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to proceed to get some induction  
 let's work with the case  $\lambda \in P$

$$\text{or } N(\lambda) \rightarrow M(\lambda) \longrightarrow V(\lambda) \rightarrow 0$$

Verma  
irr. f.d.  
inf-dim

$N(\lambda) =$  submodule generated by elements  
 $(x_i^-)^{\lambda(h_i)+1} m_\lambda$ .

element has weight  $\vartheta - (\lambda(h_i)+1)\alpha_i$ .

$\frac{x_{\alpha_i}^+}{x_{\alpha_i}^-} m_\lambda = 0 \quad \forall i$

i.e.  $M(\vartheta + (\lambda(h_i)+1)\alpha_i) \rightarrow M(\lambda)$

$$\vartheta - \lambda(h_i)\alpha_i = s_{\alpha_i}(\lambda) \quad , \quad \rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \quad s_{\alpha_i}\rho = \rho - \alpha_i$$

$$\begin{aligned} \lambda - (\lambda(h_i)+1)\alpha_i &= s_{\alpha_i}(\vartheta + \rho) - \rho \quad \cancel{\text{at}} \rightarrow 0 \\ &= s_{\alpha_i} \cdot \} \end{aligned}$$

dot action of  $w$  on  $\mathfrak{g}^*$  is (22)

$$(w \cdot \lambda)(\beta) = w(\lambda + \rho) - \rho.$$

Lemma:  $\lambda \in \mathfrak{h}^*$ ,  $(\lambda + \rho)(h_\alpha) \in \mathbb{Z}_+$

$$\Rightarrow M(s_{\alpha_i} \cdot \lambda) \hookrightarrow M(\lambda)$$

What happens when if instead of  $\alpha_i$  we work with  $s_\alpha$ . this time  $(\lambda + \rho, h_\alpha) \in \mathbb{Z}$

Thm:  $(s_\alpha \cdot \lambda)(h_\alpha) = m_\alpha \neq 0$

$$s_{\alpha_j} \cdot (s_\alpha^{-1})^{(\lambda(h_\alpha)+1)} = m_\alpha \neq 0$$

$$\begin{aligned} (s_\alpha \cdot \lambda)(h_\alpha) &= (s_\alpha (\lambda + \rho) - \rho)(h_\alpha) \\ &= \lambda - (\lambda(h_\alpha) + \rho(h_\alpha)) \alpha \end{aligned}$$

Thm: <sup>Lemma</sup>  $\alpha \in R_1^+ \Rightarrow s_\alpha \cdot \lambda \leq \lambda$

$$\text{i.e. } \lambda(h_\alpha) + \rho(h_\alpha) - (\lambda + \rho)(h_\alpha) > 0$$

$$\text{then } M(s_\alpha \cdot \lambda) \hookrightarrow M(\lambda)$$

strongly linked:  $\mu, \lambda$  suppose that

$\exists$  roots  $\gamma_1, \dots, \gamma_r \in \mathbb{C}^+$  s.t.

$$\mu \uparrow_{s_{\gamma_1}} \mu \uparrow \dots \uparrow_{s_{\gamma_r}} \mu = \lambda$$

$\mu \uparrow_{s_{\gamma_i}} \mu$  if  $\mu \not\geq s_{\gamma_i} \cdot \mu$  then  $M(\lambda) \subset M(\mu)$

and so  $[M(\mu) : V(\lambda)] > 0$

BGG: converse is true.

$$\cancel{M(\mu)} \neq 1$$

BGG:  $\mu$  strongly linked  $\Rightarrow [M(\mu) : V(\lambda)] > 0$

$\Rightarrow \mu$  strongly linked to  $\lambda$ .

Cor:  $[M(\mu) : V(\lambda)] > 0 \Leftrightarrow \text{tor}_{\mathfrak{g}}(M(\lambda), M(\mu)) \neq 0$

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Review: cat<sub>g</sub>  $\Theta$ ,  $\mathcal{L} \in \mathcal{G}^*$ ,  $M(\lambda)$ ,  $V(\lambda)$

$M^r \in \Theta$

- $V(\lambda)$  def<sup>\*</sup> are all simple objects upto iso morphism.
  - $M(\lambda)$  has a ! irr. quotient
  - ★ • objects in  $\Theta$  have JH series
  - $\dim \text{Hom}_{\mathfrak{A}}(M(\mu), M(\lambda)) \leq 1$  and so
  - $M(\lambda)$  has a ! irr. submodule which must be a norm module
  - strongly linked  $\mu \uparrow \lambda$  if  $\exists \alpha \in \Delta^+$  s.t  $\mu = \sum \alpha$  and  $(\lambda + \rho) \alpha (\lambda + \rho)(\alpha) \in \mathbb{Z}^{>0}$   
i.e.  $\lambda + \mu$  and  $\lambda - \mu \in Q^+$ .
- then  $\text{Hom}_\mathfrak{A}(M(\mu), M(\lambda)) \neq 0 \Leftrightarrow [M(\lambda) : V(\mu)] \neq 0$   
 $\Leftrightarrow \mu$  strongly linked to  $\lambda$

hard thm but if we assume  $\lambda \nmid \mu$   $\lambda + \rho \in P^+$   
then  $\mu$  strongly linked  $\nmid \lambda \Rightarrow M(\mu \circ \mu) \hookrightarrow M(\lambda)$

Proof: by induction on  $\ell(w)$

$$\ell(w) = 1, \quad w = s_i \quad \left( \tilde{x}_{\alpha_i} \right)^{m_{\tilde{x}} * 1} \xrightarrow{\lambda(h_i) \neq 1}$$

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$$W = s_{i_1} s_{i_2} (\lambda) \quad (x_{\alpha_{i_1}}^-) \quad (x_{\alpha_{i_2}}^-)$$

$$(s_{i_1} \nmid h_i) \quad \lambda(h_{i_2})$$

$$\tilde{s}_{i_2} \lambda = (s_{i_2} (\lambda + \rho) \cancel{+}) (h_{i_2})$$

$$(\lambda + \rho) (\underbrace{s_{i_2} h_{i_2}}_{\cancel{+}}) \neq 0$$

so rk.

$$w_0 = s_{i_1} \dots s_{i_N} \quad \text{longest element of } W$$

$$M(w_0 \lambda) \hookrightarrow M(\lambda)$$

Thm: Prop.  $M(w_0 \lambda)$  irreducible if  $\lambda \in \rho^+$

$w_0 \lambda$  is antidominant

Prop special case of a by thm

Thm:  $M(\lambda), \lambda \in \rho^*$  is irred iff  $(\lambda + \rho)(h_\alpha) \notin \mathbb{Z}^{>0}$

for any  $\alpha \in \mathbb{I}^+$ .

at this point, I can't go on any further  
without saying something about original  
methods of proof

One of the main tools then in rep. theory is the theory of characters. - for category  $\mathcal{O}$ , we'll encounter two kinds, formal character of  $M \in \mathcal{O}$ , theory of central characters.

$$M \in \mathcal{O} \quad \text{ch } M : \mathfrak{h}^* \rightarrow \mathbb{Z}_+$$

$$\text{ch}(M \otimes N) = \lambda \mapsto \dim_{\lambda} M \cdot \text{ch } N \quad \text{ch}(M \otimes N) = \text{ch } M \text{ ch } N$$

provided  $M \otimes N \in \mathcal{O}$

well-defined one year prove.

$$\text{Lemma: } M \in \mathcal{O} \Rightarrow \dim M_{\lambda} < \infty \quad \forall \lambda \in \mathfrak{h}^*$$

Pf. enough to prove when  $M$  is generated by a single element. say  $m_1 \in M_m$

$$\text{by PBW thm: } M = U(n^-) \underbrace{U(n^+) m_1}_{\text{finite dimension}}$$

say has basis consisting of element  $m_1, \dots, m_r, m_i \in M_m$

$U(n^-) m_i$  — clear

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$$\text{char } \text{ch } M = \sum_{\lambda \in \mathfrak{h}^*} (\dim M_\lambda) e(\lambda)$$

$$e(\lambda) : \mathfrak{h}^* \rightarrow \mathbb{Q}, \quad \lambda \mapsto 1$$

$$\text{ch } M(\lambda) = e_\lambda + \sum_{\mu \in \lambda - Q^+} \dim M(\lambda)_\mu e_\mu \quad M \rightarrow G \xrightarrow{\text{ad}}$$

character contains <sup>as</sup> information <sup>about</sup> the module

but does not usually determine the module.

$$M \cong N \Rightarrow \text{ch } M = \text{ch } N$$

Converse is usually false.  $M$  and  $M'$  have same char.  
but ~~not~~ <sup>not</sup> isomorphism

Special cases: Suppose  $M, N$  are f.d. then

$$M \cong N \Leftrightarrow \text{ch } M = \text{ch } N.$$

$$\text{ch } V(\lambda) \quad - \text{ weight character formula.}$$

$\lambda \in P^+$

character of representations are also not too hard to write down

$$\text{ch } M(\lambda) = \sum_{\eta \in Q^+} (\dim O(n^-)_\eta) \text{ch}(\lambda - \eta)$$

$$\text{Kostant partition function} = \prod_{\lambda \in P^+} \frac{e(\lambda)}{(1 - e(\lambda))}$$

$\text{char } V(\lambda)$  - hard problem uses

K-L theory.

note a simple fact now. if  $M \in \Theta$  has TH sum

$$\text{ch } M = \sum_{\mu \in \mathbb{Y}^+} [M : V(\mu)] \text{ ch } V(\mu)$$

$$\text{ch } M(\lambda) = \text{ch } V(\lambda) + \sum_{\eta \in Q^+} [M(\lambda) : V(\lambda - \eta)] \text{ ch } V(\lambda - \eta)$$

$$= \sum_{w \in W} [M(\lambda) : V(w \cdot \lambda)] \text{ ch } V(w \cdot \lambda)$$

$w \cdot \lambda \uparrow \lambda$

strong links

kinds of arguments carry

over to  $g$  of Kac-Moody algebras

Anot

- central characters special to  $g$  simple

in K-M case, center too small.

but variations exist which work for certain

classes of objects - different flavor

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$\mathcal{Z}(\mathfrak{g})$  = center of universal enveloping algebra of  $\mathfrak{g}$

well-known element Casimir  $\Omega$

$\Omega$  Killing form  $n_\alpha^\pm \alpha \bar{\alpha}^+$ ,  $h_i | 1 \leq i \leq n$ .

pick a dual basis  $x_\alpha^\pm$   $H_i$

$$\Omega = \sum n_\alpha^\pm x_\alpha^\pm + \sum h_i H_i$$

Theorem:  $\mathcal{Z}(\mathfrak{g})$  is a polynomial of  $\mathfrak{g}$  in  $n$ -variables

$$\text{and } \mathcal{Z}(\mathfrak{g}) \cong \mathcal{S}(\mathfrak{g})^W$$

$W \times \mathfrak{g} \rightarrow \mathfrak{g}$   $S(\mathfrak{g})^W = \text{alg. of invariants}$

a central character is just an algebra homomorphism from  $\chi: \mathcal{Z}(\mathfrak{g}) \rightarrow \mathbb{C}$

Also

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$M \in \Omega$        $\mathfrak{z}(\mathfrak{g}) \rightarrow \text{end } M$  commutes

operators so can write

$M = \bigoplus_x M_x$  generalized

asym space for the action of  $\mathfrak{z}(\mathfrak{g})'$

$$M_x = \left\{ m \in M : (\mathfrak{z} - \chi(\lambda))^r m = 0 \text{ for some } r = r(m, \lambda) \right\}$$

~~so~~  $M_x$  is a  $\mathfrak{g}$ -submodule and so  $M_x \in \Omega$

$$(M_x)_\alpha$$

Say  $M$  admits a central character  $\chi$

$$\text{if } zm = \chi(z)m.$$

Lemma:  $\underline{\text{• }} M(\lambda)$  <sup>admits</sup> a central character

$\cdot M \in \Omega$  is indecomposable  $\Rightarrow M = M_\lambda$

Pf.  $M(\lambda) = U(\mathfrak{g})m_\lambda$

$$\mathfrak{z}(\mathfrak{g}) : M(\lambda) \rightarrow M(\lambda) \quad zm_\lambda = \chi_\lambda(z)m_\lambda$$

$$\chi_\lambda : \mathfrak{z}(\mathfrak{g}) \rightarrow \mathbb{C} \text{ alg hom.}$$

Thm (Harish-Chandra)

- Any central character  $\chi : \mathcal{Z}(G) \rightarrow \mathbb{C}$  is of the form  $\chi_\lambda$  for some  $\lambda \in \mathfrak{h}^*$
- $\chi_\lambda = \chi_\mu \Leftrightarrow \text{wt}(\lambda + \rho) = \mu + \rho$  for some  $w \in W$

$$\Theta_\chi = \{M \in \mathcal{O} : M = M_\chi\}$$

Lemma: (i)  $\text{Hom}_{\mathcal{O}}(\Theta_{\chi}, \Theta_{\chi'}) \neq 0 \Rightarrow \chi = \chi'$

(ii) Suppose  $0 \rightarrow M \rightarrow N \rightarrow U \rightarrow 0$

$M \in \Theta_\chi, N \in \Theta_{\chi'}$ , then  $\chi \neq \chi' \Rightarrow$  ses is split

$$M = \bigoplus_{\substack{\lambda \in \mathfrak{h}^* \\ /W}} M_\lambda$$

$\Theta_\chi$  has only finitely many ~~principal~~ <sup>single objects</sup> finite-dimensional

so one can reduce study of  $\mathcal{O}$  to  $\mathcal{O}_x$

- still too many  $\mathcal{O}_x$  - translation lindars &

Jantzen which studied how the  $\mathcal{O}_x$  were related.

principal block  $\mathcal{O}_0 \quad \chi_g: g(\mathbb{Q}) \rightarrow \mathbb{C} \quad \theta = 0$

so now lets use theory of centred characters to prove.

Thm: Any object  $M \in \mathcal{O}$  has finite length

Pf: two steps

① prove for  $\mathfrak{m}$  modules (or more generally quotients of  $\mathfrak{m}$  modules)

② generalize to  $\mathfrak{m}$  ab  $M$ .

Pf  $0 \rightarrow N(\lambda) \rightarrow M(\lambda) \rightarrow V(\lambda) \rightarrow 0$

$$V = \sum_{\lambda \in \mathfrak{w}} M(\lambda)$$

Suppose  $N_i \subseteq N(\lambda)$  submod  $\Rightarrow \chi_i$  has central character  
on other hand  $N_i$  contains a max vector  $\mu \Rightarrow \chi_i = \chi_\mu$

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$$\Rightarrow \mu = w_0 \lambda \Rightarrow N_i \cap U \neq \{0\}$$

$$M(\lambda) \supseteq N(\lambda) \supseteq N_i$$

No, i.e. done otherwise repeat chain and  
stop because  $\dim U < \infty$

$\Rightarrow M(\lambda)$  contains an irr. module

All this argument uses is  $M(\lambda)$  has central charach.

$\forall \lambda$  so repeat with  $M(\lambda)/_{\text{irr. module}}$

now you increasing chain with source - q.e.d

irr, noetherian, must stop

arbitrary  $M$ ? follows from the next lemma.

Lemma:  $M \in \mathcal{O}$  then  $M$  admits a (finitely)

increasing filtration

$$0 \subseteq M^0 \subseteq M^1 \subseteq \dots \subseteq M^r = M \quad \text{with } M^i /_{i \in I} \text{ a quotient}$$

of a  $\mathbb{K}R\Gamma$  module

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Pf of Lemma  $M \in \Theta$   $\exists \alpha \neq m_1 \in M_{m_1}$  s.t.

$$n^+ m_1 = 0 \Rightarrow U(g)m_1 \text{ is a factor of } M/m$$

take  $M' = U(g)m_{m_1}$

$$0 \rightarrow M' \rightarrow M \rightarrow \frac{M}{M'} \rightarrow 0$$

find  $m_2 \in M$  s.t.  $\frac{U(g)m_2 + M'}{M'}$  is a

generator of  $M(m_2)$

so now we have  $0 \subseteq M' \subseteq M + U(g)m_2 \subseteq$

repeat process stops because  $U(g)$  nothing done.

=

this brings us to a def<sup>n</sup>.

def: Say  $M \in \Theta$  has a room flag if  $\exists$  a decreasing/increasing chain of submodules

$$0 \subseteq M_1 \subseteq M_2 \subseteq \dots \subseteq M_\delta = M$$

$\Delta 1.$   $M_i \subseteq M / (\mu_i)$  for some  $\mu_i \in h^*$

$$\frac{M}{\mu_{i-1}}$$

$[M : M / (\mu_i)] = \# \text{ of times } M / (\mu_i) \text{ occurs}$   
in  $M$

Ex: use theory of formal characters to see

that  $[M : M / (\mu_i)]$  is well defined when if

$M$  admits a 'reindeer flag'

are there examples of modules with many

flags? Should we care? turns out yes we

should, because ~~so~~ these are classical techniques  
in rep. theory of associative algebras

lets look at a small example first

$\mathfrak{g}_2 = \mathfrak{sl}_2$  dim  $\mathfrak{h}^*$  = 4 (take identity)

$\mathfrak{h}^*$  with  $\mathbb{C}$   $\lambda \in \mathbb{C}$   $M(\lambda)$

consider:  $M(\lambda)$   $\otimes V(1)$

$V(1)$  - fd. irrep of  $\mathfrak{sl}_2$  with  $h\omega_1$

= flat module of  $\mathfrak{sl}_2$

$\dim V(1) = 2$  basis  $v_0, v_1$

$$\begin{matrix} x_\alpha^+ v_0 = 0 & h_\alpha v_0 = b, & (x_\alpha^-)^2 v_0 = 0 \end{matrix}$$

$$v_0, \quad x_\alpha^+ v_0 = v_1$$

look at  $m_{\lambda} \otimes v_0$  then  $x_\alpha^+ m_\lambda = 0 \quad h_m = \lambda h \quad m_\lambda$

$M(\lambda+1) \rightarrow M(\lambda) \otimes V(1)$

$$(x_\alpha^-)^s m_{\lambda+1} \rightarrow (x_\alpha^-)^s (m_\lambda \otimes v_0)$$

$$= (x_\alpha^-)^s m_\lambda \otimes v_0 + (x_\alpha^-)^s m_\lambda \otimes x_\alpha^- v_1$$

$\neq 0$

No mapping isjective

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look at question

$$\frac{M(\lambda) \otimes V(1)}{M(\lambda+1)}$$

generated  $m_{\lambda} \otimes v_1$ ,  $m^*(m_{\lambda} \otimes v_0) \in M(\lambda+1) \cap M(\lambda-1)$

by image of

$$0 \rightarrow M(\lambda+1) \rightarrow M(\lambda) \otimes V_1 \rightarrow \frac{M(\lambda) \otimes V(1)}{M(\lambda+1)} \rightarrow 0$$

use characters

$$\text{ch } M(\lambda) \otimes V(1) = \text{ch } M(\lambda+1) + \text{ch } M(\lambda-1)$$

So have a s.p.s.

$$0 \rightarrow M(\lambda+1) \rightarrow M(\lambda) \otimes V(1) \rightarrow M(\lambda-1) \rightarrow 0$$

and its non-split.

Prop:  $M(\lambda) \otimes V(m)$  has a verma flag if

$$\mu \in P^+$$

## B6h reciprocity and filtrations

If not semisimple, one is interested in understand homological properties, for this one needs to know about projectives and injectives

Prop:  $\lambda + \rho$  is dominant  $\Rightarrow$   $(\lambda + \rho) \not\in Q^+$

then  $M(\lambda)$  is projective

$$\text{Pf: } 0 \rightarrow U \rightarrow M \rightarrow M(\lambda) \rightarrow 0$$

want to show sequence splits. can assume

now  $\exists$   $U, M \in \mathcal{O}$  because  $M(\lambda) \in Q^+$

$$\begin{matrix} & M(\lambda) \\ \cong & \downarrow \\ U & M \end{matrix}$$

or equivalently from  $M \rightarrow N \rightarrow 0$

$$m \rightarrow m_\lambda$$

$$\Rightarrow m_\lambda = 0 \Rightarrow \exists \mu \text{ s.t. } \mu = \lambda + \eta \quad M(\mu) \rightarrow M$$

$$\Rightarrow \lambda = \mu - \eta \Rightarrow \mu = w, \lambda \neq \mu$$

contradicts  $\lambda$  dominant done.

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so we have some projectiles

$\Delta t^*$

$\lambda_1 \eta_P$

$n >> 0$

$M/\lambda + n_P$