ALGEBRAS IN MONOIDAL CATEGORIES

Motivation

MESSAGE: algebras in monoidal categories are natural and nice

Motivation

POSSIBLE MOTIVATIONS:

- express similarities:

  traditionally: Hopf algebras as generalizations of group algebras

  in fact: group $=$ Hopf algebra in $\mathcal{S}et$

- disclose fake peculiarities: e.g. non-associativity:

  - octonions $=$ *associative* algebra in $\mathbb{Z}_2^{\times 3}$-$\mathcal{V}ect$ ALBUQUERQUE-MAJID 1999

- increase degree of familiarity:

  - Beilinson-Drinfeld chiral algebra $=$ Lie algebra in some category of $\mathcal{D}$-modules

    SOIBELMAN $\gtrsim$1997

  - vertex algebra ($=$ B-D chiral algebra on the formal disk)

    $=$ singular commutative associative algebra in certain functor category

    BORCHERDS 1998, 2001

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POSSIBLE MOTIVATIONS:

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  in fact: group $=$ Hopf algebra in $\mathcal{S}et$

- ☞ disclose fake peculiarities: e.g. non-associativity:

  ↯ octonions $=$ *associative* algebra in $\mathbb{Z}_2^{\times 3}$-$\mathcal{V}ect$ ALBUQUERQUE-MAJID 1999

- ☞ present in applications:

  ↯ coend Hopf algebras for 3-mf invariants and MCG-reps LYUBASHENKO 1995

  ↯ coalgebras for cloning operations in quantum mechanics COECKE-DUNCAN 2011

  ↯ Frobenius algebras for classifying full CFTs associated with given chiral CFT

  JF-RUNKEL-SCHWEIGERT 2002

SPECIFIC MOTIVATION:

- ☞ bulk state space of a full CFT as $\cdots$ Frobenius algebra in $\mathcal{Z}(\mathcal{C}) \simeq \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$

  CLAIM: allows to construct bulk fields correlation functions on any closed world sheet

Plan

PLAN

- ☞ algebras

  Frobenius algebras

  Hopf algebras

  weak Hopf algebras

  Lie algebras

- ☞ sample results
- ☞ handle Hopf algebras
- ☞ conformal field theory
- ☞ bulk state spaces in CFT
- ☞ bulk field correlation functions
- ☞ appendix

$\Bbbk$-algebras

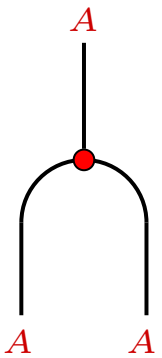
- $\Bbbk$-algebra $=$ vector space with multiplication and unit element
- category $\mathcal{V}ect_{\Bbbk}$:
  - objects $=$ vector spaces over $\Bbbk$
  - morphisms $=$ linear maps
  - monoidal structure :
    tensor product $\otimes_{\Bbbk}$ of vector spaces with tensor unit $\mathbf{1} = \Bbbk$
- interpret
  - multiplication $\equiv$ *bi* linear map $A \times A \to A$ $\rightsquigarrow$ linear map $m\colon A \otimes_{\Bbbk} A \to A$
  - unit element $1_A \in A$ $\rightsquigarrow$ linear map $\eta\colon \Bbbk \to A$ $\qquad \eta(c) = c\,1$

$\Longrightarrow$ $\Bbbk$-algebra $=$ algebra object $(A, m, \eta) \in \mathcal{V}ect_{\Bbbk}$

Algebras

☞ associative algebra $=$ object A + morphism

$$m: \quad A \otimes A \longrightarrow A$$



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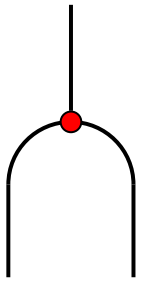
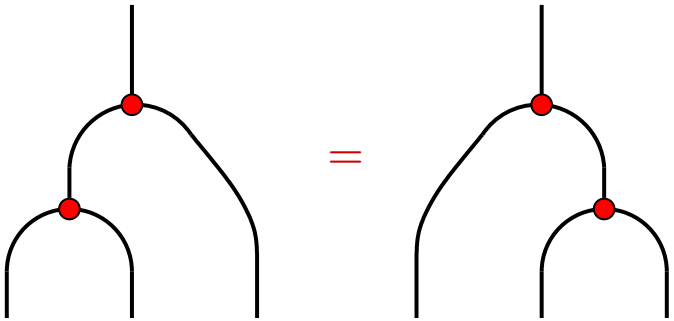


such that

$$m \circ (m \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes m)$$



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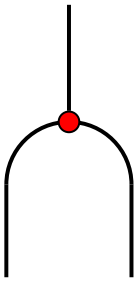
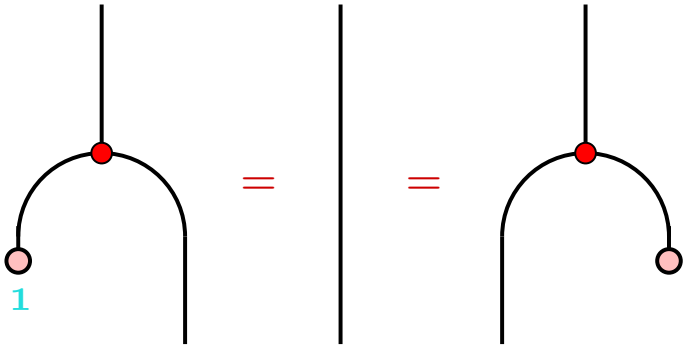


- unital algebra (A, m, η):

$$\eta: \quad \mathbf{1} \to A$$

$$m \circ (\eta \otimes \mathrm{id}) = \mathrm{id} = m \circ (\mathrm{id} \otimes \eta)$$



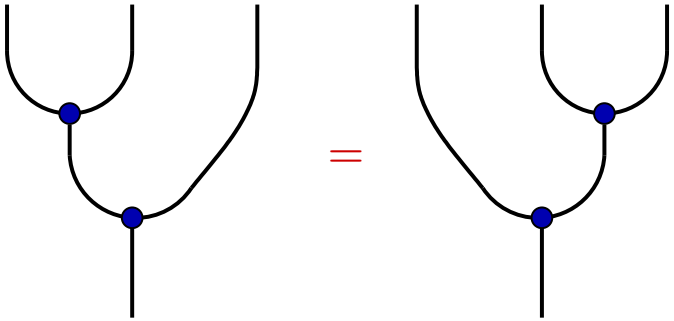
- natural setting: strict monoidal categories $(\mathcal{C}, \otimes, \mathbf{1})$

Coalgebras

- coassociative coalgebra (C, Δ) :

$$\Delta : \quad A \to A \otimes A$$

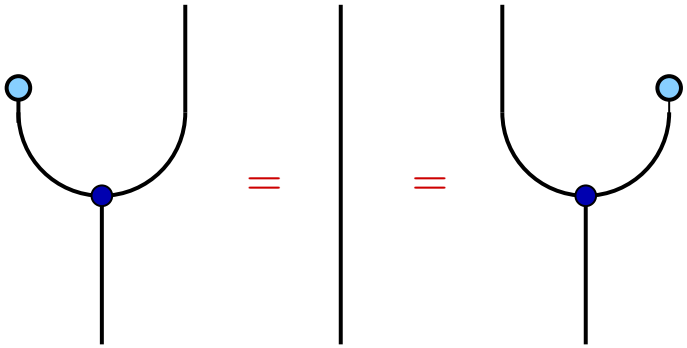
$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$$



- co-unital coalgebra (C, Δ, ε) :

$$\varepsilon : \quad A \to \mathbf{1}$$

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta$$



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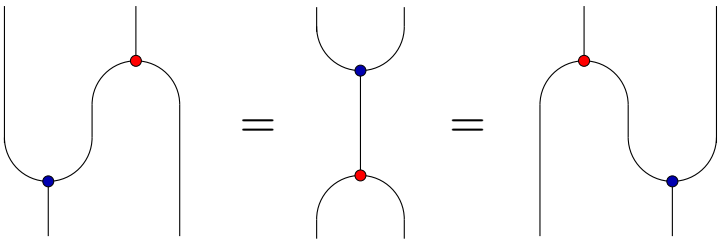
Frobenius algebras

- Algebra:

- Coalgebra:

- Frobenius algebra: algebra and coalgebra and coproduct a bimodule morphism:



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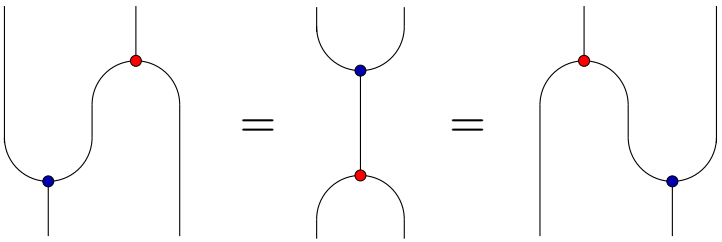
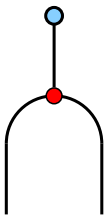


equivalent to existence of non-degenerate invariant form



if $\mathcal{C}$ is rigid

Frobenius algebras

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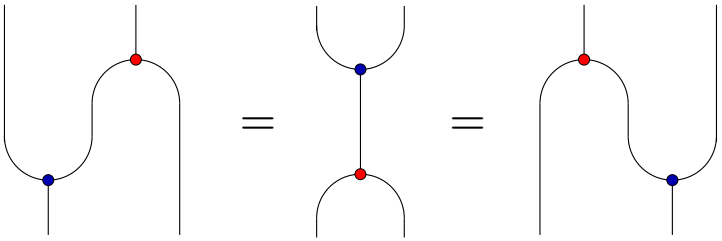
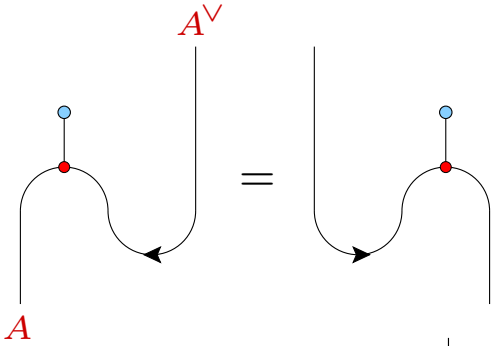
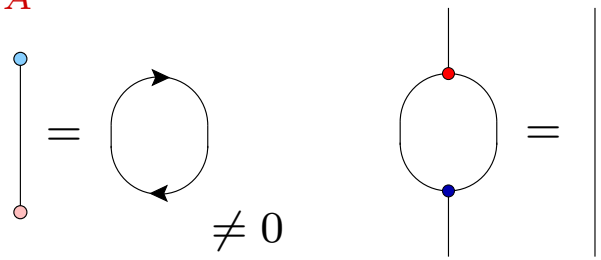


natural setting: monoidal categories $(\mathcal{C}, \otimes, \mathbf{1})$

- symmetric Frobenius algebra:



special Frobenius algebra:

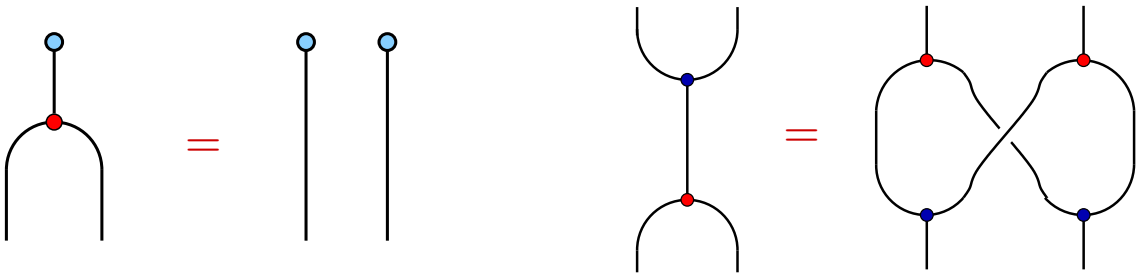


natural setting: sovereign rigid monoidal categories

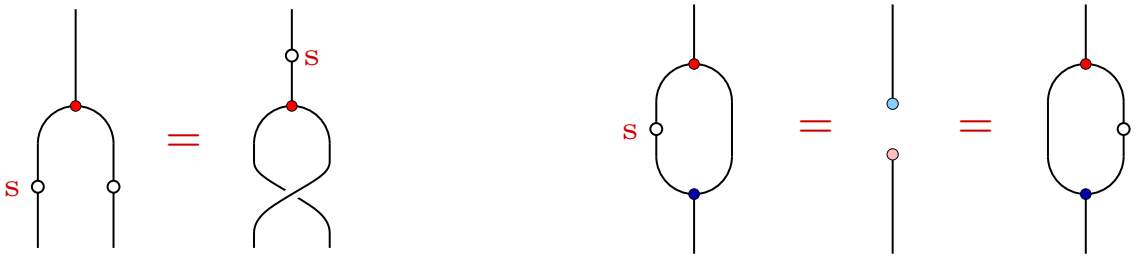
Hopf algebras

- bi algebra : algebra and coalgebra

  and coproduct and counit algebra morphisms :



- Hopf algebra : bialgebra with antipode



natural setting : braided monoidal categories

Hopf algebras

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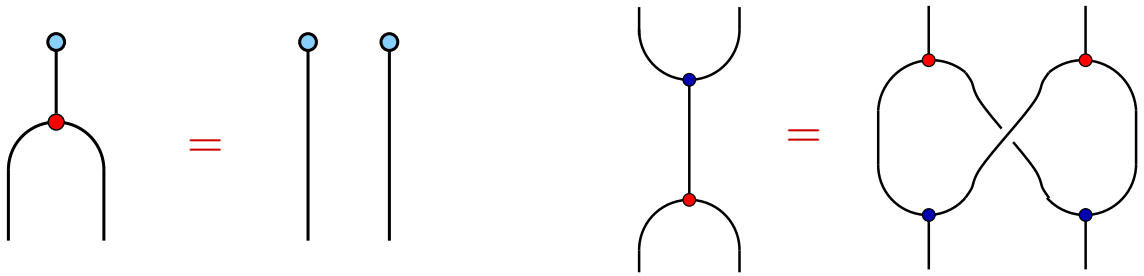

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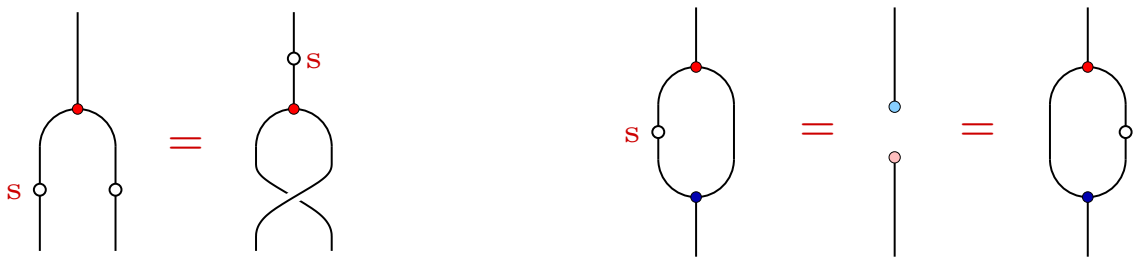
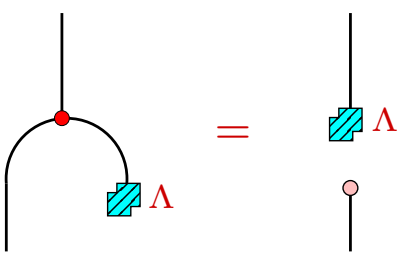


- left integral on H: $\Lambda \in \mathrm{Hom}(\mathbf{1}, H)$ s.t. $m \circ (\mathrm{id}_A \otimes \Lambda) = \Lambda \circ \varepsilon$



Hopf algebras

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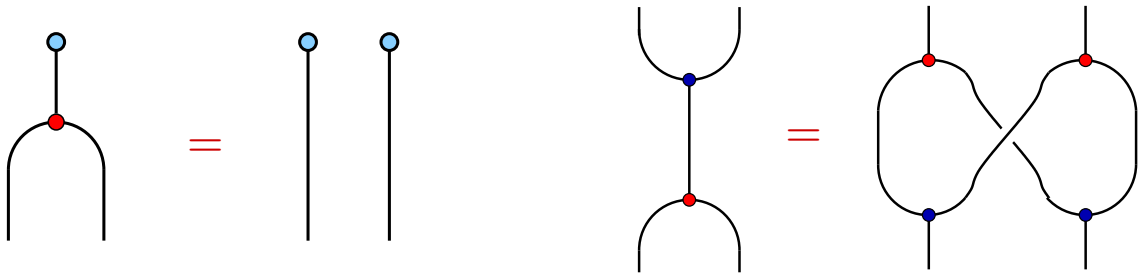

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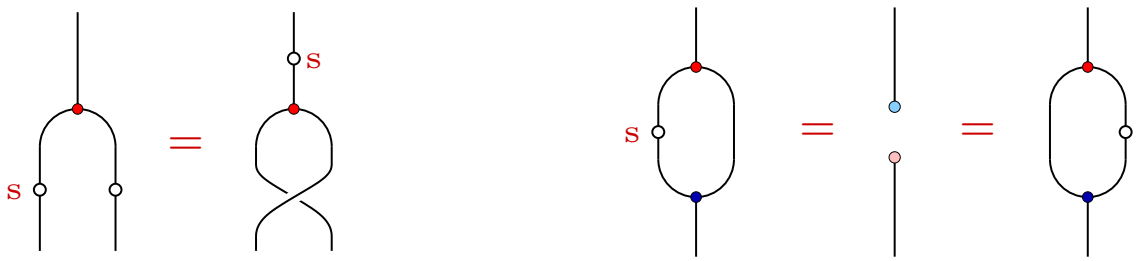
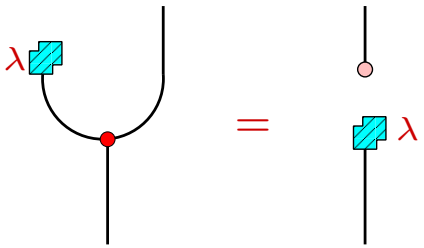


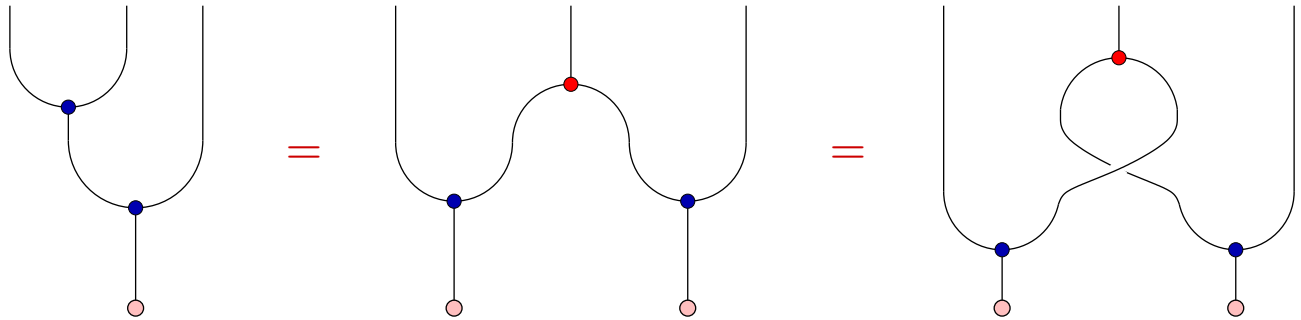
- right cointegral on H: $\lambda \in \mathrm{Hom}(H, \mathbf{1})$ s.t. $(\lambda \otimes \mathrm{id}_H) \circ \Delta = \eta \circ \lambda$



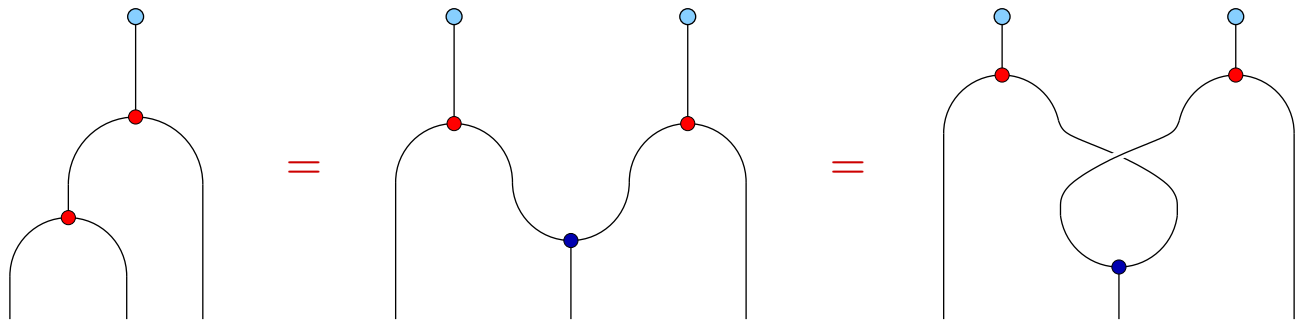
NB : weak Hopf algebras

☞ Weak Hopf algebra $(H, m, \eta, \Delta, \varepsilon, \mathrm{s})$:

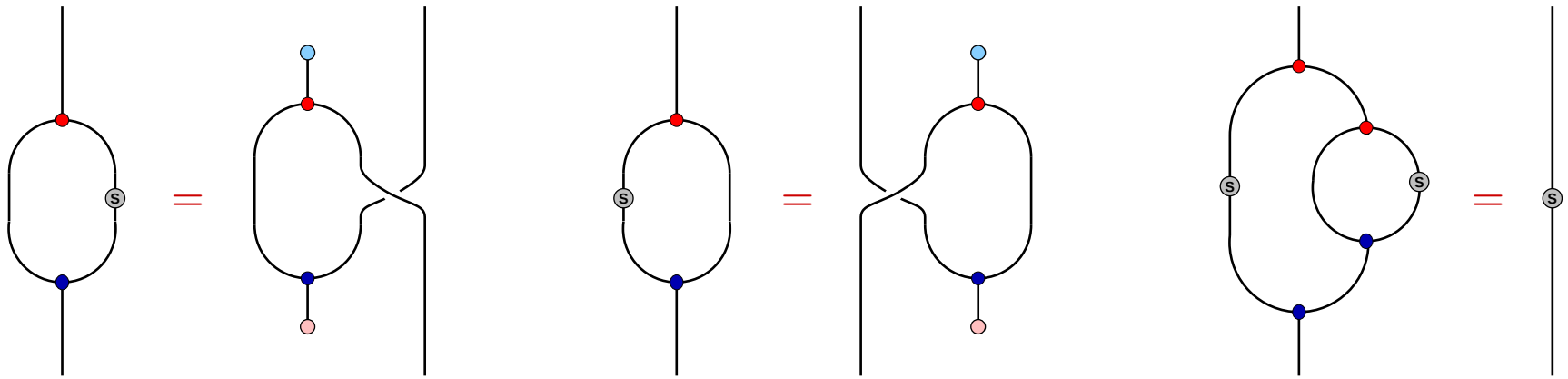
Δ not nec. unital



ε not nec. algebra morphism

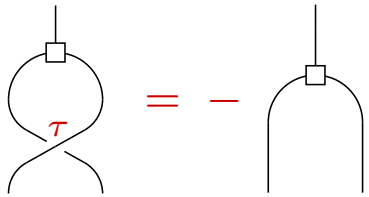


weak antipode

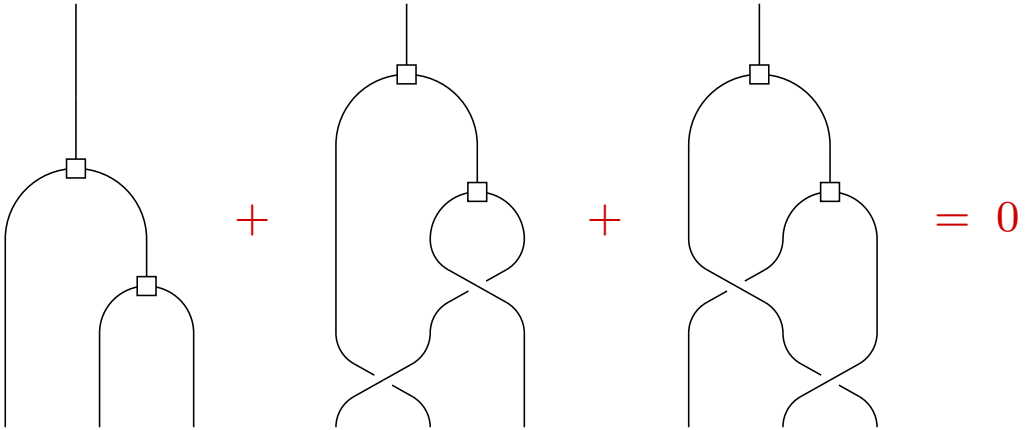


Lie algebras

- Lie algebra: object L with morphism $\ell \in \mathrm{Hom}_{\mathcal{C}}(L \otimes L, L)$ s.t.
  - antisymmetry



  - Jacobi identity



GUREVICH 1986

- natural setting: symmetric monoidal additive categories $(\mathcal{C}, \otimes, \mathbf{1}, \tau)$
- examples:
  - Lie superalgebra $=$ Lie algebra in $\mathcal{SV}ect = \mathbb{Z}_2\text{-}\mathcal{V}ect$
  - color Lie algebra $=$ Lie algebra in $\Gamma\text{-}\mathcal{V}ect$ (Γ abelian group with skew bicharacter)

Lie algebras

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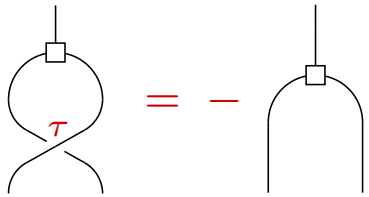

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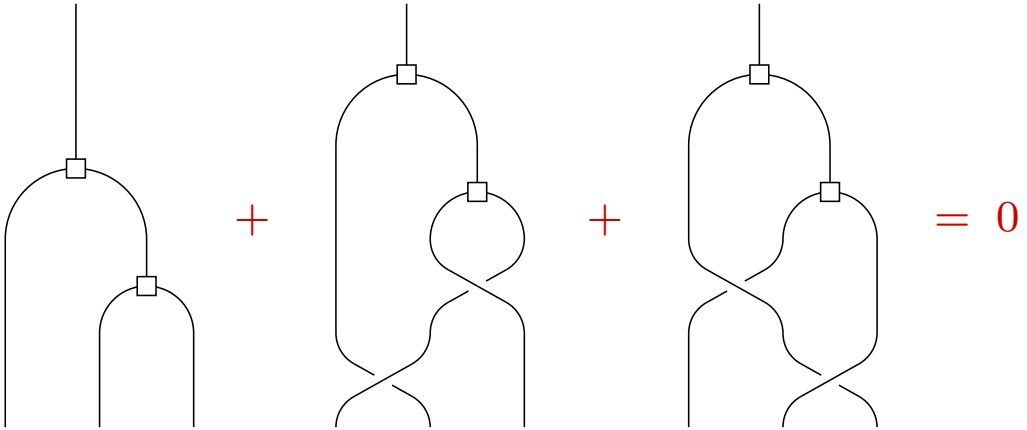

GUREVICH 1986

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- examples:
  - Hom-Lie algebra in $\mathcal{C}$ (Jacobi identity modified by an automorphism)
    $=$ Lie algebra in $\mathcal{HC}$

CAENEPEEL-GOYVAERTS 2011

Lie algebras

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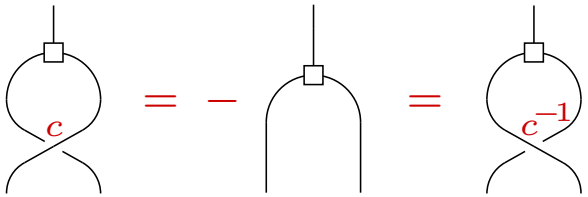
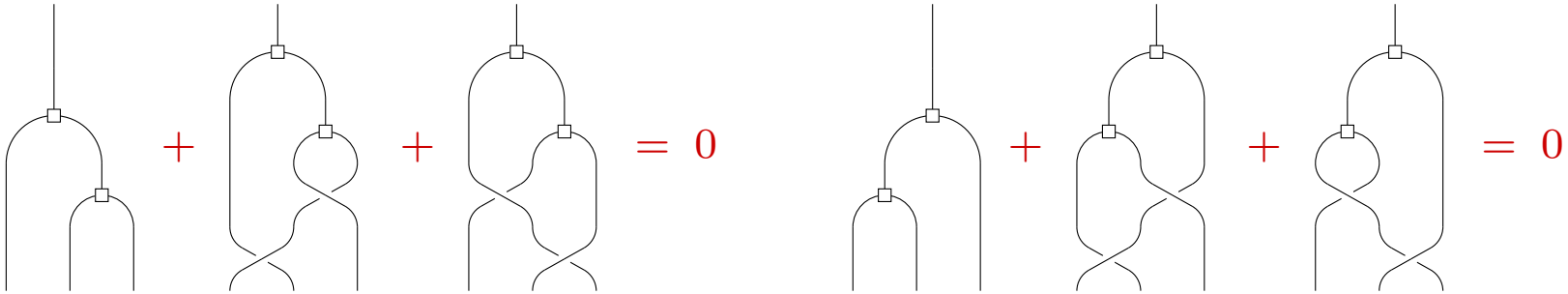


  - two Jacobi identities



ZHANG-ZHANG 2004

- natural setting: braided monoidal additive categories $(\mathcal{C}, \otimes, \mathbf{1}, c)$
- example: commutator algebras satisfying antisymmetry

Lie algebras

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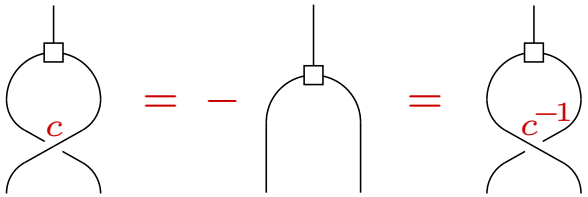

  - Jacobi identity / ies ($\mathcal{C}$ symmetric / braided)

- more conceptual approaches for braided $\mathcal{C}$:
  - quantum Lie bracket related to adjoint action for Hopf algebras in $\mathcal{C}$
    compatible with a coalgebra structure of L MAJID 1994
    satisfies a generalized Jacobi identity

Lie algebras

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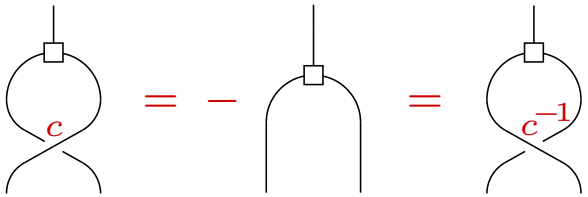
  - antisymmetry



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PAREIGIS 1997
KHARCHENKO 1998

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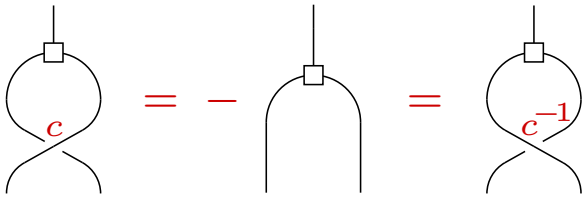

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    PAREIGIS 1997

    e.g. primitive elements of a Hopf algebra in Γ-$\mathcal{V}ect$

    e.g. derivations of an algebra in Γ-$\mathcal{V}ect$

    reduces to color Lie algebras for skew bicharacter

Lie algebras in symmetric $\mathcal{C}$

consider $\mathcal{C}$ symmetric rigid monoidal idempotent-complete $\Bbbk$-linear abelian

☞ FACTS :

- trivial abelian Lie algebra $(\mathbf{1}\,,0)$
- commutator algebras $(A\,,m-m\circ\tau)$
- Frobenius algebra $(U\otimes U^\vee,\ \mathrm{id}_U\otimes d_U\otimes \mathrm{id}_{U^\vee})$
  Morita equivalent to $(\mathbf{1}\,,\mathrm{id}_{\mathbf{1}})$
- $U\otimes U^\vee\cong\mathbf{1}\oplus R$ and $\dim(U)\neq 0 \implies R$ inherits Lie algebra structure
  e.g. adjoint representation $\in \mathfrak{sl}_n(\Bbbk)$-mod
  e.g. $R_{\mathbf{2024}}\in \mathrm{M}_{23}$-mod or M_{24}-mod
  e.g. for $U=\mathrm{st}\in\mathfrak{gl}(m|n)$-mod for $m\neq n$
- L simple as object (and non-trivial) $\implies$ simple as Lie algebra
- $U\otimes U^\vee\cong\mathbf{1}\oplus R$ and $\dim(U)\neq 0$ and U generating $\mathcal{C}$
  $\implies$ every $V\in\mathcal{C}$ carries natural R-action (but no handle on all of R-mod)

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☞ desirable application : group-theoretical coefficients for Feynman diagrams

CVITANOVIĆ 1976

Sample results

- $\mathcal{C}$ fusion category and $\mathcal{M}$ semisimple indecomposable $\mathcal{C}$-module category
  $\Longrightarrow$ $\mathcal{M} \simeq A\text{-mod}_{\mathcal{C}}$ for algebra $A \in \mathcal{C}$ OSTRIK 2003
- $\mathcal{M}$ in addition endowed with *module trace* $\Longrightarrow$ A Frobenius SCHAUMANN 2013
- $\mathcal{C}$ modular tensor category
  $\Longrightarrow$ finite number of Morita classes of simple symmetric special Frobenius algebras
  JF-RUNKEL-SCHWEIGERT 2004
- $\mathcal{C}$ ribbon and A commutative symmetric special Frobenius
  $\Longrightarrow$
  - category $A\text{-mod}_{\mathcal{C}}^{\ell oc}$ of *local* A-modules ribbon
  - $\mathcal{C}$ semisimple $\Longrightarrow$ $A\text{-mod}_{\mathcal{C}}^{\ell oc}$ semisimple
  - $\mathcal{C}$ modular and A simple $\Longrightarrow$ $A\text{-mod}_{\mathcal{C}}^{\ell oc}$ modular
- $\mathcal{C}$ rigid monoidal and A special Frobenius
  $\Longrightarrow$ every $M \in A\text{-mod}_{\mathcal{C}}$ is submodule of an induced module $(A \otimes U, m \otimes \mathrm{id}_U)$
- $\mathcal{C}$ modular and A symmetric special Frobenius:
  $\Longrightarrow$ every $X \in A\text{-bimod}_{\mathcal{C}}$ is sub-bimodule of a braided-induced module $U \otimes^{+} A \otimes^{-} V$

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- $\mathcal{C}$ ribbon and A Azumaya
  $\Longrightarrow$ $1 \to \mathrm{Inn}(A) \to \mathrm{Aut}(A) \to \mathrm{Pic}(\mathcal{C})$ exact — VAN OYSTAEYEN-ZHANG 1998
- $\mathcal{C}$ modular and A symmetric special Frobenius:
  A Azumaya $\Longleftrightarrow$ $\dim_{\mathbb{C}} \mathrm{Hom}_{A|A}\big(U_i \otimes^+ A \otimes^- U_j, A\big)_{ij}$ permutation matrix
  $\Longrightarrow$ exact sequence $1 \to \mathrm{Inn}(A) \to \mathrm{Aut}(A) \to \mathrm{Pic}(\mathcal{C})$
- $\mathcal{C}$ modular and A simple symmetric special Frobenius
  $\Longrightarrow$ bimodule fusion rules $K_0(A\text{-bimod}_{\mathcal{C}}) \otimes_{\mathbb{Z}} \mathbb{C}$
  isomorphic as $\mathbb{C}$-algebra to $\bigoplus_{i,j \in I} \mathrm{End}_{\mathbb{C}}\big(\mathrm{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_j, A)\big)$

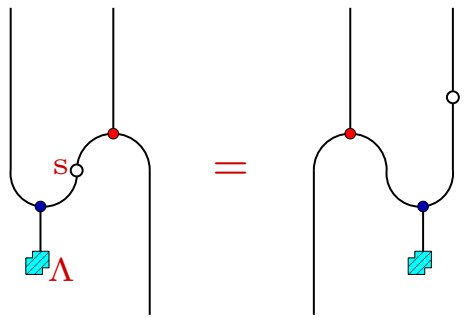
Sample results

- $\mathcal{C}$ fusion category and $\mathcal{M}$ semisimple indecomposable $\mathcal{C}$-module category
  $\Longrightarrow$ $\mathcal{M} \simeq A\text{-mod}_{\mathcal{C}}$ for algebra $A \in \mathcal{C}$ — OSTRIK 2003
- $\mathcal{M}$ in addition endowed with *module trace* $\Longrightarrow$ A Frobenius — SCHAUMANN 2013
- $\mathcal{C}$ modular tensor category
  $\Longrightarrow$ finite number of Morita classes of simple symmetric special Frobenius algebras
  JF-RUNKEL-SCHWEIGERT 2004
- H Hopf algebra with s invertible and (co)integral s.t. $\lambda \circ \Lambda \in \Bbbk^\times$
  $\Longrightarrow$ H Frobenius with same algebra structure

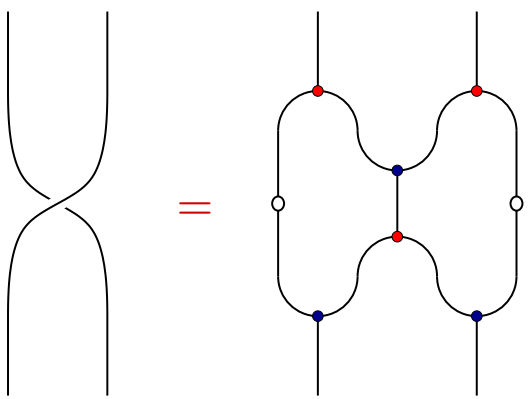
PAREIGIS 1971
KADISON-STOLIN 2000
JF-SCHWEIGERT 2010

$\varepsilon_F = \lambda$

$\Delta_F =$



- Hopf algebras have symmetric self-braiding:
  SCHAUENBURG 1998



Handle Hopf algebras

☞ example : 1-holed torus in category of 3-cobordisms with corners
(1-morphism in bicategory)

YETTER 1995

CRANE-YETTER 1999

Handle Hopf algebras

- example: 1-holed torus in category of 3-cobordisms with corners
- example:

  coend $\boldsymbol{L} = \int^{\boldsymbol{U}\in\boldsymbol{\mathcal{D}}} \boldsymbol{U}\otimes\boldsymbol{U}^\vee$ in any finite abelian $\Bbbk$-linear ribbon category $\mathcal{D}$

  - exists and carries a natural structure of a Hopf algebra in $\mathcal{D}$
  - endowed with left integral and Hopf pairing ϖ_L

MAJID, LYUBASHENKO, KERLER, ... ∼1995

$$m_L \circ (\iota_U \otimes \iota_V) := \iota_{V\otimes U} \circ (\gamma_{U,V} \otimes \mathrm{id}_{V\otimes U}) \circ (\mathrm{id}_{U^\vee} \otimes c_{U,V^\vee\otimes V})$$

$$\Delta_L \circ \iota_U := (\iota_U \otimes \iota_U) \circ (\mathrm{id}_{U^\vee} \otimes b_U \otimes \mathrm{id}_U)$$

$$\eta_L := \iota_{\mathbf{1}} \qquad \varepsilon_L \circ \iota_U := d_U$$

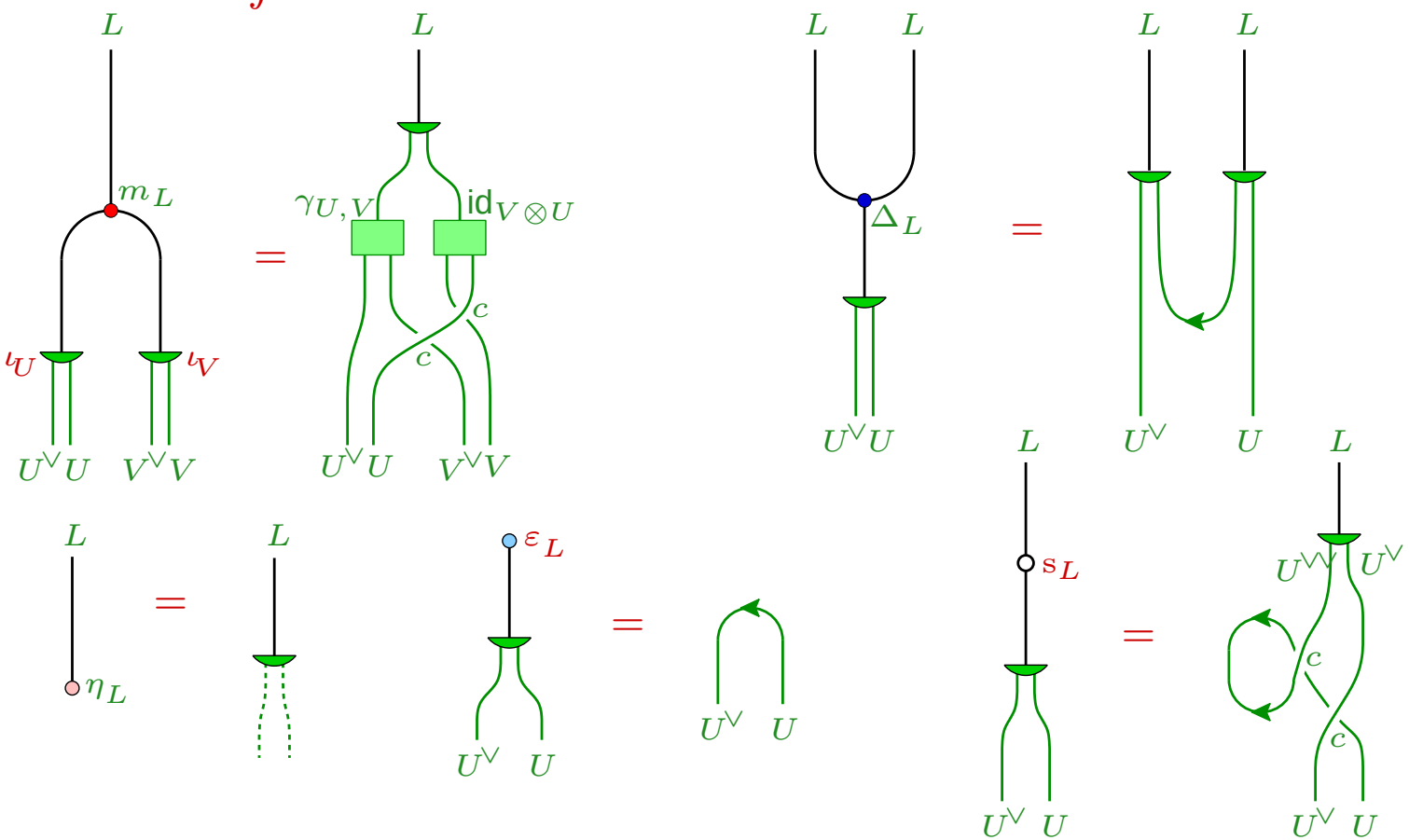
$$\mathrm{s}_L \circ \iota_U := (d_U \otimes \iota_{U^\vee}) \circ (\mathrm{id}_{U^\vee} \otimes c_{U^{\vee\vee},U} \otimes \mathrm{id}_{U^\vee}) \circ (b_{U^\vee} \otimes c_{U^\vee,U})$$

$$\varpi_L \circ (\iota_U \otimes \iota_V) := (d_U \otimes d_V) \circ \big[\mathrm{id}_{U^\vee} \otimes (c_{V^\vee,U} \circ c_{U,V^\vee} \otimes \mathrm{id}_V)\big]$$

- special case: $\mathcal{D}$ modular $\Longrightarrow$ $L = \bigoplus_{i\in\mathcal{I}} S_i^\vee \otimes S_i$

- example : 1-holed torus in category of 3-cobordisms with corners
- example :

  coend $\boldsymbol{L} = \int^{\boldsymbol{U}\in\boldsymbol{\mathcal{D}}} \boldsymbol{U}\otimes\boldsymbol{U}^\vee$ in any finite abelian $\Bbbk$-linear ribbon category $\mathcal{D}$



Handle Hopf algebras

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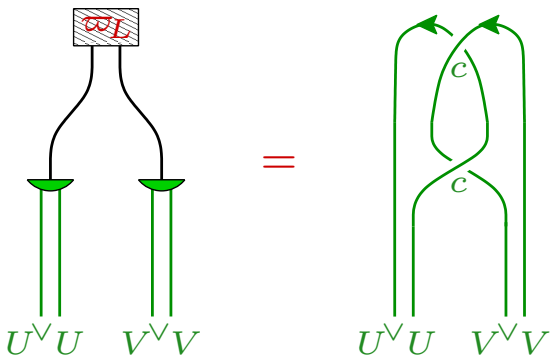

Handle Hopf algebras

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  coend $\boldsymbol{L} = \int^{\boldsymbol{U} \in \boldsymbol{\mathcal{D}}} \boldsymbol{U} \otimes \boldsymbol{U}^\vee$ in any finite abelian $\Bbbk$-linear ribbon category $\mathcal{D}$
- for $\mathcal{D}$ *factorizable* : (e.g. $\mathcal{D} = H$-mod for factorizable ribbon Hopf algebra H)
  - integral two-sided
  - Hopf pairing non-degenerate
  - projective rep $\pi_{g,m}$ of Map_Σ on $\mathrm{Hom}_{\mathcal{D}}(L^{\otimes g}, U_1 \otimes \cdots \otimes U_m)$

LYUBASHENKO 1995

Handle Hopf algebras

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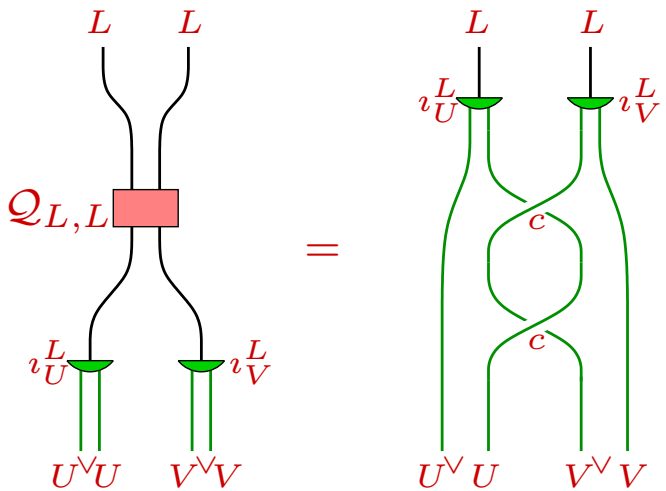
  - e.g. modular S-transformation $=$ composition with automorphism

$$S_L := (\varepsilon_L\otimes\mathrm{id}_L)\circ\mathcal{Q}_{L,L}\circ(\mathrm{id}_L\otimes\Lambda_L)$$



Handle Hopf algebras

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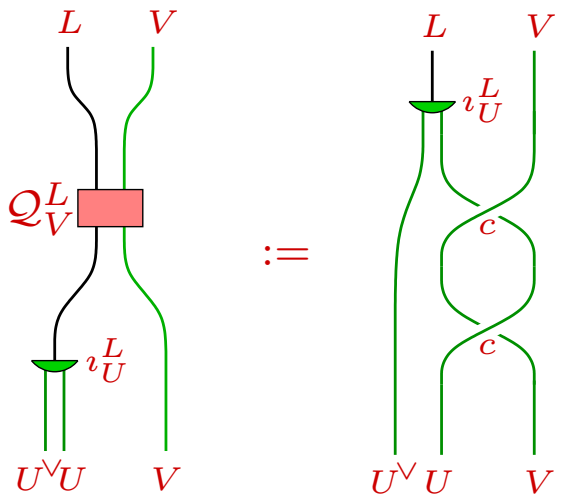
  - Hopf pairing non-degenerate
  - projective rep $\pi_{g,m}$ of Map_Σ on $\mathrm{Hom}_{\mathcal{D}}(L^{\otimes g}, U_1\otimes\cdots\otimes U_m)$
- every $V\in\mathcal{D}$ has natural L-module structure $(\boldsymbol{V}, \varkappa_{\boldsymbol{V}}^{\boldsymbol{L}})$

$$\varkappa_V^L := (\varepsilon_L\otimes \mathrm{id}_V)\circ \mathcal{Q}_V^L$$



JF-SCHWEIGERT-STIGNER 2013

Conformal field theory

2d conformal quantum field theory (CFT) in a nutshell :

- chiral CFT :
  - chiral symmetry algebra – conformal vertex algebra $\mathfrak{V}$ (say)
  - category $\mathcal{C}$ of $\mathfrak{V}$-representations
  - for sufficiently nice $\mathfrak{V}$: $\mathcal{C}$ *factorizable finite ribbon category* – assumed below –
  - conformal blocks – sheaves on moduli spaces of curves with marked points

[cf. Belkale's talk]

Conformal field theory

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  - conformal blocks – sheaves on moduli spaces of curves with marked points
- full local CFT :
  - conformal world sheet – possibly with boundary, possibly non-orientable
  - holomorphic factorization : bulk fields $\in \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$ WITTEN 1992
  - local correlation functions :
    specific sections in the chiral blocks for complex double of the world sheet

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  - local correlation functions :
    specific sections in the chiral blocks for complex double of the world sheet
  - consistent boundary conditions $\rightsquigarrow$ bulk fields $\in \mathcal{Z}(\mathcal{C})$

    $\Longrightarrow$ take $\mathcal{C}$ *factorizable* : $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}} \simeq \mathcal{Z}(\mathcal{C})$
  - in particular : bulk state space $F \in \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$
  - torus partition function $Z_{\mathrm{T}}(\tau) =$ Virasoro character of F

The bulk state space F

Bulk state space $F \in \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$

☞ simplest situation :

$\mathcal{C}$ modular tensor category and F charge conjugation bulk state space

$$\boldsymbol{F = F_C^{\mathrm{rat.}} \equiv \bigoplus_{i \in I} S_i^\vee \boxtimes S_i} \qquad\qquad \mathrm{Z_T} = \sum_{i \in I} \chi_{\bar{\imath}}^* \, \chi_i$$

↯ indeed appears as part of a consistent full CFT for any MTC $\mathcal{C}$

FELDER-FRÖHLICH-JF-SCHWEIGERT 2000

The bulk state space F

Bulk state space $F \in \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$

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- interpretation: combine *every* object of $\mathcal{C}$ with its dual (= charge conjugate)

  accounting for *all relations* among objects

- proper concept: coend $\boldsymbol{F = \int^{U \in \mathcal{C}} U \boxtimes U^\vee \in \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}}$

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- proper concept: coend $\boldsymbol{F = \int^{U \in \mathcal{C}} U \boxtimes U^\vee \in \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}}$

  - object together with dinatural family of morphisms $\imath_U^F : U \boxtimes U^\vee \to F$
  - defined by universal property
  - $\mathcal{C}$ semisimple (thus modular): $F = F_C^{\mathrm{rat.}}$
  - $\mathcal{C} = H$-mod for factorizable ribbon Hopf $\mathbb{C}$-algebra:

    $F =$ coregular bimodule $H^* \in H$-bimod $\simeq \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$

The bulk state space F

- $\mathcal{C}$ semisimple (RCFT):
  - $F_C^{\mathrm{rat.}} = Z(\mathbf{1})$ full center

The bulk state space F

- $\mathcal{C}$ semisimple (RCFT):
  - *any* bulk state space is of the form $Z(A)$

    for $A \in \mathcal{C}$ simple symmetric special Frobenius algebra
  - $Z(A) \cong \bigoplus_{i,j \in I} (S_i \boxtimes S_j^\vee)^{\oplus z_{ij}(A)}$ with

    $z_{ij}(A) = \dim_{\mathbb{C}}\big(\mathrm{Hom}_{A|A}(S_i \otimes^+ A \otimes^- S_j^\vee, A)\big)$

    commutative cocommutative symmetric Frobenius algebra in $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$

FJELSTAD-JF-RUNKEL-SCHWEIGERT 2008

The bulk state space F

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    commutative cocommutative symmetric Frobenius algebra in $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$
- physics terminology:
  - multiplication on F formalizes operator product of bulk fields
  - Frobenius pairing formalizes
    non-degeneracy of two-point correlators of bulk fields on the sphere
  - (co-) commutativity implements monodromy invariance of correlators
  - .....

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    commutative cocommutative symmetric Frobenius algebra in $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$
- general case:
  - set $m_F \circ (\imath_U^F \otimes \imath_V^F) := \imath_{V \otimes U}^F \circ (\gamma_{U,V} \boxtimes c_{U,V})$, $\eta_F := \imath_{\mathbf{1}}^F$ $\qquad \gamma_{U,V} : U^\vee \otimes V^\vee \xrightarrow{\cong} (V \otimes U)^\vee$
  - (F, m_F, η_F) is commutative associative unital algebra in $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$

JF-SCHWEIGERT-STIGNER 2013

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  - $(\boldsymbol{F}, \boldsymbol{m_F}, \boldsymbol{\eta_F})$ is commutative associative unital algebra in $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$
- $\mathcal{C} = H$-mod (with H not necessarily semisimple)
  - also symmetric Frobenius and cocommutative

JF-SCHWEIGERT-STIGNER 2012

The bulk state space F

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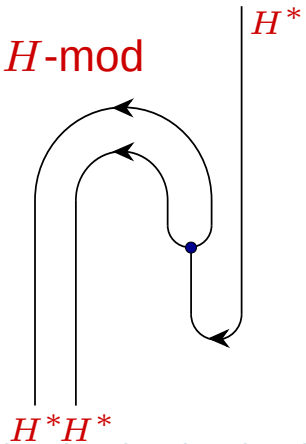
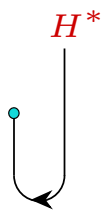
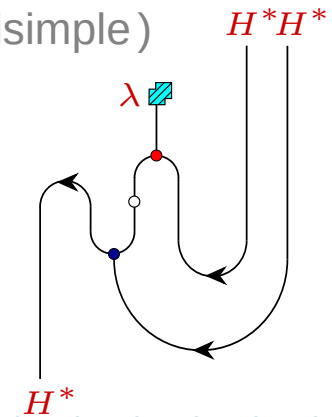
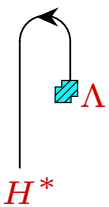
$m_F =$ 

$\eta_F =$ 

$\Delta_F =$ 

$\varepsilon_F =$ 

Bulk correlation functions

- recall : handle Hopf algebra $L = \int^{U \in \mathcal{D}} U \otimes U^\vee \in \mathcal{D}$

  recall : partial monodromy action $\varkappa_U^L$ of L on $U \in \mathcal{D}$

  recall : composition with automorphism S_L gives modular S-transformation

Bulk correlation functions

- ☞ input : bulk handle Hopf algebra $K = \int^{X \in \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}} X \otimes X^\vee \in \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$

  input : partial monodromy action $\varkappa_X^K$

Bulk correlation functions

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  input: partial monodromy action $\varkappa_X^K$

- ☞ further input: $(\boldsymbol{F}, \boldsymbol{m_F}, \boldsymbol{\eta_F}, \boldsymbol{\Delta_F}, \boldsymbol{\varepsilon_F})$ algebra - coalgebra in $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$

Bulk correlation functions

- input: bulk handle Hopf algebra $K = \int^{X \in \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}} X \otimes X^\vee$

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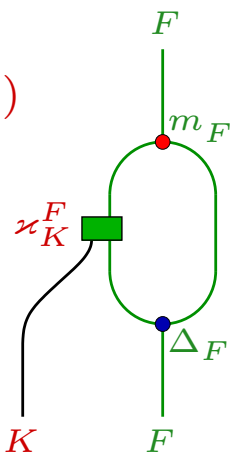
- further input: $(\boldsymbol{F}, \boldsymbol{m_F}, \boldsymbol{\eta_F}, \boldsymbol{\Delta_F}, \boldsymbol{\varepsilon_F})$ algebra - coalgebra

- define family of morphisms $\mathbf{Cor}_{\boldsymbol{g;p,q}} = \mathrm{Cor}_{g;p,q}(F)$:

$$\mathrm{Cor}_{0;1,1} := \mathrm{id}_F$$

$$\mathrm{Cor}_{1;1,1} := m_F \circ (\rho_F^K \otimes \mathrm{id}_F) \circ (\mathrm{id}_K \otimes \Delta_F)$$



$\in \mathrm{Hom}_{\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}}(K \otimes F, F)$

Bulk correlation functions

- input: bulk handle Hopf algebra $K = \int^{X \in \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}} X \otimes X^\vee$

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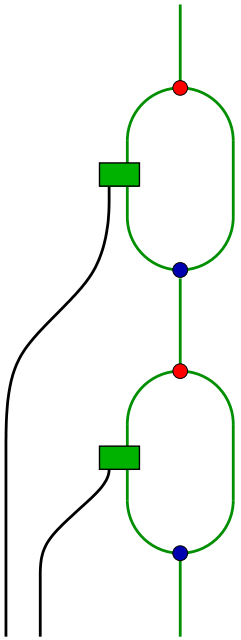
$$\mathrm{Cor}_{g;1,1} := \mathrm{Cor}_{1;1,1} \circ (\mathrm{id}_K \otimes \mathrm{Cor}_{g-1;1,1}) \quad \text{for} \quad g > 1$$

$\mathrm{Cor}_{2;1,1} =$



Bulk correlation functions

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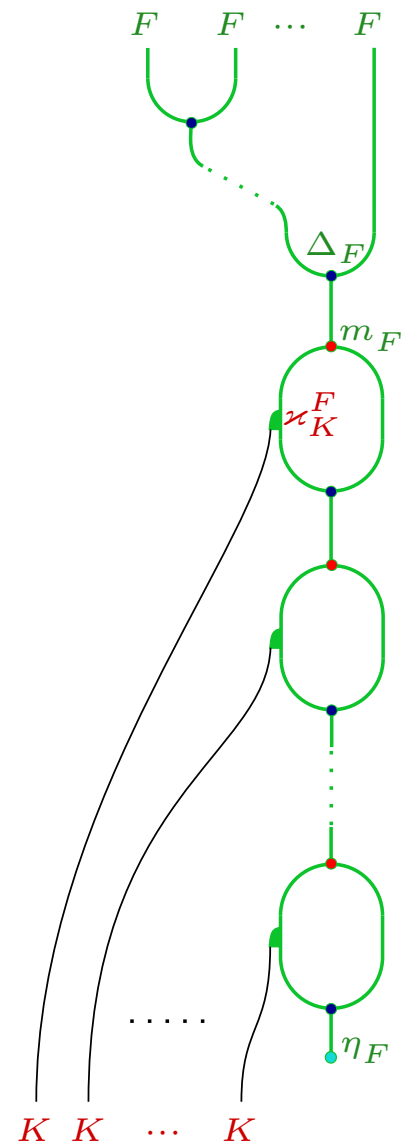
$$\mathrm{Cor}_{g;p,q} := \Delta_F^{(p)} \circ \mathrm{Cor}_{g;1,1} \circ \left(\mathrm{id}_{K^{\otimes g}} \otimes m_F^{(q)}\right)$$

$\mathrm{Cor}_{g;0,n} =$



Bulk correlation functions

- ☞ input: bulk handle Hopf algebra $K = \int^{X \in \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}} X \otimes X^\vee$

  input: partial monodromy action $\varkappa_X^K$

- ☞ further input: $(\boldsymbol{F}, \boldsymbol{m_F}, \boldsymbol{\eta_F}, \boldsymbol{\Delta_F}, \boldsymbol{\varepsilon_F})$ algebra - coalgebra

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- ☞ goal: various desirable properties of $\mathbf{Cor}(\boldsymbol{F})$ from properties of $\boldsymbol{F}$

  $\Longrightarrow$ $\mathbf{Cor}$ candidate for bulk correlation functions of a full CFT

Bulk correlation functions

PROPERTIES OF $\mathbf{Cor}(F)$

☞ action of mapping class group:

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  $\Longrightarrow$ $\mathrm{Cor}(F)$ invariant under *fusing move*
- F commutative and cocommutative (and trivial twist)
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Bulk correlation functions

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☞ Lego-Teichmüller game: systematic description of diffeomorphisms
via connected simply connected CW complex with F / B / R / S moves as 1-cells

BAKALOV-KIRILLOV 2000
BARMEIER 2007

$\Longrightarrow$ invariance of $\mathrm{Cor}(F)$ under moves
implies invariance of $\mathrm{Cor}(F)$ under whole mapping class group

PROPERTIES OF $\mathbf{Cor}(\boldsymbol{F})$

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☞ STATUS:

- general $\mathcal{C}$: arguments straightforward but not written up
- $\mathcal{C} = H$-mod: $F = \int^{U \in \mathcal{C}} U \boxtimes \omega(U^\vee)$ is $\cdots\cdots$ S-invariant Frobenius

JF-SCHWEIGERT-STIGNER 2012

Bulk correlation functions

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  and $\mathrm{Cor}(F)$ is mapping class group invariant JF-SCHWEIGERT-STIGNER 2014
  – by brute force: invariance under action of set of generators of $\mathrm{Map}(\Sigma)$ –

Bulk correlation functions

PROPERTIES OF $\mathbf{Cor}(F)$

☞ behavior under sewing $\Sigma \to \Sigma'$ of world sheets:

known: sewing / factorization relations among spaces of conformal blocks

desired: sewing relations map $\mathrm{Cor}(\Sigma; F)$ to $\mathrm{Cor}(\Sigma'; F)$

Bulk correlation functions

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- two types of sewings:

  - non-handle creating:

    result follows from simple form of sewing relations

e.g.
$$\int^{X \in \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}} \mathrm{Hom}_{\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}}(K^{\otimes g_1}, X) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}}(K^{\otimes g_2} \otimes X, Y) = \mathrm{Hom}_{\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}}(K^{\otimes g_1 + g_2}, Y)$$

with dinatural family $f \otimes g \longmapsto g \circ (\mathrm{id}_{K^{\otimes g_2}} \otimes f)$

Bulk correlation functions

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  - handle creating:

    must form coend in category of left exact functors

    LYUBASHENKO 1995

    $\Longrightarrow$ technically involved

Bulk correlation functions

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  - handle creating:

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    LYUBASHENKO 1995

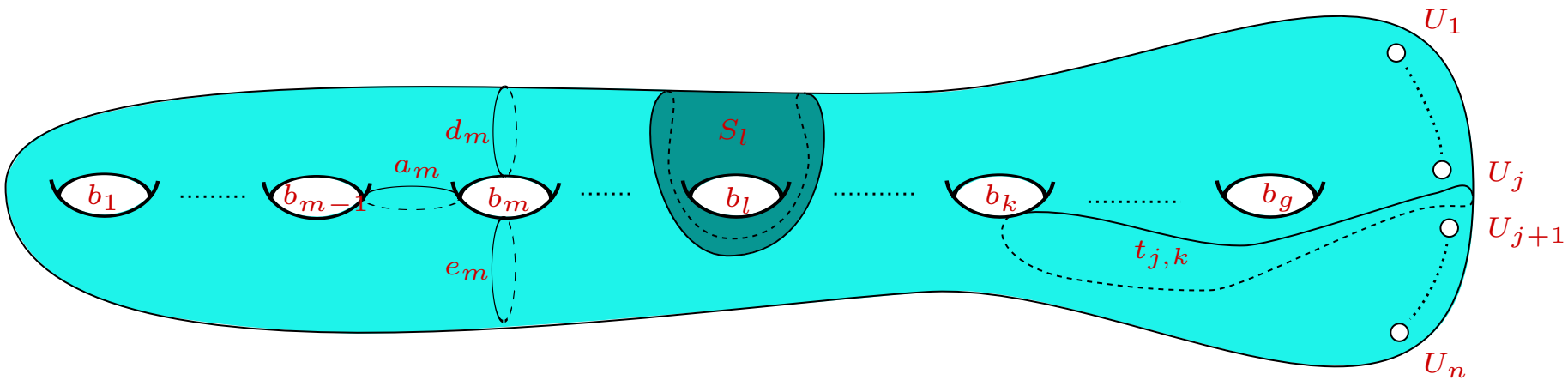
- OUTLOOK: to be written up quite soon

THANK YOU

APPENDIX

Appendix: Generators of Map_Σ

- exact sequence $1 \longrightarrow B_{g;n} \longrightarrow \mathrm{Map}_{g;n} \longrightarrow \mathrm{Map}_{g;0} \longrightarrow 1$
- thus generated by permutations of holes and Dehn twists



(relations not needed)

Appendix: Coends

- given a functor $G\colon\ \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{E}$ and an object $B \in \mathcal{E}$

  a dinatural transformation $G \Rightarrow B$ is a family of morphisms

  $\varphi_X\colon\ G(X,X) \to B$ s.t.

$$\begin{tikzcd}
G(Y,X) \arrow[r, "G(\mathrm{id}_Y, f)"] \arrow[d, "G(f, \mathrm{id}_X)"'] & G(Y,Y) \arrow[d, "\varphi_Y"] \\
G(X,X) \arrow[r, "\varphi_X"] & B
\end{tikzcd}$$

  commutes for all $f\colon X \to Y$

- *coend* (A, ι) for G:

  initial object in category of dinatural transformations $G \Rightarrow -$:

$$\begin{tikzcd}
G(Y,X) \arrow[r, "G(\mathrm{id}_Y, f)"] \arrow[d, "G(f, \mathrm{id}_X)"'] & G(Y,Y) \arrow[d, "\iota_Y"] \arrow[ddr, bend left, "\varphi_Y"] & \\
G(X,X) \arrow[r, "\iota_X"] \arrow[drr, bend right, "\varphi_X"'] & A \arrow[dr, dashed] & \\
& & B
\end{tikzcd}$$

  - notation: $(A, \iota) = \int^X G(X,X)$
  - unique up to unique isomorphism (if exists)