{O})$, $\mathcal{F}(\mathcal{D}, \mathcal{O}) := \{\chi : \mathcal{D} \to \mathcal{O} \text{ smooth}\}$
2. The usual advection equations (for the mass density, the entropy, the magnetic field, etc) need to be augmented by a new advected quantity on which the group $G$ acts by an affine representation.
Algebraic structure of the symmetry group of complex fluids:

\text{Diff}(\mathcal{D}) \text{ acts on } \mathcal{F}(\mathcal{D}, \emptyset) \text{ via the } \textbf{right} \text{ action}

\[(\eta, \chi) \in \text{Diff}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \emptyset) \mapsto \chi \circ \eta \in \mathcal{F}(\mathcal{D}, \emptyset).\]

Therefore, the group multiplication is given by

\[(\eta, \chi)(\varphi, \psi) = (\eta \circ \varphi, (\chi \circ \varphi)\psi).\]

Fix a volume form \(\mu\) on \(\mathcal{D}\), so identify densities with functions, one-form densities with one-forms, etc.

The \textbf{Lie algebra} \(g\) of the semidirect product group is

\[\mathfrak{g} = \mathfrak{X}(\mathcal{D}) \circledast \mathcal{F}(\mathcal{D}, \emptyset) \ni (u, \nu),\]

and the Lie bracket is computed to be

\[\text{ad}_{(u, \nu)}(v, \zeta) = (\text{ad}_u v, \text{ad}_\nu \zeta + d\nu \cdot v - d\zeta \cdot u),\]

where \(\text{ad}_u v = -[u, v]\), \(\text{ad}_\nu \zeta \in \mathcal{F}(\mathcal{D}, \emptyset)\) is given by \(\text{ad}_\nu \zeta(x) := \text{ad}_\nu(x)\zeta(x)\), and \(d\nu \cdot v \in \mathcal{F}(\mathcal{D}, \emptyset)\) is given by \(d\nu \cdot v(x) := d\nu(x)(v(x))\).
The dual Lie algebra is identified with

\[ g^* = \Omega^1(\mathcal{D}) \otimes \mathcal{F}(\mathcal{D}, \mathfrak{o}^*) \ni (m, \kappa), \]

through the pairing

\[ \langle (m, \kappa), (u, \nu) \rangle = \int_{\mathcal{D}} (m \cdot u + \kappa \cdot \nu) \mu. \]

The dual map to \( \text{ad}_{(u, \nu)} \) is

\[ \text{ad}^*_{(u, \nu)}(m, \kappa) = \left( \mathcal{L}_u m + (\text{div} u) m + \kappa \cdot d \nu, \text{ad}^*_{\nu} \kappa + \text{div}(u \kappa) \right). \]

**Explanation of the symbols:**

- \( \kappa \cdot d \nu \in \Omega^1(\mathcal{D}) \) denotes the one-form defined by
  \[ (\kappa \cdot d \nu)(v_x) := \kappa(x)(d \nu(v_x)) \]

- \( \text{ad}^*_{\nu} \kappa \in \mathcal{F}(\mathcal{D}, \mathfrak{o}^*) \) denotes the \( \mathfrak{o}^* \)-valued mapping defined by
  \[ (\text{ad}^*_{\nu} \kappa)(x) := \text{ad}^*_{\nu(x)}(\kappa(x)). \]

- \( u \kappa \) is the 1-contravariant tensor field with values in \( \mathfrak{o}^* \) defined by
  \[ (u \kappa)(\alpha_x) := \alpha_x(u(x)) \kappa(x) \in \mathfrak{o}^*. \]
So \( u_\kappa \) is a generalization of the notion of a vector field. \( \mathfrak{X}(\mathcal{D}, \phi^*) \) denotes the space of all \( \phi^* \)-valued 1-contravariant tensor fields.

- \( \text{div}(u) \) denotes the divergence of the vector field \( u \) with respect to the fixed volume form \( \mu \). Recall that it is defined by the condition

\[
(\text{div} u)\mu = \mathcal{L}_u\mu.
\]

This operator can be naturally extended to the space \( \mathfrak{X}(\mathcal{D}, \phi^*) \) as follows. For \( w \in \mathfrak{X}(\mathcal{D}, \phi^*) \) we write \( w = w_a \varepsilon^a \) where \( (\varepsilon^a) \) is a basis of \( \phi^* \) and \( w_a \in \mathfrak{X}(\mathcal{D}) \). We define \( \text{div} : \mathfrak{X}(\mathcal{D}, \phi^*) \to \mathcal{F}(\mathcal{D}, \phi^*) \) by

\[
\text{div} w := (\text{div} w_a)\varepsilon^a.
\]

Note that if \( w = u_\kappa \) we have

\[
\text{div}(u_\kappa) = d_\kappa \cdot u + (\text{div} u)_\kappa.
\]

Split the space of advected quantities in two: usual ones and new ones that involve affine actions and cocycles.
GEOMETRY OF THE ERICKSEN-LESLIE EQUATIONS

• **Symmetry group:** \( G = \text{Diff}(\mathcal{D}) \circledast \mathcal{F}(\mathcal{D}, SO(3)) \ni (\eta, \chi) \), macromotion and micromotion.

• **Advected variables:** \( V^* = \mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathbb{R}^3) \ni (\rho, n) \), mass density and director field.

• **Representation of \( G \) on \( V^* \):**
\[
(\rho, n) \mapsto \left( J(\eta)(\rho \circ \eta), \chi^{-1}(n \circ \eta) \right).
\]

• **Associated infinitesimal actions and diamond operations:**

\[

\nu = \nabla n \cdot u, \quad n\nu = n \times \nu, \quad m \diamond_1 n = -\nabla n^\top \cdot m \quad \text{and} \quad m \diamond_2 n = n \times m,
\]

where \( \nu, m, n \in \mathcal{F}(\mathcal{D}, \mathbb{R}^3) \).

• **No cocycle.**
• **EP equations** for \((\text{Diff}(\mathcal{D}) \circledast \mathcal{F}(\mathcal{D}, \text{SO}(3))) \circledast (\mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathbb{R}^3))\):

\[
\begin{align*}
\frac{\partial}{\partial t} \frac{\delta \ell}{\delta u} &= -\mathcal{L}_u \frac{\delta \ell}{\delta u} - \text{div } u \frac{\delta \ell}{\delta u} - \frac{\delta \ell}{\delta \nu} \cdot d\nu + \rho d \frac{\delta \ell}{\delta \rho} - \left( \nabla n^T \cdot \frac{\delta \ell}{\delta n} \right)^\flat, \\
\frac{\partial}{\partial t} \frac{\delta \ell}{\delta \nu} &= \nu \times \frac{\delta \ell}{\delta \nu} - \text{div } \left( \frac{\delta \ell}{\delta \nu} u \right) + n \times \frac{\delta \ell}{\delta n},
\end{align*}
\]

• **The advection equations** are:

\[
\begin{align*}
\frac{\partial}{\partial t} \rho &+ \text{div}(\rho u) = 0, \\
\frac{\partial}{\partial t} n &+ \nabla n \cdot u + n \times \nu = 0.
\end{align*}
\]

• **Reduced Lagrangian** for nematic and cholesteric liquid crystals:

\[
\ell(u, \nu, \rho, n) := \frac{1}{2} \int_\mathcal{D} \rho \|u\|^2 \mu + \frac{1}{2} \int_\mathcal{D} \rho J \|\nu\|^2 \mu - \int_\mathcal{D} \rho F(\rho^{-1}, n, \nabla n) \mu.
\]
• EP equations for this $\ell$: yield
\[
\begin{align*}
\rho \left( \frac{\partial}{\partial t} u + \nabla u \cdot u \right) &= \text{grad} \frac{\partial F}{\partial \rho^{-1}} - \partial_i \left( \rho \frac{\partial F}{\partial n, i} \cdot \nabla n \right), \\
\rho J \frac{D}{Dt} \nu &= h \times n, \\
\end{align*}
\]

(motion)

(advection)
\[
\begin{align*}
\frac{\partial}{\partial t} \rho + \text{div}(\rho u) &= 0, \\
\frac{D}{Dt} n &= \nu \times n, \\
\end{align*}
\]

• Recovering the Ericksen-Leslie equations:
Observation: if $\nu$ and $n$ are solutions of the EP equations then:

(i) $\|n_0\| = 1$ implies $\|n\| = 1$ for all time.

(ii) $\frac{D}{Dt} (n \cdot \nu) = 0$. Therefore, $n_0 \cdot \nu_0 = 0$ implies $n \cdot \nu = 0$ for all time.
(iii) Suppose that \( n_0 \cdot \nu_0 = 0 \) and \( \|n_0\| = 1 \). Then
\[
\frac{D}{Dt} n = \nu \times n \quad \text{becomes} \quad \nu = n \times \frac{D}{Dt} n
\]

and
\[
\rho J \frac{D}{Dt} \nu = h \times n \quad \text{becomes} \quad \rho J \frac{D^2}{Dt^2} n - 2q n + h = 0.
\]

Therefore:

If \((u, \nu, \rho, n)\) is a solution of the Euler-Poincaré equations with initial conditions \( n_0 \) and \( \nu_0 \) satisfying \( \|n_0\| = 1 \) and \( n_0 \cdot \nu_0 = 0 \), then \((u, \rho, n)\) is a solution of the Ericksen-Leslie equations.

Conversely:

if \((u, \rho, n)\) is a solution of the Ericksen-Leslie equations, define
\[
\nu := n \times \frac{D}{Dt} n \in \mathcal{F}(\mathcal{D}, \mathbb{R}^3).
\]

Then, \((u, \nu, \rho, n)\) is a solution of the Euler-Poincaré equations.

Use these equations plus add dissipation, get well posedness.
GEOMETRY OF THE ERINGEN EQUATIONS

• **Symmetry group**: same group as before \( G = \text{Diff}(\mathcal{D}) \circledast \mathcal{F}(\mathcal{D}, \mathcal{O}) \).

• **Advected variables**: \( V^* = \mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \text{Sym}(3)) \times \Omega^1(\mathcal{D}, \mathfrak{so}(3)) \ni (\rho, j, \gamma) \), mass density, microinertia tensor, strain.

• **Representation**: \( (\eta, \chi) \in \text{Diff}(\mathcal{D}) \circledast \mathcal{F}(\mathcal{D}, \text{SO}(3)) \) acts *linearly* on the advected quantities \( (\rho, j) \in \mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \text{Sym}(3)) \), by

\[
(\rho, j) \mapsto \left( J(\eta)(\rho \circ \eta), \chi^T(j \circ \eta)\chi \right), \quad \chi^T = \chi^{-1}.
\]

• **Affine representation**: \( (\eta, \chi) \in \text{Diff}(\mathcal{D}) \circledast \mathcal{F}(\mathcal{D}, \text{SO}(3)) \) acts on \( \gamma \in \Omega^1(\mathcal{D}, \mathfrak{so}(3)) \) by an *affine* representation

\[
\gamma \mapsto \chi^{-1}(\eta^*\gamma)\chi + \chi^{-1}\nabla\chi.
\]

Note that \( \gamma \) transforms as a connection.

**So, Eringen’s wryness tensor is a connection one-form.**
• The reduced Lagrangian of Eringen’s theory:

\[ \ell: \left[ \mathcal{X}(\mathcal{D}) \otimes \mathcal{F}(\mathcal{D}, \mathbb{R}^3) \right] \otimes \left[ \mathcal{F}(\mathcal{D}) \oplus \mathcal{F}(\mathcal{D}, \text{Sym}(3)) \oplus \Omega^1(\mathcal{D}, \mathfrak{so}(3)) \right] \rightarrow \mathbb{R} \]

\[ \ell(u, \nu, \rho, j, \gamma) = \frac{1}{2} \int_{\mathcal{D}} \rho \|u\|^2 \mu + \frac{1}{2} \int_{\mathcal{D}} \rho (j \nu \cdot \nu) \mu - \int_{\mathcal{D}} \rho \psi(\rho^{-1}, j, \gamma) \mu. \]

• The affine Euler-Poincaré equations for \( \ell \) are:

\[
\begin{align*}
\rho \left( \frac{\partial}{\partial t} u + \nabla_u u \right) &= \text{grad} \frac{\partial \psi}{\partial \rho^{-1}} - \partial_k \left( \rho \frac{\partial \psi}{\partial \gamma^a} \gamma^a \right), \\
j \frac{D}{Dt} \nu - (j \nu) \times \nu &= -\frac{1}{\rho} \text{div} \left( \rho \frac{\partial \psi}{\partial \gamma} \right) + \gamma^a \times \frac{\partial \psi}{\partial \gamma^a}, \\
\frac{\partial}{\partial t} \rho + \text{div}(\rho u) &= 0, \quad \frac{D}{Dt} j + [j, \nu] = 0, \\
\frac{\partial}{\partial t} \gamma + \mathfrak{L}_u \gamma + \text{d}^\gamma \nu &= 0, \quad \hat{\nu} = \nu \in \mathcal{F}(\mathcal{D}, \mathfrak{so}(3)),
\end{align*}
\]

where \( \text{d}^\gamma \) is the covariant \( \gamma \)-derivative defined by

\[ \text{d}^\gamma \nu(v) := \text{d} \nu(v) + [\gamma(v), \nu]. \]

This system is identical Eringen’s equations.
The general affine Euler-Poincaré theory applied to many other complex fluids: spin chain, Yang-Mills MHD (classical and superfluid), Hall MHD, multivelocity superfluids (classical and superfluid), HBVK dynamics for superfluid $^4\text{He}$, Volovik-Dotsenko spin glasses, microfluids, Lhuillier-Rey equations (see Gay-Balmaz & Ratiu [2009]).

**Kelvin-Noether circulation theorem for micropolar liquid crystals**

\[
\frac{d}{dt} \oint_{C_t} u^b = \oint_{C_t} \frac{\partial \psi}{\partial j} \, dj + \frac{\partial \psi}{\partial \gamma} \, i \, d\gamma - \frac{1}{\rho} \text{div} \left( \rho \frac{\partial \psi}{\partial \gamma} \right) \gamma.
\]

The $\gamma$-circulation formulated in $\mathbb{R}^3$

\[
\frac{d}{dt} \oint_{C_t} \gamma = \oint_{C_t} \nu \times \gamma
\]
Physically, the Eringen equations should imply the Ericksen-Leslie equations. *Eringen* [1993] proposes

\[ j := J(I_3 - n \otimes n), \quad \gamma := \nabla n \times n \]

to pass from his equations to the Ericksen-Leslie equations. This is FALSE! Two arguments: brute force computation and symmetry considerations. So, one needs to do something else. However, not all is wrong:

1. it is true that there is \( \Psi(j, \gamma) \) such that

\[ \Psi(J(I_3 - n \otimes n), \nabla n \times n) = F(n, \nabla n). \]

2. the definition \( j := J(I_3 - n \otimes n) \) is geometrically consistent.

**WE SHALL USE THE TOOLS OF GEOMETRIC MECHANICS TO GIVE A DEFINITIVE ANSWER.**

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Note: For simplicity, we consider motionless nematics. The present approach easily generalizes to the flowing case.

STEP I: $\gamma$-formulation of Ericksen-Leslie

The material Lagrangian for nematic motionless liquid crystals $\mathcal{L} : T\mathcal{F}(D, SO(3)) \rightarrow \mathbb{R}$, $D \subset \mathbb{R}^3$, is thus given by

$$\mathcal{L}(\chi, \dot{\chi}) = \frac{1}{2} J \int_D \|\dot{\chi}n_0\|^2 \mu - \int_D F(\chi n_0, \nabla(\chi n_0)) \mu,$$

where, usually $n_0 = \hat{z}$, $J$ is the microinertia constant, and $F$ is the Oseen-Frank free energy:

$$F(n, \nabla n) = K_2 (n \cdot \text{curl } n) + \frac{1}{2} K_{11} (\text{div } n)^2 + \frac{1}{2} K_{22} (n \cdot \text{curl } n)^2$$

$$+ \frac{1}{2} K_{33} \|n \times \text{curl } n\|^2.$$

IDEA: Apply two different EP reductions to this Lagrangian.
Write $\mathcal{L}(\chi, \dot{\chi}) = L_{n_0}(\chi, \dot{\chi})$, where the Lagrangian

$$L_{n_0} : T\mathcal{F}(\mathcal{D}, SO(3)) \to \mathbb{R}$$

is invariant under the right action

$$(\chi, n_0) \mapsto (\chi \psi, \psi^{-1} n_0)$$

of $\psi \in \mathcal{F}(\mathcal{D}, SO(3))_{n_0}$ (the $G_{a_0}$ of the general theory). So get the reduced Euler-Poincaré Lagrangian

$$\ell_1(\nu, n) = \frac{1}{2} J \int_{\mathcal{D}} \|\nu \times n\|^2 \mu - \int_{\mathcal{D}} F(n, \nabla n) \mu,$$

$\dot{\nu} = \dot{\chi} \chi^{-1}$, $n = \chi n_0$. The Euler-Poincaré equations are

$$\begin{cases} \frac{d}{dt} \frac{\delta \ell_1}{\delta \nu} = \nu \times \frac{\delta \ell_1}{\delta \nu} + n \times \frac{\delta \ell_1}{\delta n} \\ \partial_t n + n \times \nu = 0 \end{cases}$$
More explicitly, upon denoting $h = -\delta l_1 / \delta n$, one has

\[
\begin{cases}
J \partial_t \nu = h \times n \\
\partial_t n + n \times \nu = 0,
\end{cases}
\]

which are the Ericksen-Leslie equations of nematodynamics if $\|n_0\| = 1$ and $\nu_0 \cdot n_0 = 0$:

\[
J \frac{d^2 n}{dt^2} - 2 \left( n \cdot h + J n \cdot \frac{d^2 n}{dt^2} \right) n + h = 0.
\]

Since there is no macromotion, $\frac{d}{dt} = \frac{D}{Dt}$. 
Start with the same Lagrangian. If $n_0$ is constant, we can write
\[
\mathcal{L}(\chi, \dot{\chi}) = \frac{1}{2} J \int_\mathcal{D} \|\dot{\chi} n_0\|^2 \mu - \int_\mathcal{D} F(\chi n_0, \nabla(\chi n_0)) \mu
\]
\[
= \frac{1}{2} J \int_\mathcal{D} \|\dot{\chi} n_0\|^2 \mu - \int_\mathcal{D} F(\chi n_0, (\nabla \chi) \chi^{-1} \cdot \chi n_0)) \mu,
\]
and we view $\mathcal{L}$ as
\[
\mathcal{L}(\chi, \dot{\chi}) = L(n_0, \gamma_0=0)(\chi, \dot{\chi}).
\]
This Lagrangian is invariant under the right action
\[
(\chi, n_0, \gamma_0) \mapsto (\chi \psi, \psi^{-1} n_0, \psi^{-1} \gamma_0 \psi + \psi^{-1} \nabla \psi)
\]
of the isotropy subgroup $\mathcal{F}(\mathcal{D}, SO(3))(n_0, 0) = \mathcal{F}(\mathcal{D}, S^1) \cap SO(3) = S^1$
(the $G_{\alpha_0}^c$ of the general theory).
So we get the reduced affine Euler-Poincaré Lagrangian
\[ \ell_2(\nu, n, \gamma) = \frac{1}{2} J \int_D \|\nu \times n\|^2 \mu - \int_D F(n, -\gamma \times n) \mu. \]
\[ \hat{\nu} = \dot{\chi} \chi^{-1}, \quad n = \chi n_0, \quad \gamma = -(\nabla \chi) \chi^{-1} \in \Omega^1(\mathcal{D}, \mathfrak{so}(3)). \]

\[ \nabla \n = n \times \gamma \text{ and not } \gamma := \nabla n \times n. \]  
Note: \( \gamma \) is NOT determined by \( n \)! It does not matter.

**Notations:**
\[ \gamma = \gamma_i dx^i \in \Omega^1(\mathcal{D}; \mathbb{R}^3), \]  
define \( \gamma \times n \in \Omega^1(\mathcal{D}, \mathbb{R}^3) \) by \( \gamma \times n = (\gamma_i \times n) dx^i \), or
\[ (\gamma \times n)(v_x) = \gamma(v_x) \times n, \quad v_x \in T_x \mathcal{D}. \]

**Important:** \( L(\chi, \dot{\chi}, n_0, 0) \) may not be defined when \( \gamma_0 \neq 0 \). \( \ell_2 \) is only defined on the orbit of \( \gamma_0 = 0 \), i.e., if \( \gamma = -(\nabla \chi) \chi^{-1} \). However, this does not affect reduction, as long as the expression \( L(\chi, \dot{\chi}, n_0, 0) \) is invariant under the isotropy group of \( \gamma_0 = 0 \). This occurs in the reduction for molecular strand dynamics with nonlocal interactions (Ellis, Gay-Balmaz, Holm, Putkaradze, Ratiu [2010]).

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The affine Euler-Poincaré equations are

\[
\begin{align*}
\frac{d}{dt} \frac{\delta \ell_2}{\delta \nu} &= \nu \times \frac{\delta \ell_2}{\delta \nu} + \text{div} \frac{\delta \ell_2}{\delta \gamma} + \text{Tr} \left( \gamma \times \frac{\delta \ell_2}{\delta \gamma} \right) + n \times \frac{\delta \ell_2}{\delta n} \\
\partial_t n + n \times \nu &= 0 \\
\partial_t \gamma + \gamma \times \nu + \nabla \nu &= 0, \quad \gamma_0 = 0.
\end{align*}
\]

If \( \gamma_0 \neq 0 \), these reduced equations still make sense, and they are an extension of EL dynamics to account for disclination dynamics. Note that these equations consistently preserve the relation \( \nabla n = n \times \gamma \), since

\[
\left( \frac{\partial}{\partial t} - \nu \times \right) (\nabla n - n \times \gamma) = 0.
\]

EP equations for \( \ell_1 \) and AEP equations \( \ell_2 \) are equivalent since they are induced by the SAME Euler-Lagrange equations for \( \mathcal{L}(\chi, \dot{\chi}) \) on \( T\mathcal{F}(D, SO(3)) \).

Moreover, the AEP equations allow for a generalization of Ericksen-Leslie to the case with disclinations.
**STEP II:** Eringen micropolar theory contains Ericksen-Leslie director theory as a particular case

Recall:

1. Eringen’s Lagrangian (motionless case = no macro motion)

\[ \mathcal{L}(\chi, \dot{\chi}) = \frac{1}{2} \int_{\mathcal{D}} \text{Tr} \left( (i_0 \chi^{-1} \dot{\chi})^T \chi^{-1} \dot{\chi} \right) \mu - \int_{\mathcal{D}} \Psi(\chi j_0 \chi^{-1}, \chi \nabla \chi^{-1} + \chi \gamma_0 \chi^{-1}) \mu, \]

was interpreted as \( \mathcal{L} = L(j_0, \gamma_0) \), where \( i_0 := \frac{1}{2} \text{Tr}(j_0)I_3 - j_0 \). This Lagrangian is invariant under the right affine action

\[ (\chi, j_0, \gamma_0) \mapsto (\chi \psi, \psi^{-1} j_0 \psi, \psi^{-1} \gamma_0 \psi + \psi^{-1} \nabla \psi) \]

of the isotropy subgroup \( \mathcal{F}(\mathcal{D}, SO(3))_{(j_0, \gamma_0)} \).

2. Reduced Lagrangian

\[ \ell_2(\nu, j, \gamma) = \frac{1}{2} \int_{\mathcal{D}} (j \nu) \cdot \nu \mu - \int_{\mathcal{D}} \Psi(j, \gamma) \mu. \]

3. Eringen’s equation are the affine Euler-Poincaré equations for:

\[ G = \mathcal{F}(\mathcal{D}, SO(3)) \]

\[ V^* = \mathcal{F}(\mathcal{D}, \text{Sym}(3)) \times \Omega^1(\mathcal{D}, \mathfrak{s}_0(3)). \]
II.1 Rod-like assumption

Take as initial condition $j_0 = J(I - n_0 \otimes n_0)$. This definition is $\mathcal{F}(\mathcal{D}, SO(3))$-equivariant, so that $j = J(I - n \otimes n)$ for all time.

Consider $\mathcal{L}(\chi, \dot{\chi}) = L(n_0, \gamma_0)(\chi, \dot{\chi}) := L(j_0 = J(I - n_0 \otimes n_0), \gamma_0)(\chi, \dot{\chi})$. This Lagrangian is invariant under the right action

$$(\chi, n_0, \gamma_0) \mapsto (\chi \psi, \psi^{-1} n_0, \psi^{-1} \gamma_0 \psi + \psi^{-1} \nabla \psi)$$

of the isotropy subgroup $\mathcal{F}(\mathcal{D}, SO(3))(n_0, \gamma_0)$. Reduced Lagrangian

$$\ell'_2(\nu, n, \gamma) := \ell_2(\nu, J(I - n \otimes n), \gamma)$$

$$= \frac{J}{2} \int_{\mathcal{D}} \|\nu \times n\|^2 \mu - \int_{\mathcal{D}} \psi(J(I - n \otimes n), \gamma) \mu,$$

Affine Euler-Poincaré equations for $\ell'_2$ are equivalent to Eringen’s equations in which the rod-like assumption has been assumed.

It remains to show that these equations contain as particular case, the Ericksen-Leslie equations.
II.2 No disclination assumption $\gamma_0 = 0$

Same step as earlier: suppose that $n_0$ is constant and take $\gamma_0 = 0$. So the evolution of $\gamma$ is given by

$$\gamma = \theta \chi^{-1}(0) = - (\nabla \chi) \chi^{-1}.$$ 

Since $n = \chi n_0$, we get $\nabla n = n \times \gamma$.

II.3 Recovering the Oseen-Frank free energy

Recall that $\Psi = \Psi(j, \gamma)$, rod-like assumption $j = J(I - n \otimes n)$, and

$$F(n, \nabla n) = K_2 \begin{aligned} \frac{1}{2} (n \cdot \text{curl } n) + \frac{1}{2} K_{11} (\text{div } n)^2 + \frac{1}{2} K_{22} (n \cdot \text{curl } n)^2 \end{aligned}$$

$$+ \begin{aligned} \frac{1}{2} K_{33} \|n \times \text{curl } n\|^2 \end{aligned}.$$ 

$K_2 \neq 0$ for cholesterics, $K_2 = 0$ for nematics.

So we need to show that there exists $\Psi = \Psi(j, \gamma)$ such that

$$\Psi(j, \gamma) = \Psi(J(I - n \otimes n), \gamma) = F(n, n \times \gamma) = F(n, \nabla n).$$

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**Lemma** The Oseen-Frank free energy can be expressed in terms of $\Psi = \Psi(j, \gamma)$ as

$$
\Psi(j, \gamma) = \frac{K_2}{J} \text{Tr}(j\gamma) + \frac{K_{11}}{J} \left( \text{Tr}(\gamma^A)^2 \left( \text{Tr}(j) - J \right) - 2 \text{Tr}(j(\gamma^A)^2) \right) + \frac{1}{2} \frac{K_{22}}{J^2} \text{Tr}^2(j\gamma) - \frac{K_{33}}{J} \text{Tr}\left( ((\gamma j)^A - J\gamma A)^2 \right).
$$

So we can rewrite the reduced Eringen Lagrangian in the rod-like assumption

$$
\ell'_2(\nu, n, \gamma) = \ell_2(\nu, J(I - n \otimes n), \gamma) = \frac{J}{2} \int_D \|\nu \times n\|^2 \mu - \int_D \Psi(J(I - n \otimes n), \gamma) \mu
$$

as

$$
\ell'_2(\nu, n, \gamma) = \frac{J}{2} \int_D \|\nu \times n\|^2 \mu - \int_D F(n, n \times \gamma) \mu,
$$

Same substitution in the unreduced Eringen Lagrangian in the rod-like assumption yields $\mathcal{L}(\chi, \dot{\chi}) = L(n_0, \gamma_0 = 0)(\chi, \dot{\chi})$. 

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II.4 Recovering Ericksen-Leslie theory

1. Interpret now this $\mathcal{L}(\chi, \dot{\chi})$ as $L_{n_0}(\chi, \dot{\chi})$ instead of $L_{(n_0, \gamma=0)}(\chi, \dot{\chi})$.

2. Check that this Lagrangian is $\mathcal{F}(\mathcal{D}, SO(3))_{n_0}$-invariant under the action $(\chi, n_0) \mapsto (\chi \psi, \psi^{-1} n_0)$.

3. Implement Euler-Poincaré reduction associated to the action $(\chi, n_0) \mapsto (\chi \psi, \psi^{-1} n_0)$ and obtain the reduced Lagrangian

$$\ell'_1(\nu, n) = \frac{J}{2} \int_{\mathcal{D}} \|\nu \times n\|^2 \mu - \int_{\mathcal{D}} F(n, \nabla n) \mu$$

(Previously we considered affine Euler-Poincaré reduction associated to the action $(\chi, n_0, \gamma_0) \mapsto (\chi \psi, \psi^{-1} n_0, \psi^{-1} \gamma_0 \psi + \psi^{-1} \nabla \psi)$, with reduced Lagrangian $\ell'_2$).

By general reduction theory: EP equations for $\ell'_1$ and AEP equations $\ell'_2$ are equivalent since they are induced by the SAME Euler-Lagrange equations for $\mathcal{L}(\chi, \dot{\chi})$ on $T\mathcal{F}(\mathcal{D}, SO(3))$. 

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It remains to show that the EP equations for $\ell_1'$ are the Ericksen-Leslie equations. True, by direct verification.

We have thus proved:

**THEOREM**: The Eringen micropolar theory of liquid crystals contains as a particular case the Ericksen-Leslie director theory. More precisely, the Ericksen-Leslie theory is recovered by assuming rod-like molecules: $j = J(I - n \otimes n)$ and absence of disclinations $\gamma_0 = 0$.

Summary of method:

- This is shown by considering two distinct Euler-Poincaré reductions associated with distinct advected quantities.

- This allows us to replace the non-consistent definition $\gamma := \nabla n \times n$ by the relation $\nabla n = n \times \gamma$ and to solve the inconsistencies in Eringen’s approach.
The Lagrangians for different theories. Lagrangians on the center line are for the material descriptions of the models. Slanted arrows are Euler-Poincaré reduction; vertical arrows show how the theories are embedded in each other. By Euler-Poincaré reduction theory, all the Lagrangians related by a dashed arrow are equivalent. Consequence: any question in a given model can be treated, equivalently, with any of the three Lagrangians in a given triangle.
$\hat{\nu} := \dot{\chi}\chi^{-1}$
Final remarks: 1.) All the discussion here can be easily extended to moving liquid crystals. One applies EP, respectively affine EP, theory, as discussed earlier. Then the same considerations as above show that Eringen micropolar theory contains Ericksen-Leslie nematodynamics.

2.) Other inconsistencies in the micropolar description: Eringen defines a smectic liquid crystal by \( \text{Tr}(\gamma) = \gamma_1^1 + \gamma_2^2 + \gamma_3^3 = 0 \). This is not preserved by the evolution \( \gamma = \eta^* (\chi \gamma_0 \chi^{-1} + \chi \nabla \chi^{-1}) \).
Consistent with the statement: the equation
\[
\frac{\partial \gamma}{\partial t} + \mathcal{E}_u \gamma + d\nu + \gamma \times \nu = 0
\]
does not imply that if \( \text{Tr}(\gamma_0) = 0 \) then \( \text{Tr}(\gamma) = 0 \) for all time.
Is Eringen’s definition of smectic incorrect? Instead of the trace need an \( \mathcal{F}(\mathcal{D}, SO(3)) \)-invariant function (of \( \gamma \)) under the action
\[
v \mapsto \chi^{-1}v + \chi^{-1}\nabla \chi, \quad v \in \mathcal{F}(\mathcal{D}, \mathbb{R}^3), \quad \chi \in \mathcal{F}(\mathcal{D}, SO(3)).
\]
We do not know how to choose a physically reasonable function of this type.

3.) Other difficulties in liquid crystals dynamics may be solved by using the tools of geometric mechanics (disclinations, defects,...)
References


