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Category 0 - important catg of modules
for a simple finite-dimensional lie-algebra. ^{study} goes back
^{late} to 1960's and it continues to be a very active
research area with increasing ^{numbers} applications and more
new methods ~~beginning~~ to play. There are other
important module categories which bear some relation to
to 0, mod rep. of algebras in char p, cherednik
algebras, catg of finite-dimensional reps of quantum
loop alg / Yangians.

As in week 1, I'll give an overview of some of the
new modules in catg 0 which are important, methods used
to study these modules and some important
results. Going to try and show how these
all come up in some sense naturally
and I'll introduce inflation as we go along and
meet them. Next week, look at what are called ^{quantized} loop algebras

2 simple Lie alg, with CSA \mathfrak{h} fixed
(fin.dim, complex) D

Φ = set of roots of $(\mathfrak{g}, \mathfrak{h})$, $\Phi \subseteq \mathfrak{h}^*$, $\# \Phi$

$\Delta = \{\alpha_1, \dots, \alpha_n\}$, $n = \dim \mathfrak{h}$ fixed simple system

$\mathbb{I}^+ = \mathbb{Z}_+ \Delta \cap \Phi$ positive roots.

root space decom. $\mathfrak{g} = \mathfrak{h} \oplus_{\alpha \in \Phi^+, \pm \alpha} \mathfrak{g}_{\pm \alpha}$,

$$m^+ = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{\pm \alpha}, \quad \mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$$

$$\dim \mathfrak{g}_\alpha = 1 \quad \alpha \in \Phi$$

Chevalley basis. $\tilde{x}_\alpha^\pm \in \mathfrak{g}_{\pm \alpha}$, $\alpha \in \Phi^+$, $\{h_1, \dots, h_n\}$ basis of

$$[h_i, x] = \alpha(h_i) x, \quad [h_i, h_j] = 0 \quad [\tilde{x}_\alpha^+, \tilde{x}_\beta^-] = \sum_i h_i$$

Serre rel's: $(\text{ad } \tilde{x}_\alpha^+)^{|\alpha|} \tilde{x}_\beta^+ = 0$

$$W = \text{wyl gpl } (\mathfrak{g}, \mathfrak{h}) \cong \text{Aut } S$$

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Generated by reflection $s_\alpha(\lambda) = \lambda - \lambda h_\alpha^\vee \alpha$, $\alpha \in \Phi^+$

examples $\underline{g} = \text{SL}_{n+1}(\mathbb{C})$ trace zero $(n+1) \times (n+1)$ matrices

\mathfrak{H} = diagonal matrices.

$$e_i \in \mathfrak{H}^* \quad e_i \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_{n+1} \end{pmatrix} = a_i$$

$$\Phi^+ = \{e_i - e_j : 1 \leq i < j \leq n+1\}$$

$$\Delta = \{e_i - e_{i+1} : 1 \leq i \leq n\}$$

$$\Phi^+ = \{e_i - e_j : 1 \leq i < j \leq n\}$$

$$\mathcal{W} \cong S_{n+1}$$

$$\underline{g}, h \quad \alpha \in \Phi^+ \quad \underline{x}_\alpha^\pm, h_\alpha = \begin{bmatrix} \underline{x}_\alpha^\pm & \underline{x}_\alpha^- \end{bmatrix}, \quad h_\alpha = \sum_{i=1}^n s_i h_i$$

$h_\alpha \in \mathbb{Z}_{+}^{\text{-span } \Phi^+}$

$$\text{then } \text{SL}(\underline{x}_\alpha^\pm, h_\alpha) = \text{SL}(\mathbb{C})$$

Representations of \underline{g} . $f: \underline{g} \rightarrow \text{End } V$ V complex or

Lie alg map $f([x, y]) = f(x)f(y) - f(y)f(x)$

① trivial rep

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$$\forall x \in \mathfrak{g} \quad \text{ad } x = 0$$

$$\text{ad}: \mathfrak{g} \rightarrow \text{end } \mathfrak{g} \quad (\text{ad } x)(y) = [x, y]$$

lets think of constucting more: direct sums & tensor

tensor products V_1, V_2 two reps of \mathfrak{g}

\mathfrak{g} acts on $V_1 \otimes V_2$

$$x(V_1 \otimes V_2) = xV_1 \otimes V_2 + V_1 \otimes xV_2$$

$$V = \mathfrak{g} \quad V \otimes V = S^2(V) \oplus \Lambda^2 V$$

$$\mathfrak{sl}_2: \dim V=3 \Rightarrow \dim \Lambda^2 V=3$$

$$\text{ex: } \Lambda^2 V \cong \mathfrak{sl}_2 \quad x \mapsto \frac{1}{2}(x \otimes 1 - 1 \otimes x)$$

$$y \mapsto x \otimes y - y \otimes x$$

$$z \mapsto \frac{1}{2}(y \otimes z - z \otimes y)$$

$$S^2(V) \quad x \otimes x$$

ex: \mathfrak{sl}_2 -module generated by $x \otimes x$ is irreducible
and 5-dim. & we have a new rep

so knowing gives new examples and its ③

it has to see that repeating this process gives large number of examples of rep's

- thus bsp ms naturally to do: can we classify
irred. fin. dim rep's of simple-algebras and this is

what we're going to discuss for a bit

$$\rho: \frac{\mathfrak{g}}{J} \rightarrow \text{end } V, \quad \dim V < \infty$$

\downarrow
CSA abelian

\Rightarrow V is a direct sum of generalized eigenvectors for action on \mathfrak{h} , in particular $\exists 0 \neq v \in V$ and

$$\nexists \text{ meh st} \quad hv = \mu(h)v$$

now, $\rho(\alpha)v$ has eigenvalue $\mu + \alpha$

as $\rho(\alpha^+) \rho(\beta^+)v = \mu + \alpha + \beta$ these are all linear
vectors $\mu, \mu + \alpha, \mu + \alpha + \beta$ are distinct

⇒ at some stage process stops i.e. \exists

a vector $v_0 \in V$ with $hv_0 = \lambda(h)v_0$ for some $h \in H$
 $v^+ v_0 = 0$

lets work with \bar{v}_0 $\alpha \in \mathbb{Q}^+$

$$v_0, \bar{x}_\alpha^{-1} v_0, (\bar{x}_\alpha^{-1})^2 v_0, \dots, (\bar{x}_\alpha^{-1})^\alpha v_0$$

$$1 \quad \alpha \quad \alpha - 1, \quad \dots$$

again process stops $\exists \bar{x}_\alpha \text{ s.t.}$

$$(\bar{x}_\alpha^{-1})^{r_\alpha} v_0 \neq 0 \quad \text{and} \quad (\bar{x}_\alpha^{-1})^{r_\alpha + 1} v_0 = 0$$

what is the relation between r_α and $\lambda(h_\alpha)$

$$(\bar{x}_\alpha^{-1})^{r_\alpha + 1} (\bar{x}_\alpha^{-1}) v_0 = (\bar{x}_\alpha^{-1})^{r_\alpha} (h_\alpha - r_\alpha) v_0 = 0$$

$$= (\bar{x}_\alpha^{-1})^{r_\alpha} (\lambda(h_\alpha) - r_\alpha) v_0 = 0$$

$\Rightarrow r_\alpha = \lambda(h_\alpha)$. So we're not picking any h_0

If we need λ s.t. $\lambda(h_i) \in \mathbb{Z}_{\geq 0}^*$ & i

n-thm: $P^+ \subseteq \mathfrak{h}^*$ subset of \mathfrak{g}^* (8)

s.t. $\lambda(h_i) \in \mathbb{Z}$ & $1 \leq i \leq n$. weight lattice

$$P^+ \quad \lambda(h_i) \in \mathbb{Z}_+$$

root lattice. $Q = \mathbb{Z}\text{-span of } \{\alpha_1, \dots, \alpha_r\}$

$$Q \subseteq P$$

define a partial order on \mathfrak{h}^* as follows

$$\mu \leq \nu \Leftrightarrow \nu - \mu \in Q^+$$

is

Lemma: \forall fd rep of \mathfrak{g} then

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$$

~~$V_\mu = \mathbb{C}$~~

$$hv_0 = \lambda(h)v_0 \quad \lambda \in P^+$$

~~$V \ni v_0$~~

$$\mu^+ v_0 = 0$$

$$\begin{pmatrix} \chi \\ \alpha \end{pmatrix}^{(\lambda(h))H} v_0 = 0$$

Spanning set for V - ~~introduce~~ ^{recall} $U(g)$ - universal
enveloping alg of \mathfrak{g}

alg $\tilde{U}(\mathfrak{g})$ is an ass alg. Lie alg has \mathfrak{G}

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\quad} & \tilde{U}(\mathfrak{g}) \\ \downarrow & & \searrow F \\ \text{universal} & & A \\ \text{par} & & \end{array}$$

associatively

rep. theory of \mathfrak{g} \rightsquigarrow rep theory of ass. alg
by $U(\mathfrak{g})$
by alg maps

$\mathfrak{g} \rightarrow \text{end } V$
 $\downarrow \rightarrow$ never distinguish between the
 $U(\mathfrak{g})$ two.

let a be any lie alg.

PBW thm: $U(a)$ is a filtered alg

and associated graded alg +

isomorphic to $S(\mathfrak{g})$ symmetric algebra

• $\dim a < \infty$ then $S(\mathfrak{g})$ ~~is~~ here $U(a)$

If neither

• $U(a)$ has no zero divisors.

D

cor
lets

PBW basis. fix an order basis

$\alpha_1, \dots, \alpha_n$ of \mathfrak{g} then monomial

$\alpha_1^{r_1} \dots \alpha_n^{r_n}$ are a basis of $\mathcal{U}(\mathfrak{g})$

for \mathfrak{g} , we have a basis $\{\alpha\}$ of \mathfrak{g}

and order. Fix any enumeration

$\beta_1, \dots, \beta_N \in \mathbb{N}^+$

$$\alpha_{-\beta_1}, \alpha_{-\beta_2}, \dots, \alpha_{-\beta_N}, h_1, \dots, h_n, \alpha_{\beta_N} = \alpha_{\beta_N}$$

$\mathfrak{h}(m^\pm, \mathfrak{h}) \hookrightarrow$ lie sub alg of \mathfrak{g}

$\mathcal{U}(m^\pm) \cup (\mathfrak{h})$ are a ss. sub of $\mathcal{U}(\mathfrak{g})$

iso PBW $\mathcal{U}(\mathfrak{g}) \approx \mathcal{U}(m^\pm) \otimes \mathcal{U}(\mathfrak{h}) \otimes \mathcal{U}(n^\pm)$ as v.s.

spanning set for $V \otimes \mathfrak{v}_0$

$$V \text{ is in } \mathcal{U}(g)-\text{mod} \Rightarrow V = \mathcal{U}(g)\mathfrak{v}_0$$

$$= \{g\mathfrak{v}_0 \mid g \in \mathcal{U}(g)\}$$

$$\text{def } \mathfrak{v}_0 = \mathcal{U}(m^\pm) \otimes \mathcal{U}(\mathfrak{h}) \otimes \mathcal{U}(n^\pm) \mathfrak{v}_0 = \mathcal{U}(m^\pm) \mathfrak{v}_0 !$$

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Spanning set. $\left\{ n_{-\beta_1}^{s_1}, \dots, n_{-\beta_N}^{s_N} \mid s_i \in \mathbb{Z}_+ \right\}$

each of these elements are eigenvectors for h with
eig.

$$h \cdot n_{-\beta_1}^{s_1} \dots n_{-\beta_N}^{s_N} = (\lambda - \sum s_i \beta_i) \cdot (n_{-\beta_1}^{s_1} \dots n_{-\beta_N}^{s_N})$$

so we are proved V is fd

$$\Rightarrow V = \bigoplus_{\mu \in h^*} V_\mu$$

$$V_\mu = \left\{ v \in V : hv = \mu(h)v \right\}$$

and $\exists \lambda \in \mathbb{P}^+$ with $V_\lambda \neq 0$ and

$$\text{not } V = \left\{ \mu \in h^* : V_\mu \neq 0 \right\} \subseteq \lambda - Q^+$$

$$\dim V_\lambda = \cancel{\text{if } \lambda \in Q^+} 1$$

V irrep $\rightarrow \lambda \in \mathbb{P}^+$ well-defined

$$\mu \in P^+ \quad V_\mu \neq 0 \quad n^+ V_\mu = 0$$

$$U(g) v_\mu \subseteq V \quad \text{or } v_\mu \in V_\mu$$

submodule of V and vts are in $\mu-Q^+$

$$\Rightarrow U(g)v_\mu = V \Rightarrow \lambda \in \mu-Q^+, \mu \in \lambda-Q^+$$

$$\Rightarrow \mu = \lambda$$

$$V \cong V' \text{ check } \lambda \neq \lambda'$$

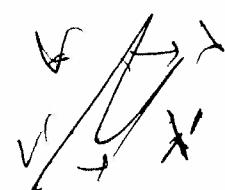
$$\varphi \rightarrow \phi(\varphi) \text{ has same prop} \Rightarrow \lambda = \lambda'$$

well-defined map

iso classes of id $\longleftrightarrow R^+$

ref.

also MA



Thm:

(i) the map is a bijection

classification thm

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next part of the ~~thm~~

~~Th~~ V is fd rep λ

$$hv_x = \lambda(h)v_x, \quad n^+v_x = 0 \quad \binom{x^-}{\alpha} \quad v_x = 0$$

ie. V is a quotient of $U(\mathfrak{g})$ by
 $\mathcal{J}(\lambda)$

the left ideal generated by setⁿ elements

$$h - \lambda(h), \quad n^+ \quad \binom{x^-}{\alpha}^{\lambda(h)+1}$$

$$U(\mathfrak{g}) \rightarrow V$$



$$\begin{array}{c} U(\mathfrak{g}) \\ \downarrow \\ \mathcal{J}_\lambda \end{array}$$

Then: $\dim \frac{U(\mathfrak{g})}{\mathcal{J}_\lambda} \neq \infty$ ~~is not irreducible~~

~~Some~~ dom. integral condition comes in
 because \mathcal{J}_λ has finite-dimensional opposite of the left

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take any $v \in \mathfrak{h}^*$ what this construct
 gives us
 is an idea for natural const. β in Q.L.

$$M(v) = U(\mathfrak{g}) / I_v = h_{-2}(v) : v \in \mathfrak{h}^*$$

these are called Verma modules

$$m_\lambda = \text{image of } 1 \quad \text{so} \quad m + m_\lambda = h_m = \lambda(h)m$$

PBW basis: $M(v) = U(\mathfrak{h})m_\lambda$

repeat everything we did before

$$\Rightarrow M(v) = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu$$

$$M_\mu \neq 0 \Rightarrow \mu \in V - Q^+$$

$$\dim M(v)_\mu = 1$$

def: Verma module for \mathfrak{g} if

$$M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu, \quad M_\mu = \{m \in M: (h_m - \mu(h))m = 0\}$$

$$\text{wt}(M) = \{\mu: M_\mu \neq 0\}$$

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Verma modules are very nice

~~M(H)~~ → Free $\mathcal{U}(n)$ -modules of rank 1
Ladge left mult by

$$M(\lambda) \underset{\text{Ladge}}{\sim} \mathcal{U}(g) \otimes \mathbb{C}_\lambda \quad n^+ g = 0 \\ \mathcal{U}(h) \otimes \mathbb{C}_{h\lambda}$$

$$\underset{n-m}{\sim} \mathcal{U}(n) \otimes \mathbb{C}$$

$$M(\lambda)_{\lambda-\eta} \simeq \mathcal{U}(n)_{\lambda-\eta} = \text{Span} \left\{ x_{-\beta_1}^{s_1} \cdots x_{-\beta_N}^{s_N} : \eta = \sum s_i \beta_i \right\}$$

Prop: $\phi \in \text{Hom}_{\mathcal{U}(g)}(M(\lambda), M(\mu))$. Then ϕ is injective

$$\phi(m_\lambda) = y m_\mu \quad y \in \mathcal{U}(n)$$

$$y \in M(\lambda) \quad m = y m_\lambda \quad y \in \mathcal{U}(n)$$

$$\phi(m) = u y v_m \neq 0 \quad \text{since } u \neq 0 \\ \mathcal{U} \text{ has no zero div}$$

$$\text{Hom}_{\mathcal{U}(g)}(M(\lambda), M(\mu)) = 0 \text{ if } \lambda \neq \mu.$$

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Ques: Who is $\text{Hom}(M(\mathbb{Z}), M(\mathbb{Z})) \cong 0$

Can we give a necessary and sufficient condition.

If $M(\mathbb{Z})$ is irreducible? Does it have \mathbb{Z}/\mathbb{Z} as

→ answering these questions is equivalent

other objects more completely, understand

their submodules - want to work

in a broader framework, keeping in mind some of

the features of these modules, now encountered

- cyclic modules → also known
submodules of cyclic modules are not cyclic
- finitely generated modules
- art modules
- if $m \in M(\mathbb{Z})$, $\dim M(m^{\perp}) = \infty$ because after
definitely many steps you close up back at \mathbb{Z}
and now we have our category, \mathcal{O}

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\mathcal{O} categ'g and M s.t.

- (i) M is finitely generated
- (ii) $\forall m \in M \quad \dim_{\mathbb{C}} U(m^+)_m < \infty$
- (iii) M weight module -

morphisms in categ. are \mathcal{O} -maps

$\lambda \in \mathbb{H}^*$, $M(\lambda) \in \mathcal{O}$ / duality in \mathcal{O}

$$\begin{aligned} \pi: z_2^+ &\rightarrow z_2^- \\ \pi(a_\alpha) &= b_\alpha \end{aligned}$$

(non-zero) $M(\lambda) \not\cong M(\lambda)^*$

Lemma: $M(\lambda)$ has a † for quotient $V(\lambda)$

Pf: $N_1, N_2 \subseteq M(\lambda)$ proper submod.

weight module claim $(N_i)^\perp = 0$

$$N_1 = \bigoplus_{\mu \in \mathbb{H}^*} N_1 \cap M(\lambda)_\mu \quad (N_1)^\perp = 0 \Rightarrow N_1 \perp M(\lambda)$$

$$\text{similarly } (N_2)^\perp = 0 \Rightarrow (N_1 + N_2)^\perp = 0 \text{ so } N_1 \perp M(\lambda)$$

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Lemma: any tor. module in \mathcal{O} is $\cong V(\lambda)$

of: $v \in V_\mu$ $hv = \mu(h)v \rightarrow v^+ \in U(\mathfrak{n}^+)v \subset \mathcal{O}$
 - look back same proof

any quotient of $M \otimes \mathcal{O}$ is in \mathcal{O}

any sub of $M \otimes \mathcal{O}$ is in \mathcal{O} : cardm need to prove
 submodule b of $M \otimes \mathcal{O}$ is finitely generated
 - true because $U(\mathfrak{g})$ is noetherian

(so this fact is very important and it is lost
 when passing to the Kac-Moody algebra form)

Opic finite direct sums of objects in \mathcal{O}

abelian category

What about tensor products?

Ex: $g = sl_2$ prove that $M(\lambda) \otimes M(\mu)$ is
 not finitely generated

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however, a very important role is played by SL_2 .

Prop: $M \in \mathcal{O}$, $V(\lambda)$ is fin dim.

$\Rightarrow M \otimes V(\lambda) \in \mathcal{O}$

B: check that if m_1, \dots, m_r generate M and v_1, \dots, v_N basis of $V(\lambda)$ then

$m_i \otimes v_j$ generate $M \otimes V(\lambda)$

so as long as we have fin dim m_i 's & 0 m_i is good

now what our sketch shows

Thm $\lambda \in P^+ \Rightarrow V(\lambda)$ is ~~not~~ finite-dim.

$V(\lambda)$ fin.dim $\Rightarrow \lambda \in P^+$ ✓ already proved

One thing to make explicit now is that $M(\lambda)$

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is an indecomposable and often irreducible

module for \mathbb{Z} i.e. category \mathcal{O} is not semi-simple
and its homological properties are interesting

Ex: $\text{sl}_2 \otimes \mathbb{C}[t^\pm]$ Find a necessary and sufficient
condition for $M(\lambda)$ to be irreducible.

Find JH series of the irreducible module for sl_2

ques. does \mathcal{O} have the finite length
property? i.e. does every decreasing seq $N_2 \supseteq N_1 \supseteq \dots \supseteq N_1$ stop?

Thm: Yes any obj $M \in \mathcal{O}$ has a JH sum
and what $[M, V(\mu)] = \#$ of times $V(\mu)$ occurs

in a JH series for M

- say something about how this is probably
mult

Later ques: wh JH series of $M(\lambda)$

answering this gives led to many developments

-KL polynomials, key org theory, linkage principle
and will see some of these later

Suppose we wanted to know when $[M(\lambda), V(\mu)] \neq 0$

well, one way would be to see if

$\text{Hom}_S(M(\mu), M(\lambda)) \neq 0$ because then

of $\varphi: M(\mu) \rightarrow M(\lambda)$ φ is injective

so $M(\mu) \subseteq M(\lambda)$, $[M(\mu), V(\mu)] = 1$

$\Rightarrow [M(\lambda), V(\mu)] \neq 0$

So gives under what cond. on μ, λ is $\text{Hom}(M(\mu), M(\lambda)) \neq 0$
and then you could ask converse questions

$(M(\lambda), V(\mu)) \neq 0 \Rightarrow \text{Hom}_S(M(\mu), M(\lambda)) \neq 0$

~~Theorem (Kostant)~~ $\alpha \in \mathbb{P}^+ \setminus \lambda + \rho$

$\check{\alpha}$

(b)

first we see that $M(\lambda)$ or $\Rightarrow \mathbb{F}[\mu + h]^\infty$

with $\phi: M(\mu) \rightarrow M(\lambda)$

$M(\lambda)$ has a TH. $M(\lambda) \supseteq M, 2, 24, \dots$

$$M_\nu \cong V(\nu) \quad \mu \not\in \gamma^*$$

$$\phi: M(\mu) \rightarrow V(\mu) \rightarrow M(\lambda)$$

$$\phi \text{ injective} \Rightarrow M(\mu) \cong V(\mu)$$

Veoma: $M(\lambda)$ has a ? irreducible submodule

M_1, M_2 ior then $M_1 \cap M_2 = \emptyset$

$$M_1 \cong M(\mu_1), \quad M_2 \cong M(\mu_2)$$

as $V(\bar{\mu}) - \text{mod}$. ~~and also $b(\bar{\mu})$~~

$$\begin{array}{ccc} M(\lambda) & \cong & I_1 \quad I_2 \\ M(\mu) & \cong & I_1 \cap I_2 = \{0\} \\ & \cong & I_1 \neq I_2 \end{array}$$

contradiction

$$\forall I_2 \subsetneq I_1 \quad I_1 \subseteq I_1 \oplus I_2 \subseteq I_1 + I_2 = I_1$$

Next induction step $I_1 \cap I_2$

$$I_1 \cap I_2 \subseteq I_1 + I_2$$

\vdash
 $I_1 \cap I_2$ no zero divisors
 neither an

• $M(\lambda)$ has a simple quilt

and a ! simple subquilt.

Vera: $\dim \text{Hom}_{\mathbb{Z}}(M(\mu), M(\lambda)) \leq 1$

$\varphi_1: M(\mu) \rightarrow M(\lambda)$ injective

$\varphi_2: M(\mu) \rightarrow M(\lambda)$

let $V(\nu)$ be irr. sub $\varphi_1, \varphi_2: M(\mu) \rightarrow M(\lambda)$

$\varphi_1|_{V(\nu)}: V(\nu) \rightarrow V(\nu)$ s.t.
 $(\varphi_1 - \varphi_2)|_{V(\nu)} = 0$