

28 Feb. 2014

- Today.
- I. Reconstruction Thm
 - II. Goddard Uniqueness Thm
 - III Sugawara Construction
- III. Vertex Lie Algebras
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Last time we saw the Heisenberg vertex algebra:

$$H = \mathbb{C}[h_{-1}, h_{-2}, \dots]$$

$$\mathbb{1} = 1$$

$$T : H \rightarrow H \text{ with } T(\mathbb{1}) = 0, [T, h_n] = -nh_{n+1}.$$

$$Y : H \longrightarrow \text{End } H[[z^{\pm 1}]]$$

$$h_{-1} \longmapsto h(z) = \sum_{n \in \mathbb{Z}} h_n z^{-n-1}$$

$$\text{where } h_n \sim H \text{ by } \left\{ \begin{array}{l} \text{left mult by } h_n, n < 0 \\ \frac{\partial}{\partial h_{-n}}, n \geq 0. \end{array} \right.$$

$$Y(h_{j_1}, h_{j_2}, \dots, h_{j_k}, z) = : \partial_z^{(j_1-1)} h(z) \dots \partial_z^{(j_k-1)} h(z) :$$

We proved that ~~H~~ H is a vertex alg.

In fact, once $Y(h_{-1}, z)$ was defined, we had no choice in the defn of the vertex operators.

I. Reconstruction Thm If $(V, \mathbb{1}, T, Y)$ is a vertex algebra,

$$\text{then } Y(a_m b, z) = : (\partial_z^{(m-1)} Y(a, z)) Y(b, z) : \quad \forall a, b \in V \text{ and } m < 0.$$

Proof. Let $A(z), B(z)$ be two mutually local fields.

$$\text{Then } [A(z), B(w)] = \sum_{j=0}^{N-1} c^j(w) z^{-1} \partial_w^{(j)} \delta\left(\frac{w}{z}\right) \text{ for some } c^j(w).$$

$$\begin{aligned} \text{Thus } A(z)B(w) - : A(z)B(w) : &= (A(z)_+ B(w) + A(z)_- B(w)) - (A(z)_+ B(w) + B(w)A(z)_-) \\ &= [A(z)_-, B(w)] = [A(z), B(w)]_- = \sum_{j=0}^{N-1} c^j(w) z^{-1} \partial_w^{(j)} \sum_{k=0}^{\infty} \left(\frac{w}{z}\right)^k + B(w)A(z)_-. \end{aligned}$$

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~~But $\frac{1}{z-\omega}$~~

$$\begin{aligned} \text{But } \frac{1}{z-\omega} &= (z-\omega)^{-1} = \sum_{k=0}^{\infty} \binom{-1}{k} z^{-1-k} (\omega)^k \\ &= \sum_{k=0}^{\infty} (-1)^k z^{-1} \left(\frac{\omega}{z}\right)^k \\ &= z^{-1} \sum_{k=0}^{\infty} \left(\frac{\omega}{z}\right)^k \end{aligned}$$

Now ^{can} induction on j to show that

$$\frac{1}{(z-\omega)^{j+1}} = z^{-1} \partial_{\omega}^{(j)} \sum_{k=0}^{\infty} \left(\frac{\omega}{z}\right)^k$$

$$\begin{aligned} \text{Thus } A(z)B(\omega) &= :A(z)B(\omega): + \sum_{j=0}^{N-1} \frac{c_j(\omega)}{(z-\omega)^{j+1}} \\ &\quad \text{No pole at } \underset{z=\omega}{\uparrow} \qquad \qquad \qquad \text{singular at } z=\omega. \end{aligned}$$

Taylor expansion in $z-\omega$:

$$A(z) = \sum_{m=0}^{\infty} \partial_{\omega}^{(m)} A(\omega) (z-\omega)^m,$$

$$\text{so } :A(z)B(\omega): = : \left(\sum_{m=0}^{\infty} \partial_{\omega}^{(m)} A(\omega) (z-\omega)^m \right) B(\omega):$$

~~Ass~~ Associativity Property: $Y(a, z)Y(b, \omega) = Y(Y(a, z-\omega)b, \omega)$

$$= \sum_{n \in \mathbb{Z}} \frac{Y(a_n b, \omega)}{(z-\omega)^{n+1}}$$

(Equality in ~~End~~ $Y(a_n b, \omega) = \text{End}(Y(a, z-\omega))((z-\omega)b)$)

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$$\text{Thus } \sum_{n \in \mathbb{Z}} \frac{Y(a_n, b, \omega)}{(z-\omega)^{n+1}} = Y(a, z) Y(b, \omega) = \sum_{m=0}^{\infty} : \partial_{\omega}^{(m)} Y(a, \omega) \cdot Y(b, \omega) : (z-\omega)^m + \sum_{j=0}^{N-1} \frac{c_j(\omega)}{(z-\omega)^{j+1}}$$

Comparing coeffs of $(z-\omega)^{-m-1}$ ~~for $m < 0$~~ ,

we get $Y(a_m, b, \omega) = : \partial_{\omega}^{(-m-1)} Y(a, \omega) \cdot Y(b, \omega) : \text{ for } m < 0. \square$

II. Goddard Uniqueness Thm ~~Very good~~

III. Sugawara Construction

Insert page (III B) here.

Return to Heisenberg.

Must find conformal vector ω in $H_2 = \text{Span} \{ h_{-1}, h_{-1}, h_2 \}$.

So if ω exists, then $\omega = \lambda(h_{-1})^2 + \mu h_{-2}$ for some $\lambda, \mu \in \mathbb{C}$.

(This is natural since L_0 is a Hamiltonian, and Newtonian mechanics gives us $\frac{1}{2}mv^2 + ma$ for Hamiltonian.) Since $h_{-2} = T(h_{-1})$, it is like a

~~and V is like h_{-1} , and $m=-1$)~~

$$\begin{aligned} \text{Then } Y(\omega, z) &= \lambda T(h_{-1}^2, z) + \mu Y(h_{-2}, z) \\ &= \lambda : h(z) h(z) : + \mu \partial_z h(z) \end{aligned}$$

Direct calculation gives that the conf. vectors are ~~are~~

$$\omega = \frac{1}{2} h_{-1}^2 + \mu h_{-2} \quad \forall \mu \in \mathbb{C}, \text{ with central charge } c_{\omega} = 1 - 12\mu^2.$$

We take $\omega = \frac{1}{2} h_{-1}^2$. Thus H is a VOA!

28 Feb. (IV)

This is called the Sugawara Construction:

$$\begin{aligned} Y(\omega, z) &= \frac{1}{z} : h(z) h(z) : \\ &= \frac{1}{2} \sum_{m,n} : h_m h_n : z^{-n-2} \\ &= \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} : h_m h_{n-m} : z^{-n-2} \end{aligned}$$

and the $L_n = \omega_{(n+1)} = \frac{1}{2} \sum_{m \in \mathbb{Z}} : h_m h_{n-m} :$ satisfy the Virasoro relations:

$$[L_n, L_m] = (n - m) L_{n+m} + \frac{n^3 - n}{12} \delta_{n,-m} \text{id}$$

as ops on $H.$

The ~~construction~~ of Heisenberg vertex algebra is an example of a much more general construction of vertex algebras.

Def. A Lie alg \mathcal{L} is a vertex Lie alg terminology of Dong-Li-Mao
Kac studies slightly more general
formal distribution Lie algs

if it is spanned by $\{u(n) : n \in \mathbb{Z}, u \in U\} \cup \{c(-1) : c \in C\}$
for some index sets U and C s.t.

(1) $c(-1)$ is central in $\mathcal{L} \quad \forall c \in C$

(2) For all $u, v \in U \exists f^j(z) \in \mathcal{F}$ s.t. $[u(z), v(w)] = \sum_{j=0}^{\text{finite}} f^j(u) z^{-j} d_w^{(j)} \delta(z)$
where $u(z) = \sum_{n \in \mathbb{Z}} u(n) z^{-n-1}$

$$c(z) = c(-1) z^0$$

$$\mathcal{F} = \text{Span} \{ d_w^{(j)} u(z), c(z) : j \geq 0, u \in U, c \in C \}$$

$$\subseteq \mathcal{L}[[z, z^{-1}]]$$

28 Feb. (X)

Example 1. Heisenberg Lie alg

$\mathcal{M} = \text{Span}\{h(n), c(-1) : n \in \mathbb{Z}\}$, where $h(n) = h_n$
 $c(1) = \emptyset$

$$\mathcal{U} = \{h\}$$

$$C = \{c\}$$

$$[h(z), h(w)] = c(w) z^{-1} \partial_w \delta\left(\frac{w}{z}\right).$$

Example 2. Untwisted affine Lie algs

g f.d. simple Lie alg.

$$\hat{\mathfrak{g}} = (g \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}\alpha, \text{ bracket } [x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m} + (x|y)n \delta_{n, -m} \alpha$$

\mathcal{U} = any basis of g
 $C = \{c\}$

~~REMARKS~~

$$x(n) = x \otimes t^n$$

$$c(-1) = \emptyset$$

$$\begin{aligned} [x(z), y(w)] &= \sum_{n, m} [x(n), y(m)] z^{-n-1} w^{-m-1} \\ &= \sum_{n, m} [x, y](n+m) z^{-n-1} w^{-m-1} + (x|y)c(-1) \sum_n n z^{-n-1} w^{n-1} && \downarrow \text{Killing form} \\ &= \sum_{a, b} [x, y](b) z^{-a-1} w^{-b-1+a} + (x|y)c(-1) z^{-1} \sum_n n \left(\frac{w}{z}\right)^{n-1} \\ &= \sum_b [x, y](b) w^{-b-1} \sum_a z^{-a-1} w^{a-a} + (x|y)c(-1) z^{-1} \partial_w \delta\left(\frac{w}{z}\right) \\ &= [x, y](w) z^{-1} \delta\left(\frac{w}{z}\right) + (x|y)c(-1) z^{-1} \partial_w \delta\left(\frac{w}{z}\right) \end{aligned}$$

$a + b = n$
 $b = n+m$, \rightarrow
 $\therefore m = b - a$

28 Feb. ~~(V)~~ (VI)

Example 3. Virasoro Lie alg.

$$Vir = \text{Span} \{ L_n : n \in \mathbb{Z} \} \oplus \mathbb{C} c$$

$$[L_n, L_m] = (n-m)L_{n+m} + \delta_{n,-m} \frac{n^3 - n}{12} c$$

$$\mathcal{U} = \{ L \}, \quad L(n) = L_{n-1}$$

$$C = \{ c \}, \quad c(-1) = c$$

Can calculate

$$[L(z), L(w)] = \partial_w L(w) z^{-1} \delta\left(\frac{w}{z}\right) + 2L(w) z^{-1} \partial_w \delta\left(\frac{w}{z}\right) + \frac{1}{2} c(w) z^{-1} \partial_w^{(3)} \delta\left(\frac{w}{z}\right).$$

How to build vertex algebras out of vertex Lie alg.

\mathcal{L} vertex Lie alg.

$$\mathcal{L}_+ = \text{Span} \{ u(n) : u \in \mathcal{U}, n \geq 0 \}$$

$$\mathcal{L}_- = \text{Span} \{ u(n), c(-1) : u \in \mathcal{U}, c \in C, n < 0 \}$$

Results

$$\text{Left: } \mathcal{L}_+ \text{ and } \mathcal{L}_- \text{ are Lie subalgs}$$

$$\text{Right: } [u(n), v(m)] = \sum_{n,m} [u(n), v(m)] z^{-n} w^{-m-1}$$

$$\text{Future Exercise: } \mathcal{L}_+, \mathcal{L}_- \subseteq \mathcal{L} \text{ are Lie subalgs}$$

28 Feb. (III B)

Another powerful fact:

Goddard Uniqueness Thm \forall vertex alg

$A(z) \in \text{End } V[[z, z^{-1}]]$ a field that is mutually local with $\Upsilon(b, z) \quad \forall b \in V$.

If $\exists a \in V$ s.t. $A(z) \mathbb{I} = \Upsilon(a, z) \mathbb{I}$, then $A(z) = \Upsilon(a, z)$.

Pf. Let $b \in V$. Then for $N \gg 0$,

$$\begin{aligned}(z-w)^N A(z) \Upsilon(b, w) \mathbb{I} &= (z-w)^N \Upsilon(b, w) A(z) \mathbb{I} \\ &= (z-w)^N \Upsilon(b, w) \Upsilon(a, z) \mathbb{I} \\ &= (z-w)^N \Upsilon(a, z) \Upsilon(b, w) \mathbb{I}\end{aligned}$$

Now use vacuum axiom to evaluate both sides at $w=0$.

Then $z^N A(z) b = z^N \Upsilon(a, z) b$, so $A(z) b = \Upsilon(a, z) b \quad \forall b \in V$
and $A(z) = \Upsilon(a, z)$.

□