

26 Feb. 2014 (o)

Today: Heisenberg VOA

The following defines the Heisenberg VOA:

(space of states)  $H = \mathbb{C}[h_{-1}, h_{-2}, \dots]$   
 $= \bigoplus_{n=0}^{\infty} H_n$ , with  $\deg(h_{-k}) = k$ .

(vacuum)  $|1\rangle = 1$

(translation)  $T : H \rightarrow H$  s.t.  $T(A) = 0$ ,  $[T, h_n] = -nh_{n-1}$

Define  $h(z) = \sum_{n \in \mathbb{Z}} h_n z^{-n-1} \in (\text{End } V)[[z, z^{-1}]]$ ,

where  $h_n \rightsquigarrow H$  by  $\left\{ \begin{array}{ll} \text{left mult by } h_n, & n < 0 \\ n \frac{\partial}{\partial z} & , n \geq 0 \end{array} \right.$

(vertex operators)  $\gamma(h_{j_1}, h_{j_2}, \dots, h_{j_k}, z) = : \partial_z^{(j_1-1)} h(z) \dots \partial_z^{(j_k-1)} h(z) :$

(conformal vector)  $\omega = (h_{-1})^2$ .

↑ normally ordered product  
 (to explain later)

Goals today: (1) Verify this is a VOA.

(2) Explain why these defns are all natural.

(3) Develop general theory along the way:

normal ordering,

Dong's Lm,

Reconstruction Thm

Sugawara Construction.

Recall Heisenberg Lie alg:  $\mathfrak{H} = \text{Span}_{\mathbb{C}} \{h_n : n \in \mathbb{Z}\} \oplus \mathbb{C}\mathbf{c}$

$$[h_n, h_m] = n \delta_{m, -n} \mathbf{c}$$

$$[\mathbf{c}, \mathfrak{H}] = 0$$

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$$\begin{aligned} \mathcal{H}_+ &= \text{Span } \{ h_n : n \geq 0 \} \\ \mathcal{H}_- &= \text{Span } \{ h_n : n < 0 \} \oplus \mathbb{C}\mathbf{1} \end{aligned} \quad \left. \begin{array}{c} \} \\ \} \end{array} \right\} \text{ abelian subalgs!}$$

$$H = U(\mathcal{H}) \otimes_{U(\mathcal{H}_+ \oplus \mathbb{C}\mathbf{1})} \mathbb{C}\mathbf{1}, \text{ where } \mathcal{H}_+ \cdot \mathbf{1} = 0 \\ \mathbf{1} \cdot \mathbf{1} = \mathbf{1}.$$

By the usual PBW-type arguments,

$H$  has a basis of monomials  $\underbrace{\{h_{j_1} h_{j_2} \dots h_{j_k} : j_1 \leq j_2 \leq \dots \leq j_k < 0, k \geq 0\}}$

$$\deg = - \sum_{i=1}^k j_i$$

(Identify  $h_{j_1} \dots h_{j_k} \longleftrightarrow h_{j_1} \dots h_{j_k} \otimes \mathbf{1}$ .)

$$\Rightarrow H = \bigoplus_{n=0}^{\infty} H_n, \text{ with } \dim H_n = p(n) = \# \text{ of partitions of } n$$

Note  $h_n \sim H$  by  $\begin{cases} \text{left mult by } h_n, & n < 0 \\ n \frac{d}{dh_{-n}}, & n \geq 0 \end{cases}$

What is VOA structure on  $H$ ?

vacuum  $\mathbf{1} = \text{empty product}$

translation  $T: V \rightarrow V$  defined by  $T(\mathbf{1}) = 0$

$$[T, h_n] = -nh_{n-1}$$

(Motivation:  $T \longleftrightarrow -\partial_t$ )  
 $h_n \longleftrightarrow t^n$ )

This defines  $T$  by induction.

$$\begin{aligned} T(h_{j_1} h_{j_2} \dots h_{j_k}) &= h_{j_1} T(h_{j_2} \dots h_{j_k}) + [T, h_{j_1}] h_{j_2} \dots h_{j_k} \\ &= h_{j_1} T(h_{j_2} \dots h_{j_k}) - j_1 h_{j_1-1} h_{j_2} \dots h_{j_k}. \end{aligned}$$

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How to get vertex operators?

Idea: Use generating field  $h(z) = \sum_{n \in \mathbb{Z}} h_n z^{-n-1} \in \text{End } H[[z^{\pm 1}]]$  action by  $h_n$  on  $H$

Then  $h_n : V_m \rightarrow V_{m-n}$ , so if  $Y(x, z) = h(z)$ ,

we'd get  $x_{(n)} = h_n : V_m \rightarrow V_{m+\deg(x)-n-1} = V_{m-n}$   
 $\Rightarrow \deg(x) = 1$  and  $x \sim h_{-1}$ .

Let  $Y(h_{-1}, z) = h(z)$ .

By a future exercise!,  $Y(T_+, z) = \partial_z Y(+, z)$ ,  
 so  $Y(h_{-2}, z) = Y(T h_{-1}, z)$   
 $= \partial_z Y(h_{-1}, z) = \partial_z h(z)$ .

Since  $T^k h_{-1} = k! h_{-1-k}$ , we have

$$\begin{aligned} Y(h_{-k}, z) &= \frac{1}{(k-1)!} Y(T^{k-1} h_{-1}, z) \\ &= \partial_z^{(k-1)} h(z). \end{aligned}$$

How to define  $Y$  on products?

$$Y(h_{-1}^2, z) = h(z)h(z) ?$$

$$\text{But } h(z)h(z)_v = \left( \sum_n h_n z^{-n-1} \sum_m h_m z^{-m-1} \right)_v$$

$$= \sum_{m,n} h_n h_m v z^{-n-m-2}$$

$$= \sum_{a \in \mathbb{Z}} \left[ \sum_{m,n} h_{a-m-1} h_m v z^{-a-1} \right]$$

$h_n$  acts as  $\frac{\partial}{\partial h_n}$  for  $n > 0$  and  $\{h_{a-m-1}, h_m\} = c$  for  $a \neq 1$   $\Rightarrow$  these sums

$-n-m-2 = -a-1 \Rightarrow n = a-m-1$   $\Rightarrow$  fin when  $a \neq 1$

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But when  $a = 1$ , we get

Fix problem by "subtracting divergences" and ~~making~~  $h_m$  act before  $h_m$  when  $m < 0$

Let  $\hat{h}_k \hat{h}_\ell := \begin{cases} h_k h_\ell & \text{if } k = -\ell \geq 0 \\ h_k h_\ell & \text{otherwise.} \end{cases}$

$$\text{Let } :h(z)h(\bar{z}): = \sum_{n \in \mathbb{Z}} \left( \sum_{k+\ell=n} :h_k h_\ell:\right) z^{-n-2}$$

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More generally, given two fields,

$$A(z) = \sum_{n \in \mathbb{Z}} A_{(n)} z^{-n-1}$$

$$B(w) = \sum_{m \in \mathbb{Z}} B_{(m)} w^{-m-1},$$

Let  $A(z)_+ = \sum_{n < 0} A_{(n)} z^{-n-1}$

$\uparrow$  non-negative powers of  $z$

$$A(z)_- = \sum_{n \geq 0} A_{(n)} z^{-n-1}$$

$\uparrow$  negative powers of  $z$ .

Then define :  $A(z)B(w) := A(z)_+ B(w) + B(w) A(z)_-$ .

(1) Note :  $A(z)B(w) = \int_{w=z} \dots$  is a well-defined field.

(2)  $:A(w)B(w): = \text{Res}_z \left[ \left( z^{-1} \delta\left(\frac{w}{z}\right) \right)_- A(z)B(w) + \left( z^{-1} \delta\left(\frac{w}{z}\right) \right)_+ B(w) A(z) \right]$

(Exercise 1 in Set 2)

(3) Normally ordered product is neither associative nor commutative, in general.

Let :  $A(z)B(w)C(\zeta) := :A(z) \left( :B(w)C(\zeta): \right)$ :

Now define vertex operator for  $H$  to be

$$Y(h_{j_1}, h_{j_2}, \dots, h_{j_k}, z) = : \partial_z^{(-j_1-1)} h(z) \dots \partial_z^{(-j_k-1)} h(z) :$$

Easy to check all axioms but locality.

## Locality

$$[h(z), h(w)] = \sum_{n,m} [h_n, h_m] z^{-n-1} w^{-m-1}$$

$$= \sum_{n \in \mathbb{Z}} n \not\in z^{-n-1} w^{n-1}$$

$$= \not\in w^0 z^{-1} \sum_{n \in \mathbb{Z}} n \left( \frac{w^{n-1}}{z^n} \right)$$

$$= \not\in w^0 z^{-1} \partial_w \delta\left(\frac{w}{z}\right), \text{ so } (z-w)^2 [h(z), h(w)] = 0.$$

$$[h(z), \partial_w^{(k)} h(w)] = \cancel{\partial_z^{(k)} h(z)} \partial_w^{(k)} [h(z), h(w)]$$

$$= \frac{1}{k+1} \not\in w^0 z^{-1} \partial_w^{(k+1)} \delta\left(\frac{w}{z}\right),$$

so  $h(z)$  and  $\partial_z^{(k)} h(z)$  are mutually local.

Similar arguments show that  $\partial_z^{(k)} h(z)$  and  $\partial_z^{(l)} h(z)$  are mutually local  $\forall k, l \geq 0$ .

Dong's Lm If  $A(z), B(z), C(z)$  are mutually local,  
then  $:A(z)B(z):$  and  $C(z)$  are mutually local.

Pf.  $\exists r > 0$  s.t.

$$(w-z)^n A(z) B(w) = (w-z)^n B(w) A(z)$$

$$(u-z)^n A(z) C(u) = (u-z)^n C(u) A(z)$$

$$(u-w)^n B(w) C(u) = (u-w)^n C(u) B(w)$$

We want  $(w-u)^N :A(w)B(w):C(u) = (w-u)^N C(u) :A(u)B(u):$  for some  $N$ .

Take  $N = 3r$ .

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Then  $(\omega - u)^{3r} \rightarrow (\omega - u)^r \sum_{s=0}^{2r} \binom{2r}{s} (\omega - z)^s (z - u)^{2r-s}$

Now use future exercise 2:

$$A(\omega)B(\omega) = \text{Res}_z \left[ \left( z^{-1} \delta\left(\frac{\omega}{z}\right) \right)_- A(z)B(\omega) + \left( z^{-1} \delta\left(\frac{\omega}{z}\right) \right)_+ B(\omega)A(z) \right],$$

so we consider

+

$$P(\omega, z, u) = (\omega - u)^{3r} (*)$$

The terms in  $(\omega - u)^{3r} = (\omega - u)^r \sum_{s=0}^{2r} \binom{2r}{s} (\omega - z)^s (z - u)^{2r-s}$

with  $r < s \leq 2r$  in  $P(\omega, z, u)$  vanish:

~~Since~~  $(\omega - z)^s B(\omega) A(z) = (\omega - z)^r A(z) B(\omega)$ ,

~~so~~ the terms will be of the form

$$\binom{2r}{s} (\omega - u)^r (z - u)^{2r-s} (\omega - z)^{s-r} z^{-1} \delta\left(\frac{\omega}{z}\right) \cancel{(\omega - z)^r A(z) B(\omega)} = 0$$

Thus  $P(\omega, z, u) C(u) = (\omega - u)^r \sum_{s=0}^r \binom{2r}{s} (\omega - z)^s (z - u)^{2r-s} \stackrel{(*)}{=} 0$  since  $s-r \geq 1$

$$= (\omega - u)^r \sum_{s=0}^r \binom{2r}{s} (\omega - z)^s (z - u)^{2r-s} C(u) (*)$$

Since we have factors of  $(\omega - u)^r$  and  $(z - u)^r$ ,

$$= C(u) P(\omega, z, u), \text{ so}$$

$$(\omega - u)^{3r} : A(\omega)B(\omega) : C(u) = \text{Res}_{z=u} P(\omega, z, u) = \text{Res} C(u) P(\omega, z, u)$$

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$$= (\omega - u)^{3r} ((u) : A(\omega) B(\omega) :)$$

□

Conclusion:  $Y(a, z)$  and  $Y(b, z)$  are mutually local  $\forall a, b \in H$ , and  $H$  is a vertex algebra.

In fact, once  $Y(h_{-}, z)$  was defined, we had no choice.

~~Defn~~

Reconstruction Thm

If  $(V, \mathbb{I}, T, Y)$  is a vertex algebra, then

$$Y(a_m b, z) = : (\partial_z^{(m)}) Y(a, z) Y(b, z) :$$

$\forall a, b \in V$  and  $m < 0$ .

Proof. Use Associativity Property:

$$\begin{aligned} Y(a, z) Y(b, w) &= Y(Y(a, z-w)b, w) \\ &= \sum_{n \in \mathbb{Z}} \frac{Y(a_n b, w)}{(z-w)^{n+1}}. \end{aligned}$$

Note that if

$$[A(z), B(w)] = \sum_{j=0}^{N-1} C^j(w) \partial_w^{(j)} B(z),$$

then

$$A(z) B(w) - : A(z) B(w) :$$

$$\binom{-1}{k} = \frac{(-1)(-1-1)\cdots(-1-k+1)}{k!}$$

$$= \frac{k! (-1)^k}{k!} = (-1)^k$$

$$= (A(z)_+ B(w) + A(z)_- B(w)) - (A(z)_+ B(w)$$

$$+ B(w) A(z)_-)$$

$$= [A(z)_-, B(w)] = [A(z), B(w)]_-$$

$$= \sum_{j=0}^{N-1} C^j(w) z^{-1} \partial_w^{(j)} \sum_{k=0}^{\infty} \left(\frac{w}{z}\right)^k$$

it can be shown by induction  
that this is

$$\begin{aligned} (z-w)^{j+1} &= \sum_{\ell=0}^{j+1} \binom{j+1}{\ell} z^{j+1-\ell} (-w)^{\ell} = \sum_{\ell=0}^{j+1} \binom{j+1}{\ell} z^{j+1-\ell} \left(\frac{w}{z}\right)^{\ell} = \frac{(j+1)!}{\ell!(j+1-\ell)!} z^{j+1-\ell} \left(\frac{w}{z}\right)^{\ell} = \frac{(j+1)!}{\ell!(j+1-\ell)!} z^{j+1-\ell} w^{\ell} \end{aligned}$$