ALGEBRAS IN MONOIDAL CATEGORIES
MESSAGE: algebras in monoidal categories are natural and nice
Possible motivations:

- Express similarities:
  - Traditionally: Hopf algebras as generalizations of group algebras
  - In fact: group = Hopf algebra in Set

- Disclose fake peculiarities: e.g. non-associativity:
  - Octonions = associative algebra in \( \mathbb{Z}_2^3 \)-\text{Vect}

- Increase degree of familiarity:
  - Beilinson-Drinfeld chiral algebra = Lie algebra in some category of \( \mathcal{D} \)-modules

  - Vertex algebra (\( = \) B-D chiral algebra on the formal disk)
    = singular commutative associative algebra in certain functor category
POSSIBLE MOTIVATIONS:

- express similarities:
  
  traditionally: Hopf algebras as generalizations of group algebras
  
  in fact: group = Hopf algebra in Set

- disclose fake peculiarities: e.g. non-associativity:
  
  octonions = associative algebra in $\mathbb{Z}_2 \times 3\text{-}Vect$

- present in applications:
  
  coend Hopf algebras for 3-mf invariants and MCG-reps
  
  coalgebras for cloning operations in quantum mechanics
  
  Frobenius algebras for classifying full CFTs associated with given chiral CFT

SPECIFIC MOTIVATION:

- bulk state space of a full CFT as ... Frobenius algebra in $\mathbb{Z}(C) \simeq C \boxtimes C^{rev}$

CLAIM: allows to construct bulk fields correlation functions on any closed world sheet
Plan

Algebras in monoidal categories

PLAN

- algebras
  - Frobenius algebras
  - Hopf algebras
  - weak Hopf algebras
  - Lie algebras
- sample results
- handle Hopf algebras
- conformal field theory
- bulk state spaces in CFT
- bulk field correlation functions
- appendix
\( k \)-algebra = vector space with multiplication and unit element

\[ \text{category } \mathcal{V}ect_k : \]
- objects = vector spaces over \( k \)
- morphisms = linear maps
- monoidal structure:
  - tensor product \( \otimes_k \) of vector spaces with tensor unit \( 1 = k \)

interpret
- multiplication \( \equiv \) bi linear map \( A \times A \to A \) \( \sim \) linear map \( m : A \otimes_k A \to A \)
- unit element \( 1_A \in A \) \( \sim \) linear map \( \eta : k \to A \quad \eta(c) = c \cdot 1 \)

\[ \implies \text{\( k \)-algebra} = \text{algebra object } (A, m, \eta) \in \mathcal{V}ect_k \]
associative algebra = object $A$ + morphism

$m : A \otimes A \to A$
A

associative algebra = object $A$ + morphism

$m : A \otimes A \rightarrow A$

such that

$m \circ (m \otimes \text{id}) = m \circ (\text{id} \otimes m)$
**Algebras**

- **associative algebra** = object $A$ + morphism
  
  $$m : \ A \otimes A \rightarrow A$$

- **unital algebra** $(A, m, \eta)$:
  
  $$\eta : \ 1 \rightarrow A$$

  $$m \circ (\eta \otimes \text{id}) = \text{id} = m \circ (\text{id} \otimes \eta)$$

- **natural setting**: strict monoidal categories $(C, \otimes, 1)$
Coalgebras

Algebras in monoidal categories

- **coassociative coalgebra** \((C, \Delta)\):

  \[
  \Delta : \ A \to A \otimes A
  \]

  \[
  (\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta
  \]

- **co-unital coalgebra** \((C, \Delta, \varepsilon)\):

  \[
  \varepsilon : \ A \to 1
  \]

  \[
  (\varepsilon \otimes \text{id}) \circ \Delta = \text{id} = (\text{id} \otimes \varepsilon) \circ \Delta
  \]

- **natural setting**: strict monoidal categories \((C, \otimes, 1)\)
**Frobenius algebras**

Algebras in monoidal categories

- **Algebra**:
  - Diagrams showing algebraic structures.

- **Coalgebra**:
  - Diagrams showing coalgebraic structures.

- **Frobenius algebra**: algebra and coalgebra and coproduct a **bimodule morphism**:
  - Diagrams illustrating the properties of Frobenius algebras and the associated bimodule morphism.
Frobenius algebras

**Algebra**: 

\[
\begin{align*}
\begin{array}{ccc}
\vdots & = & \vdots \\
\end{array}
\end{align*}
\]

**Coalgebra**: 

\[
\begin{align*}
\begin{array}{ccc}
\vdots & = & \vdots \\
\end{array}
\end{align*}
\]

**Frobenius algebra**: algebra and coalgebra

and coproduct a **bimodule morphism**:

\[
\begin{align*}
\begin{array}{ccc}
\vdots & = & \vdots \\
\end{array}
\end{align*}
\]

equivalent to existence of non-degenerate invariant form

if $C$ is rigid
**Frobenius algebras**

- **Frobenius** algebra: algebra and coalgebra and coproduct a **bimodule morphism**:

  \[
  Z \quad \text{isymmetric Frobenius algebra:}
  \]

  \[
  \text{natural setting: monoidal categories } (C, \otimes, 1)
  \]

- **symmetric** Frobenius algebra:

  \[
  A^\lor
  \]

- **special** Frobenius algebra:

  \[
  \neq 0
  \]

  \[
  \text{natural setting: sovereign rigid monoidal categories}
  \]
**Hopf algebras**

**Algebras in monoidal categories**

- **bi** algebra: algebra and coalgebra
  and coproduct and counit algebra morphisms:

![Diagram of algebra morphisms]

- **Hopf** algebra: bialgebra with antipode

![Diagram of Hopf algebra morphisms]

natural setting: braided monoidal categories
Hopf algebras

- **bi** algebra: algebra and coalgebra
  and coproduct and counit algebra morphisms:

- **Hopf** algebra: bialgebra with antipode

- left integral on $H$: $\Lambda \in \text{Hom}(1, H)$ s.t. $m \circ (\text{id}_A \otimes \Lambda) = \Lambda \circ \varepsilon$
Hopf algebras

**bi** algebra: algebra and coalgebra

and coproduct and counit algebra morphisms:

\[
\begin{align*}
\text{coproduct} & : \quad (\Delta \otimes \text{id}) \circ \lambda = \eta \circ \lambda \\
\text{counit} & : \quad \lambda \in \text{Hom}(H, 1) \quad \text{s.t.} \quad (\lambda \otimes \text{id}_H) \circ \Delta = \eta \circ \lambda
\end{align*}
\]
NB: weak Hopf algebras

A weak Hopf algebra \((H, m, \eta, \Delta, \varepsilon, s)\):

- \(\Delta\) not nec. unital
- \(\varepsilon\) not nec. algebra morphism
- Weak antipode

Note: \(\varepsilon\) not nec. unital and \(\Delta\) not nec. morphism.
Lie algebra: object $L$ with morphism $\ell \in \text{Hom}_C(L \otimes L, L)$ s.t.

- antisymmetry

\[ \tau = -\]

- Jacobi identity

\[ + + = 0 \]

natural setting: symmetric monoidal additive categories $(C, \otimes, 1, \tau)$

examples:

- Lie superalgebra = Lie algebra in $S\text{Vect} = \mathbb{Z}_2\text{-Vect}$
- color Lie algebra = Lie algebra in $\Gamma\text{-Vect}$ ($\Gamma$ abelian group with skew bicharacter)
**Lie algebra**: object $L$ with morphism $\ell \in \text{Hom}_C(L \otimes L, L)$ s.t.

- antisymmetry

  \[ \tau = - \]

- Jacobi identity

  \[ + + = 0 \]

**natural setting**: symmetric monoidal additive categories $(C, \otimes, 1, \tau)$

**examples**:

- Hom-Lie algebra in $C$ (Jacobi identity modified by an automorphism)

  \[ = \text{Lie algebra in } \mathcal{H}C \]
**Lie algebra**: object $L$ with morphism $\ell \in \text{Hom}_C(L \otimes L, L)$ s.t.

- **antisymmetry**
  
  ![antisymmetry_diagram]

- **two Jacobi identities**
  
  ![two_jacobi_diagrams]

**natural setting**: braided monoidal additive categories $(C, \otimes, 1, c)$

**example**: commutator algebras satisfying antisymmetry
Lie algebra: object $L$ with morphism $\ell \in \text{Hom}_C(L \otimes L, L)$ s.t.

- antisymmetry

- Jacobi identity/ies ($C$ symmetric/braided)

more conceptual approaches for braided $C$:

- quantum Lie bracket related to adjoint action for Hopf algebras in $C$
  compatible with a coalgebra structure of $L$
  satisfies a generalized Jacobi identity
Lie algebra: object $L$ with morphism $\ell \in \text{Hom}_C(L \otimes L, L)$ s.t.

- antisymmetry

- Jacobi identity/ies ($C$ symmetric/braided)

- more conceptual approaches for braided $C$:
  - quantum Lie bracket related to adjoint action for Hopf algebras in $C$
  - family of $n$-ary products in $\Gamma\text{-Vect}$ ($\Gamma$ abelian group with bicharacter)

References:
- Pareigis 1997
- Kharchenko 1998
**Lie algebra**: object $L$ with morphism $\ell \in \text{Hom}_C(L \otimes L, L)$ s.t.

- **antisymmetry**

- **Jacobi identity/ies** ($C$ symmetric/braided)

**more conceptual approaches for braided $C$**:

- **quantum Lie bracket** related to adjoint action for Hopf algebras in $C$

- **family of $n$-ary products in $\Gamma\text{-Vec}$** ($\Gamma$ abelian group with bicharacter)

  e.g. primitive elements of a Hopf algebra in $\Gamma\text{-Vec}$
  e.g. derivations of an algebra in $\Gamma\text{-Vec}$

  reduces to color Lie algebras for skew bicharacter

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**PAREIGIS 1997**
consider $C$ symmetric rigid monoidal idempotent-complete $\mathbb{k}$-linear abelian

\begin{itemize}
  \item F Facts:
    \begin{itemize}
      \item trivial abelian Lie algebra $(1, 0)$
      \item commutator algebras $(A, m - m \circ \tau)$
      \item Frobenius algebra $(U \otimes U^\vee, id_U \otimes dU \otimes id_{U^\vee})$
        Morita equivalent to $(1, id)$
      \item $U \otimes U^\vee \cong 1 \oplus R$ and $\text{dim}(U) \neq 0$ $\implies$ $R$ inherits Lie algebra structure
        e.g. adjoint representation $\in \mathfrak{sl}_n(\mathbb{k})$-mod
        e.g. $R_{2024} \in M_{23}$-mod or $M_{24}$-mod
        e.g. for $U = st \in \mathfrak{gl}(m|n)$-mod for $m \neq n$
      \item $L$ simple as object (and non-trivial) $\implies$ simple as Lie algebra
      \item $U \otimes U^\vee \cong 1 \oplus R$ and $\text{dim}(U) \neq 0$ and $U$ generating $C$
        $\implies$ every $V \in C$ carries natural $R$-action (but no handle on all of $R$-mod)
    \end{itemize}
\end{itemize}
consider $\mathcal{C}$ symmetric rigid monoidal idempotent-complete $\mathbb{k}$-linear abelian

**Facts:**

- trivial abelian Lie algebra $(\mathbf{1}, 0)$
- commutator algebras $(A, m - m \circ \tau)$
- Frobenius algebra $(U \otimes U^\vee, \text{id}_U \otimes d_U \otimes \text{id}_{U^\vee})$

  Morita equivalent to $(\mathbf{1}, \text{id}_1)$

- $U \otimes U^\vee \cong \mathbf{1} \oplus R$ and $\dim(U) \neq 0$ $\implies$ $R$ inherits Lie algebra structure
  
  e.g. adjoint representation $\in \mathfrak{sl}_n(\mathbb{k})$-$\text{mod}$
  
  e.g. $R_{2024} \in M_{23}$-$\text{mod}$ or $M_{24}$-$\text{mod}$
  
  e.g. for $U = \text{st} \in \mathfrak{gl}(m|n)$-$\text{mod}$ for $m \neq n$

- $L$ simple as object (and non-trivial) $\implies$ simple as Lie algebra

- $U \otimes U^\vee \cong \mathbf{1} \oplus R$ and $\dim(U) \neq 0$ and $U$ generating $\mathcal{C}$

  $\implies$ every $V \in \mathcal{C}$ carries natural $R$-action (but no handle on all of $R$-$\text{mod}$)

- desirable application: group-theoretical coefficients for Feynman diagrams

Cvitanović 1976
Sample results

Algebras in monoidal categories

- $\mathcal{C}$ fusion category and $\mathcal{M}$ semisimple indecomposable $\mathcal{C}$-module category
  $\implies \mathcal{M} \cong A\text{-mod}_C$ for algebra $A \in \mathcal{C}$
  [Ostrik 2003]

- $\mathcal{M}$ in addition endowed with module trace $\implies A$ Frobenius
  [Schaumann 2013]

- $\mathcal{C}$ modular tensor category
  $\implies$ finite number of Morita classes of simple symmetric special Frobenius algebras
  [F-Runkel-Schweigert 2004]

- $\mathcal{C}$ ribbon and $A$ commutative symmetric special Frobenius
  $\implies$
  - category $A\text{-mod}^\ell_{\text{oc}}$ of local $A$-modules ribbon
  - $\mathcal{C}$ semisimple $\implies A\text{-mod}^\ell_{\text{oc}}$ semisimple
  - $\mathcal{C}$ modular and $A$ simple $\implies A\text{-mod}^\ell_{\text{oc}}$ modular

- $\mathcal{C}$ rigid monoidal and $A$ special Frobenius
  $\implies$ every $M \in A\text{-mod}_C$ is submodule of an induced module $(A \otimes U, m \otimes \text{id}_U)$

- $\mathcal{C}$ modular and $A$ symmetric special Frobenius:
  $\implies$ every $X \in A\text{-bimod}_C$ is sub-bimodule of a braided-induced module $U \otimes^+ A \otimes^- V$
Sample results

- $C$ fusion category and $\mathcal{M}$ semisimple indecomposable $C$-module category
  $\implies \mathcal{M} \cong A\text{-mod}_C$ for algebra $A \in C$  
  \text{Ostrik 2003}

- $\mathcal{M}$ in addition endowed with module trace $\implies A$ Frobenius  
  \text{Schaumann 2013}

- $C$ modular tensor category
  $\implies$ finite number of Morita classes of simple symmetric special Frobenius algebras  
  \text{J-Runkel-Schweigert 2004}

- $C$ ribbon and $A$ Azumaya
  $\implies 1 \to \text{Inn}(A) \to \text{Aut}(A) \to \text{Pic}(C)$ exact  
  \text{Van Oystaeyen-Zhang 1998}

- $C$ modular and $A$ symmetric special Frobenius:
  $A$ Azumaya $\iff \dim C \text{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_j, A)_{ij}$ permutation matrix
  $\implies$ exact sequence $1 \to \text{Inn}(A) \to \text{Aut}(A) \to \text{Pic}(C)$

- $C$ modular and $A$ simple symmetric special Frobenius
  $\implies$ bimodule fusion rules $K_0(A\text{-bimod}_C) \otimes \mathbb{Z} C$
  isomorphic as $C$-algebra to $\bigoplus_{i,j \in I} \text{End}_C(\text{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_j, A))$
Algebras in monoidal categories

- \( C \) fusion category and \( \mathcal{M} \) semisimple indecomposable \( C \)-module category
  \[\Rightarrow \mathcal{M} \simeq A\text{-mod}_C \text{ for algebra } A \in C\]

- \( \mathcal{M} \) in addition endowed with module trace \[\Rightarrow A \text{ Frobenius}\]

- \( C \) modular tensor category
  \[\Rightarrow \text{finite number of Morita classes of simple symmetric special Frobenius algebras}\]

- \( H \) Hopf algebra with \( s \) invertible and (co)integral s.t. \( \lambda \circ \Lambda \in k^\times \)
  \[\Rightarrow H \text{ Frobenius with same algebra structure}\]

- Hopf algebras have symmetric self-braiding:
  \[\text{Schauenburg } 1998\]
example: 1-holed torus in category of 3-cobordisms with corners
(1-morphism in bicategory)

Yetter 1995
Crane-Yetter 1999
example: 1-holed torus in category of 3-cobordisms with corners

example:

\[
\text{coend } L = \int_{U \in \mathcal{D}} U \otimes U^\vee
\]

in any finite abelian \(k\)-linear ribbon category \(\mathcal{D}\)

exists and carries a natural structure of a Hopf algebra in \(\mathcal{D}\)

endowed with left integral and Hopf pairing \(\varpi_L\)

\[
m_L \circ (\iota_U \otimes \iota_V) := \iota_V \otimes \iota_U \circ (\gamma_{U,V} \otimes \text{id}_V \otimes \text{id}_U) \circ (\text{id}_U \otimes c_{U,V} \otimes V) \\
\Delta_L \circ \iota_U := (\iota_U \otimes \iota_U) \circ (\text{id}_U \otimes b_U \otimes \text{id}_U) \\
\eta_L := \iota_1 \quad \varepsilon_L \circ \iota_U := d_U \\
s_L \circ \iota_U := (d_U \otimes \iota_U) \circ (\text{id}_U \otimes c_{U,V} \otimes U \otimes \text{id}_U) \circ (b_U \otimes c_{U,V} \otimes U) \\
\varpi_L \circ (\iota_U \otimes \iota_V) := (d_U \otimes d_V) \circ [\text{id}_U \otimes (c_{V,U} \circ c_{U,V} \otimes \text{id}_V)]
\]

special case: \(\mathcal{D}\) modular \(\implies L = \bigoplus_{i \in \mathcal{I}} S_i^\vee \otimes S_i\)
**Example**: 1-holed torus in category of 3-cobordisms with corners.

\( \text{coend } L = \int_{U \in \mathcal{D}} U \otimes U^\vee \) in any finite abelian \( \mathbb{k} \)-linear ribbon category \( \mathcal{D} \)
example: 1-holed torus in category of 3-cobordisms with corners

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\[ \text{coend } L = \int_{U \in \mathcal{D}} U \otimes U^\vee \] in any finite abelian \( \mathbb{k} \)-linear ribbon category \( \mathcal{D} \)
**Example:** 1-holed torus in category of 3-cobordisms with corners

**Example:**

$$\text{coend } L = \int_{U \in \mathcal{D}} U \otimes U^\vee$$

in any finite abelian \(k\)-linear ribbon category \(\mathcal{D}\)

- **for \(\mathcal{D} \text{ factorizable:}** (e.g. \(\mathcal{D} = H\text{-mod} \) for factorizable ribbon Hopf algebra \(H\))
  - integral two-sided
  - Hopf pairing non-degenerate
  - projective rep \(\pi_{g,m}\) of \(\text{Map}_\Sigma\) on \(\text{Hom}_\mathcal{D}(L \otimes g, U_1 \otimes \cdots \otimes U_m)\)

**Lyubashenko 1995**
example: 1-holed torus in category of 3-cobordisms with corners

example:

$$\text{coend } L = \int^{U \in \mathcal{D}} U \otimes U^\vee$$

in any finite abelian $\mathbb{k}$-linear ribbon category $\mathcal{D}$

for $\mathcal{D}$ factorizable: (e.g. $\mathcal{D} = H\text{-mod}$ for factorizable ribbon Hopf algebra $H$)

- integral two-sided
- Hopf pairing non-degenerate
- projective rep $\pi_{g,m}$ of $\text{Map}_\Sigma$ on $\text{Hom}_\mathcal{D}(L \otimes g, U_1 \otimes \cdots \otimes U_m)$

- e.g. modular $S$-transformation $= \text{composition with automorphism}$

$$S_L := (\varepsilon_L \otimes \text{id}_L) \circ Q_{L,L} \circ (\text{id}_L \otimes \Lambda_L)$$
example: 1-holed torus in category of 3-cobordisms with corners

example:

\[
\text{coend } L = \int_{U \in \mathcal{D}} U \otimes U^\vee \quad \text{in any finite abelian } \mathbb{k}-\text{linear ribbon category } \mathcal{D}
\]

for \( \mathcal{D} \) factorizable: (e.g. \( \mathcal{D} = H\text{-mod} \) for factorizable ribbon Hopf algebra \( H \))

- integral two-sided
- Hopf pairing non-degenerate
- projective rep \( \pi_{g,m} \) of \( \text{Map}_\Sigma \) on \( \text{Hom}_\mathcal{D}(L \otimes g, U_1 \otimes \cdots \otimes U_m) \)

every \( V \in \mathcal{D} \) has natural \( L \)-module structure \( (V, \mathcal{L}_V) \)

\[
\mathcal{L}_V := (\varepsilon_L \otimes \text{id}_V) \circ Q^L_V
\]
2d conformal quantum field theory (CFT) in a nutshell:

- chiral CFT:
  - chiral symmetry algebra – conformal vertex algebra $\mathcal{V}$ (say)
  - category $C$ of $\mathcal{V}$-representations
  - for sufficiently nice $\mathcal{V}$: $C$ factorizable finite ribbon category
  - conformal blocks – sheaves on moduli spaces of curves with marked points

[cf. Belkale’s talk]
2d CFT in a nutshell:

- **chiral CFT:**
  - chiral symmetry algebra – conformal vertex algebra $\mathcal{V}$ (say)
  - category $\mathcal{C}$ of $\mathcal{V}$-representations
  - for sufficiently nice $\mathcal{V}$: $\mathcal{C}$ factorizable finite ribbon category
  - conformal blocks – sheaves on moduli spaces of curves with marked points

- **full local CFT:**
  - conformal world sheet – possibly with boundary, possibly non-orientable
  - **holomorphic factorization**: bulk fields $\in \mathcal{C} \otimes \mathcal{C}^{rev}$
  - local correlation functions:
    specific sections in the chiral blocks for complex double of the world sheet
2d CFT in a nutshell:

- **Chiral CFT:**
  - chiral symmetry algebra – conformal vertex algebra \( \mathcal{V} \) (say)
  - category \( \mathcal{C} \) of \( \mathcal{V} \)-representations
  - for sufficiently nice \( \mathcal{V} \): \( \mathcal{C} \) factorizable finite ribbon category
  - conformal blocks – sheaves on moduli spaces of curves with marked points

- **Full local CFT:**
  - conformal world sheet – possibly with boundary, possibly non-orientable
  - holomorphic factorization: bulk fields \( \in \mathcal{C} \otimes \mathcal{C}^{\text{rev}} \)
  - local correlation functions:
    - specific sections in the chiral blocks for complex double of the world sheet
  - consistent boundary conditions \( \leadsto \) bulk fields \( \in \mathcal{Z}(\mathcal{C}) \)
    - take \( \mathcal{C} \) factorizable: \( \mathcal{C} \otimes \mathcal{C}^{\text{rev}} \simeq \mathcal{Z}(\mathcal{C}) \)
  - in particular: bulk state space \( \mathcal{F} \in \mathcal{C} \otimes \mathcal{C}^{\text{rev}} \)
  - torus partition function \( Z_T(\tau) = \text{Virasoro character of } \mathcal{F} \)
The bulk state space $F$

Bulk state space $F \in C \otimes C^{\text{rev}}$

- simplest situation:
  
  - $C$ modular tensor category and $F$ charge conjugation bulk state space

  \[
  F = F_C^{\text{rat.}} \equiv \bigoplus_{i \in I} S_i^\vee \boxtimes S_i
  \]

  \[Z_T = \sum_{i \in I} \chi_i^* \chi_i\]

- indeed appears as part of a consistent full CFT for any MTC $C$

Felder-Fröhlich-J-Schweigert 2000
The bulk state space $F$

Bulk state space $F \in C \otimes C^{rev}$

- simplest situation:

  $C$ modular tensor category and $F$ charge conjugation bulk state space

  $$F = F_{C}^{rat.} \equiv \bigoplus_{i \in I} S_{i}^{\vee} \boxtimes S_{i}$$

  $$Z_{T} = \sum_{i \in I} \chi_{i}^{*} \chi_{i}$$

- interpretation: combine every object of $C$ with its dual (\textast\, charge conjugate) accounting for *all relations* among objects

- proper concept: coend $F = \int^{U \in C} U \boxtimes U^{\vee} \in C \otimes C^{rev}$
Bulk state space $F \in C \boxtimes C^{\text{rev}}$

- simplest situation:
  $C$ modular tensor category and $F$ charge conjugation bulk state space

$$F = F_C^{\text{rat.}} \equiv \bigoplus_{i \in I} S_i^\vee \boxtimes S_i$$

$$Z_T = \sum_{i \in I} \chi_i^* \chi_i$$

- interpretation: combine every object of $C$ with its dual ( = charge conjugate ) accounting for all relations among objects

- proper concept: coend $F = \int^{U \in C} U \boxtimes U^\vee \in C \boxtimes C^{\text{rev}}$

  object together with dinatural family of morphisms $\nu_U^F : U \boxtimes U^\vee \to F$

  defined by universal property

  $C$ semisimple (thus modular): $F = F_C^{\text{rat.}}$

  $C = H\text{-mod}$ for factorizable ribbon Hopf $C$-algebra:

  $F =$ coregular bimodule $H^* \in H\text{-bimod} \simeq C \boxtimes C^{\text{rev}}$
$\mathcal{C}$ semisimple (RCFT):

$F_C^{\text{rat.}} = Z(1)$ full center
The bulk state space $F$

$\mathcal{C}$ semisimple (RCFT):

- Any bulk state space is of the form $Z(A)$ for $A \in \mathcal{C}$ simple symmetric special Frobenius algebra.

$$Z(A) \cong \bigoplus_{i,j \in I} (S_i \otimes S_j^\vee) \oplus z_{ij}(A)$$

with

$$z_{ij}(A) = \dim_{\mathbb{C}}(\text{Hom}_{\mathcal{A}|\mathcal{A}}(S_i \otimes^+ A \otimes^\vee S_j, A))$$

commutative cocommutative symmetric Frobenius algebra in $\mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$
The bulk state space $F$

$\mathcal{C}$ semisimple (RCFT):

- any bulk state space is of the form $Z(A)$ for $A \in \mathcal{C}$ simple symmetric special Frobenius algebra

\[
Z(A) \cong \bigoplus_{i,j \in I} (S_i \boxtimes S_j^\vee) \oplus z_{ij}(A)
\]

with

\[
z_{ij}(A) = \dim \mathbb{C}(\text{Hom}_{\mathcal{A}} A (S_i \otimes^+ A \otimes^- S_j^\vee, A))
\]

commutative cocommutative symmetric Frobenius algebra in $\mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$

Physics terminology:

- multiplication on $F$ formalizes operator product of bulk fields
- Frobenius pairing formalizes non-degeneracy of two-point correlators of bulk fields on the sphere
- (co-)commutativity implements monodromy invariance of correlators

\[
\text{· · · · · ·}
\]
The bulk state space $F$

**C** semisimple (RCFT):

- **any** bulk state space is of the form $Z(A)$ for $A \in C$ simple symmetric special Frobenius algebra

$$Z(A) \cong \bigoplus_{i,j \in I} \left(S_i \boxtimes S_j^\vee \right) \otimes z_{ij}(A)$$

with

$$z_{ij}(A) = \dim_C(\text{Hom}_{A|A}(S_i \otimes^+ A \otimes^- S_j^\vee, A))$$

commutative cocommutative symmetric Frobenius algebra in $C \boxtimes C^\text{rev}$

**general case**:

- set $m_F \circ (\iota_U^F \otimes \iota_V^F) := \iota_{U^\vee \otimes U} \circ (\gamma_{U,V} \boxtimes c_{U,V})$ with $\gamma_{U,V} : U^\vee \otimes V^\vee \xrightarrow{\cong} (V \otimes U)^\vee$

$\eta_F := \iota_{\mathbb{1}}^F$

$\langle F, m_F, \eta_F \rangle$ is commutative associative unital algebra in $C \boxtimes C^\text{rev}$
The bulk state space $F$

- **$C$** semisimple (RCFT):
  - any bulk state space is of the form $Z(A)$ for $A \in C$ simple symmetric special Frobenius algebra
  - $Z(A) \cong \bigoplus_{i,j \in I} (S_i \boxtimes S_j^\vee) \oplus z_{ij}(A)$ with $z_{ij}(A) = \dim_C(\text{Hom}_{A|A}(S_i \otimes^+ A \otimes^− S_j^\vee, A))$

- commutative cocommutative symmetric Frobenius algebra in $C \boxtimes C^{\text{rev}}$

- general case:
  - set $m_F \circ (\iota_U^F \otimes \iota_V^F) := \iota_V^F \otimes_U \circ (\gamma_{U,V} \boxtimes c_{U,V})$
  - $\eta_F := \iota_1^F$
  - $(F, m_F, \eta_F)$ is commutative associative unital algebra in $C \boxtimes C^{\text{rev}}$

- **$C = H$-mod** (with $H$ not necessarily semisimple)
  - also symmetric Frobenius and cocommutative

$\text{Schweigert-Stigner 2012}$
The bulk state space $F$

- $C$ semisimple (RCFT):
  - any bulk state space is of the form $Z(A)$ for $A \in C$ simple symmetric special Frobenius algebra
  - $Z(A) \cong \bigoplus_{i,j \in I} (S_i \boxtimes S_j^\vee) \oplus z_{ij}(A)$ with $z_{ij}(A) = \dim \mathbb{C} \left( \text{Hom}_{A|A}(S_i \otimes^+ A \otimes^- S_j^\vee, A) \right)$
    commutative cocommutative symmetric Frobenius algebra in $C \boxtimes C^{\text{rev}}$

- general case:
  - set $m_F \circ (\iota_U^F \otimes \iota_V^F) := \iota_U^F \otimes U \circ (\gamma_{U,V} \boxtimes c_{U,V})$
    $\gamma_{U,V} : U^\vee \otimes V^\vee \xrightarrow{\cong} (V \otimes U)^\vee$
  - set $\eta_F := \iota_1^F$
  - $(F, m_F, \eta_F)$ is commutative associative unital algebra in $C \boxtimes C^{\text{rev}}$

- $C = H \text{-mod}$ (with $H$ not necessarily semisimple)
recall: handle Hopf algebra \( L = \int_{U \in \mathcal{D}} U \otimes U^\vee \in \mathcal{D} \)

recall: partial monodromy action \( \kappa_U^L \) of \( L \) on \( U \in \mathcal{D} \)

recall: composition with automorphism \( S_L \) gives modular S-transformation
input: bulk handle Hopf algebra \( K = \int_{X \in C \boxtimes C^{rev}} X \otimes X^\vee \in C \boxtimes C^{rev} \)

input: partial monodromy action \( \varpi^K_X \)
input: bulk handle Hopf algebra \( K = \int_{X \in C \boxtimes C^{rev}} X \otimes X^\vee \in C \boxtimes C^{rev} \)

input: partial monodromy action \( \kappa^K_X \)

further input: \((F, m_F, \eta_F, \Delta_F, \varepsilon_F)\) algebra-coalgebra in \( C \boxtimes C^{rev} \)
input: bulk handle Hopf algebra \( K = \int_{X \in C \boxtimes C^\text{rev}} X \otimes X^\vee \)

input: partial monodromy action \( \kappa_X^K \)

further input: \((F, m_F, \eta_F, \Delta_F, \varepsilon_F)\) algebra - coalgebra

define family of morphisms \( \text{Cor}_{g;p,q} = \text{Cor}_{g;p,q}(F) : \)

\[
\begin{align*}
\text{Cor}_{0;1,1} & := \text{id}_F \\
\text{Cor}_{1;1,1} & := m_F \circ (\rho_F^K \otimes \text{id}_F) \circ (\text{id}_K \otimes \Delta_F)
\end{align*}
\]

\( \kappa_F^K \)

\( \Delta_F \)

\( m_F \)

\( F \)

\( \in \text{Hom}_{C \boxtimes C^\text{rev}} (K \otimes F, F) \)
input: bulk handle Hopf algebra \( K = \int_{X \in \mathcal{C}} \mathcal{C}^{rev} X \otimes X^\vee \)

input: partial monodromy action \( \kappa_X^K \)

further input: \((F, m_F, \eta_F, \Delta_F, \varepsilon_F)\) algebra-coalgebra

define family of morphisms \( \text{Cor}_{g;p,q} = \text{Cor}_{g;p,q}(F) \):

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\text{Cor}_{g;1,1} := \text{Cor}_{1;1,1} \circ (\text{id}_K \otimes \text{Cor}_{g-1;1,1}) \quad \text{for} \quad g > 1
\]
Bulk correlation functions

input: bulk handle Hopf algebra

\[ K = \int_{X \in \mathcal{C}} C_{\text{rev}} X \otimes X^\vee \]

input: partial monodromy action \( \kappa_X^K \)

further input: \((F, m_F, \eta_F, \Delta_F, \varepsilon_F)\) algebra - coalgebra

define family of morphisms \( \text{Cor}_{g;p,q} = \text{Cor}_{g;p,q}(F) \):

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\text{Cor}_{g;1,1} & := \text{Cor}_{1;1,1} \circ (\text{id}_K \otimes \text{Cor}_{g-1;1,1}) \quad \text{for} \quad g > 1 \\
\text{Cor}_{g;p,q} & := \Delta_F^{(p)} \circ \text{Cor}_{g;1,1} \circ (\text{id}_K \otimes g \otimes m_F^{(q)})
\end{align*}
\]

\[ \text{Cor}_{g;0,n} = \]

\[ K K \ldots K \]

\[ \eta_F \]

\[ F F \ldots F \]

\[ \Delta_F \]

\[ m_F \]

\[ \kappa^K_F \]
Bulk correlation functions

input: bulk handle Hopf algebra \( K = \int_{X \in C} C^{\text{rev}} X \otimes X^\vee \)

input: partial monodromy action \( \kappa^K_X \)

further input: \( (F, m_F, \eta_F, \Delta_F, \varepsilon_F) \) algebra-coalgebra

define family of morphisms \( \text{Cor}_{g;p,q} = \text{Cor}_{g;p,q}(F) : \)

\[
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\text{Cor}_{0;1,1} & := \text{id}_F \\
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\text{Cor}_{g;1,1} & := \text{Cor}_{1;1,1} \circ (\text{id}_K \otimes \text{Cor}_{g-1;1,1}) \quad \text{for } g > 1 \\
\text{Cor}_{g;p,q} & := \Delta_F^{(p)} \circ \text{Cor}_{g;1,1} \circ (\text{id}_K \otimes g \otimes m^{(q)}_F) \\
\end{align*}
\]

goal: various desirable properties of \( \text{Cor}(F) \) from properties of \( F \)

\( \implies \) \( \text{Cor} \) candidate for bulk correlation functions of a full CFT
Properties of $\text{Cor}(F)$

- Action of mapping class group:
  - $F$ Frobenius (and associative and coassociative)
    - $\Rightarrow$ $\text{Cor}(F)$ invariant under fusing move
  - $F$ commutative and cocommutative (and trivial twist)
    - $\Rightarrow$ $\text{Cor}(F)$ invariant under braiding move
  - $F$ \ldots\ldots $\Rightarrow$ $\text{Cor}(F)$ also invariant under rotation move
**Properties of** \( \text{Cor}(F) \)

- **action of mapping class group:**
  - \( F \) Frobenius (and associative and coassociative)
    \[ \implies \text{Cor}(F) \text{ invariant under fusing move} \]
  - \( F \) commutative and cocommutative (and trivial twist)
    \[ \implies \text{Cor}(F) \text{ invariant under braiding move} \]
  - \( F \) \( \cdots \cdots \implies \text{Cor}(F) \) also invariant under rotation move

**call** \( F \) **\( S \)-invariant** iff \( \text{Cor}_{1;1,0} \circ S_K = \text{Cor}_{1;1,0} \)

- \( F \) \( S \)-invariant \[ \implies \text{Cor}(F) \text{ invariant under } S \)-move \]
- \( F \) symmetric \[ \implies \text{can exchange incoming and outgoing field insertions} \]
Properties of $\text{Cor}(F)$

- action of mapping class group:
  - $F$ Frobenius (and associative and coassociative)
    $\implies$ $\text{Cor}(F)$ invariant under \textit{fusing move}
  - $F$ commutative and cocommutative (and trivial twist)
    $\implies$ $\text{Cor}(F)$ invariant under \textit{braiding move}
  - $F$ \ldots \ldots $\implies$ $\text{Cor}(F)$ also invariant under \textit{rotation move}

- call $F$ \textit{S-invariant} iff $\text{Cor}_{1;1,0} \circ S_K = \text{Cor}_{1;1,0}$

- $F$ S-invariant $\implies$ $\text{Cor}(F)$ invariant under \textit{S-move}

- $F$ symmetric $\implies$ can exchange incoming and outgoing field insertions

- Lego-Teichmüller game: systematic description of diffeomorphisms via connected simply connected CW complex with $F / B / R / S$ moves as 1-cells

$\implies$ invariance of $\text{Cor}(F)$ under moves
implies invariance of $\text{Cor}(F)$ under whole mapping class group
Properties of $\text{Cor}(F)$

- action of mapping class group:
  - $F$ Frobenius (and associative and coassociative)
    $\implies$ $\text{Cor}(F)$ invariant under fusing move
  - $F$ commutative and cocommutative (and trivial twist)
    $\implies$ $\text{Cor}(F)$ invariant under braiding move
  - $F$ symmetric $\implies$ can exchange incoming and outgoing field insertions

- call $F$ $S$-invariant iff $\text{Cor}_{1;1,0} \circ S_K = \text{Cor}_{1;1,0}$

- $F$ $S$-invariant $\implies$ $\text{Cor}(F)$ invariant under $S$-move

- $F$ symmetric $\implies$ can exchange incoming and outgoing field insertions

Status:
- general $C$: arguments straightforward but not written up
- $C = H$-mod: $F = \int^{U \in C} U \boxtimes \omega(U^\vee)$ is $\cdots$ $S$-invariant Frobenius
Properties of $\text{Cor}(F)$

- Action of mapping class group:
  - $F$ Frobenius (and associative and coassociative) $\implies \text{Cor}(F)$ invariant under fusing move
  - $F$ commutative and cocommutative (and trivial twist) $\implies \text{Cor}(F)$ invariant under braiding move
  - $F$ · · · · · · $\implies \text{Cor}(F)$ also invariant under rotation move

Call $F \underline{\text{S-invariant}}$ iff $\text{Cor}_{1;1,0} \circ S_K = \text{Cor}_{1;1,0}$

- $F$ S-invariant $\implies \text{Cor}(F)$ invariant under S-move
- $F$ symmetric $\implies$ can exchange incoming and outgoing field insertions

Status:
- General $C$: arguments straightforward but not written up
- $C = H\text{-mod}$: $F = \int^U \in C U \boxtimes \omega(U^\vee)$ is · · · · S-invariant Frobenius
  and $\text{Cor}(F)$ is mapping class group invariant
  - by brute force: invariance under action of set of generators of $\text{Map}(\Sigma)$

$\text{F-SCHWEIGERT-STIGNER 2014}$
Properties of $\text{Cor}(F)$

- behavior under sewing $\Sigma \to \Sigma'$ of world sheets:
  - known: sewing/factorization relations among spaces of conformal blocks
  - desired: sewing relations map $\text{Cor}(\Sigma; F)$ to $\text{Cor}(\Sigma'; F')$
Properties of $\text{Cor}(F)$

- behavior under sewing $\Sigma \to \Sigma'$ of world sheets:
  - known: sewing/factorization relations among spaces of conformal blocks
  - desired: sewing relations map $\text{Cor}(\Sigma; F)$ to $\text{Cor}(\Sigma'; F)$

- two types of sewings:
  - non-handle creating:
    - result follows from simple form of sewing relations
    - e.g.
      \[
      \int_{X \in C \boxtimes C^{\text{rev}}} \text{Hom}_{C \boxtimes C^{\text{rev}}}(K \otimes g_1, X) \otimes_k \text{Hom}_{C \boxtimes C^{\text{rev}}}(K \otimes g_2 \otimes X, Y) = \text{Hom}_{C \boxtimes C^{\text{rev}}}(K \otimes g_1 + g_2, Y)
      \]
      with dinatural family $f \otimes g \mapsto g \circ (\text{id}_{K \otimes g_2} \otimes f)$
Properties of $\text{Cor}(F)$

- behavior under sewing $\Sigma \to \Sigma'$ of world sheets:
  - known: sewing/factorization relations among spaces of conformal blocks
  - desired: sewing relations map $\text{Cor}(\Sigma; F)$ to $\text{Cor}(\Sigma'; F')$

- two types of sewings:
  - non-handle creating:
    - result follows from simple form of sewing relations
  
  e.g.
  \[
  \int_{X \in C \boxtimes C^{\text{rev}}} \text{Hom}_{C \boxtimes C^{\text{rev}}}(K \otimes g_1, X) \otimes_k \text{Hom}_{C \boxtimes C^{\text{rev}}}(K \otimes g_2 \otimes X, Y) = \text{Hom}_{C \boxtimes C^{\text{rev}}}(K \otimes g_1 + g_2, Y)
  \]
  with dinatural family $f \otimes g \mapsto g \circ (\text{id}_{K \otimes g_2} \otimes f)$

  - handle creating:
    - must form coend in category of left exact functors
    - technically involved
**Properties of** $\text{Cor}(F')$

- behavior under sewing $\Sigma \to \Sigma'$ of world sheets:
  - known: sewing/factorization relations among spaces of conformal blocks
  - desired: sewing relations map $\text{Cor}(\Sigma; F')$ to $\text{Cor}(\Sigma'; F')$

- two types of sewings:
  - **non-handle creating:**
    - result follows from simple form of sewing relations
    - e.g. $\int_{X \in C \boxtimes C^{\text{rev}}} \text{Hom}_{C \boxtimes C^{\text{rev}}}(K \otimes g_1, X) \otimes_k \text{Hom}_{C \boxtimes C^{\text{rev}}}(K \otimes g_2 \otimes X, Y) = \text{Hom}_{C \boxtimes C^{\text{rev}}}(K \otimes g_1 + g_2, Y)$ with dinatural family $f \otimes g \mapsto g \circ (\text{id}_{K \otimes g_2} \otimes f)$
  - **handle creating:**
    - must form coend in category of left exact functors

**Outlook**: to be written up quite soon
THANK YOU
Appendix: Generators of $\text{Map}_\Sigma$

Algebras in monoidal categories

- exact sequence $1 \to B_{g;n} \to \text{Map}_{g;n} \to \text{Map}_{g;0} \to 1$

thus generated by permutations of holes and Dehn twists

(relations not needed)
given a functor \( G : \mathcal{D}^{\text{op}} \times \mathcal{D} \to \mathcal{E} \) and an object \( B \in \mathcal{E} \)

a dinatural transformation \( G \Rightarrow B \) is a family of morphisms

\[
\varphi_X : G(X, X) \to B \quad \text{s.t.} \quad G(Y, X) \xrightarrow{G(\text{id}_Y, f)} G(Y, Y)
\]

\[
G(f, \text{id}_X) \downarrow \quad \varphi_Y \\
G(X, X) \quad \Downarrow \varphi_X \\
G(X, X) \quad \xrightarrow{\varphi_X} B
\]

commutes for all \( f : X \to Y \)

coend \((A, \iota)\) for \( G \):

initial object in category of dinatural transformations \( G \Rightarrow - : \)

\[
\text{notation:} \quad (A, \iota) = \int^X G(X, X)
\]

unique up to unique isomorphism (if exists)