

25 Feb. 2014

Course schedule:

This week: Tu 11-12:30, 3-4:30
Th 11-12:30, 3-4:30 (Exercises)
F 1-2:30

Office:

- Plan.
- I. Motivation
 - II. Calculus of formal distributions
 - III. Defn. of VOA
 - IV. First example

Reference books:

- V. Kac (Vertex Algs for Beginners)
J. Lepowsky & H. Li
E. Freudenthal & D. Ben-Zvi
I. Frenkel, J. Lepowsky, & A. Meurman

- Also:
- M. Wakimoto (Lectures on Infin.-Dim Lie Algs)
 - E. Freudenthal (Langlands for Loop Gps)
 - T. Gannon (Moonshine & the Monster)
 - T. Kohno (CFT & Topology)
 - P. DiFrancesco, P. Mathieu, & David Sénéchal (CFT)

I. Motivation

- particle physics
- reps of affine Lie algs
- modular forms

Particle physics

classical mechanics: minimize $\int \text{Lagrangian} \rightarrow \text{eqns. of motion}$

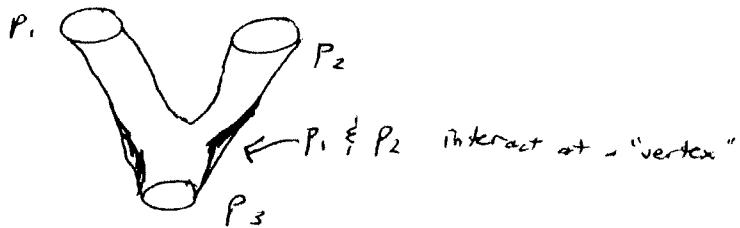
(position & momentum at time t)

quantum mechanics: particles replaced by prob density functions
state = prob density $\Psi(x, t)$ that particle is at pt x at time t
 V = phase space (vector space of all possible ψ 's)

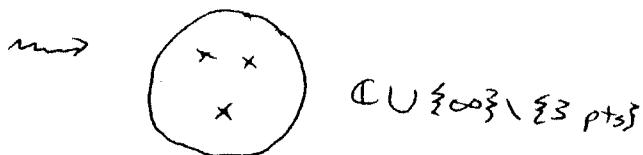
CFT rec. (iii)

- quantum field theory:
- emphasize fields to ~~allow~~ # of particles to change
 - include special relativity (action of $\mathbb{R}^4 \rtimes SO_{\text{dil}}$)
 - quantize classical versions of field theory

- (closed) string theory:
- particles are 1-dim strings, sweeping out worldsheets
 - dissect worldsheets into compositions of pairs-of-points.



CFT = identify conformally equivalent worldsheets



vertex operators are fields $Y(p, z) = \sum_{n \in \mathbb{Z}} p_n z^{-n-1}$ GEN $V[[z^{\pm 1}]]$ assoc. to each state p
 that let us calculate "scattering amplitudes":

$$\langle Y(p_3, z) | p_1, p_2 \rangle = \text{prob. density of } p_3 \text{ at } z \in \mathbb{C} \cup \{0\}.$$

Wightman axioms of 2-d QFT + conformal invariance

\rightsquigarrow defn of chiral algebras

\rightsquigarrow VOAs

25 Feb. (III)

Reps of affine Lie algs

Heisenberg Lie alg: $\mathcal{H} = \text{Span}\{h_{2n+1} : n \in \mathbb{Z}\} \oplus \mathbb{C}\alpha$

$$[h_n, h_m] = n\delta_{m+n, 0}\alpha$$

$$[\alpha, \mathcal{H}] = 0$$

Faithful irrep: $\mathcal{H} \curvearrowright \mathcal{F} = \mathbb{C}[x_1, x_3, x_5, \dots]$ bosonic Fock space

$$h_n \mapsto \begin{cases} \text{left mult by } x_{-n}, & n < 0 \\ \frac{\partial}{\partial x_n}, & n > 0 \end{cases}$$

$$\alpha \mapsto \text{id}$$

Lepowsky-Wilson 1978:

$\hat{\mathfrak{g}} = (\mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}\alpha$ has several Heisenberg subalgs,
e.g. $\tilde{\mathcal{S}} = \text{Span}\{B_{2n+1} : n \in \mathbb{Z}\} \oplus \mathbb{C}\alpha$,
 $B_{2n+1} := e \otimes t^n + f \otimes t^{n+1}$

$$[B_n, B_m] = n\delta_{m+n, 0}\alpha$$

• Fundamental module $L(\omega_0)$ ($h \in \text{wt } \hat{\mathfrak{g}} - \text{irrep } \mathcal{U}(\hat{\mathfrak{g}}) v_+$, where
 $(h \otimes 1)v_+ = 0$
stays irred on restriction to $\tilde{\mathcal{S}}$. $\alpha \cdot v_+ = v_+$)

Uniqueness of $\mathcal{F} \Rightarrow \text{Res}_{\tilde{\mathcal{S}}}^{\hat{\mathfrak{g}}} L(\omega_0) \cong_{\mathcal{S}' = \mathcal{S}' - \text{iso}} \mathcal{F}$.

• Use this iso to transfer $\hat{\mathfrak{g}}$ -action on $L(\omega_0)$ to $\hat{\mathfrak{g}}$ -action on \mathcal{F} .
Let's calculate this action!

Generating field: $X(z) := \sum_{n \in \mathbb{Z}} x_n z^{-n-1}$,

$$x_{2n+1} := -e \otimes t^n + f \otimes t^{n+1}$$

$$x_{2n} := h \otimes t^n - \frac{1}{2} \delta_{n,0} \alpha$$

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$$\text{Then } [B_n, X_m] = 2X_{m+n}$$

$$[B_n, X(z)] = 2z^n X(z).$$

If $X(z) \curvearrowright F$ via $\Gamma(z)$, some formal series w/ diff'ls ops as coeffs,
then for $n > 0$ odd,

$$\left[n \frac{\partial}{\partial x_n}, \Gamma(z) \right] = 2z^n \Gamma(z)$$

$$\left[x_n, \Gamma(z) \right] = 2z^{-n} \Gamma(z) \quad \Rightarrow \Gamma(z) \text{ looks like}$$

$$\prod_{\substack{n>0 \\ \text{odd}}} \exp\left(\frac{2x_n}{n} z^n\right) \prod_{\substack{n>0 \\ \text{odd}}} \exp\left(-2 \frac{\partial}{\partial x_n} z^{-n}\right), \quad \text{a } \underline{\text{(twisted) vertex operator!}}$$

Frenkel-Kac 1980: can do this with actual vertex operators.

There is a VOA hidden in the background:

$$\left\{ \begin{array}{l} \text{smooth reps of} \\ \text{untwisted affine Lie algs} \end{array} \right\} \xleftrightarrow{b_{ij}} \left\{ \text{reps of affine VOAs} \right\} !$$

Modular forms

$$\mathbb{H} = \{ \tau \in \mathbb{C} : \operatorname{im} \tau > 0 \}$$

$$\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$$

$$SL_2(\mathbb{C}) \curvearrowright \mathbb{H} \quad \text{via} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}.$$

$$\sum := SL_2(\mathbb{C}) \backslash \overline{\mathbb{H}} \quad \text{orbit space is a sphere}$$

$$j: \overline{\mathbb{H}} \longrightarrow \mathbb{P}^1(\mathbb{C}) \quad \text{lift of identification } \sum \rightarrow \mathbb{P}^1(\mathbb{C}).$$

Modular functions: mero fns $f: \overline{\mathbb{H}} \rightarrow \mathbb{C}$ s.t. $f(x\tau) = f(\tau) \quad \forall x \in SL_2(\mathbb{C})$
(i.e. mero fns on \sum).

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Only mero fns on \mathbb{H}^1 are rational fns $\Rightarrow f(\tau) = \frac{\text{Poly in } j(\tau)}{\text{Poly in } j(\tau)}$

Modular j -function: $j(\tau) = q^{-1} + 744 + 196,884q + 21,493,760q^2 + 864,299,970q^3 + \dots$
 $q := e^{2\pi i \tau}$
 t arbitrary const, otherwise j is unique

M : Monster simple gp

Moonshine [McKay 1978]

$$\begin{aligned} 196,884 &= 1 + 196,883 \\ 21,493,760 &= 1 + 196,883 + 21,296,876 \\ 864,299,970 &= 2 \cdot 1 + 2 \cdot 196,883 + 21,296,876 + 842,609,326 \end{aligned}$$

dimensions of smallest
irreps of $M!$

Explanation [Borcherds 1986]

M is the auto gp of a new structure, a VOA,

$$\mathbb{Z}\text{-graded v.s. } V^{\mathbb{N}} = V_0 \oplus V_1 \oplus V_2 \oplus \dots$$

$$\text{s.t. } \sum_{n \in \mathbb{Z}} (\dim V_n) t^n = j(\tau) - 744$$

Modularity in characters of affine Lie algs too...

Reason [Zhu 1999] VOAs! (SL_2 -action on space of conf. blocks)

Rough defn. A vertex operator algebra consists of the following data:

V vect space (space of states)

$|1\rangle \in V$ (vacuum)

$T: V \rightarrow V$ (translation operator)

$Y: V \rightarrow \text{End } V [[z, z^{-1}]]$ (vertex operators)
 $a \longmapsto Y(a, z)$

$w \in V$ (conformal vector)

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Satisfying some axioms:

(vacuum) $Y(\mathbb{I}, z) = (d)z^0$ and $Y(a, z) \mathbb{I} /_{z=0} = a$

(translation) $[T, Y(a, z)] = \partial_z Y(a, z)$ and $T(\mathbb{I}) = 0$

(locality) $(z-w)^N [Y(a, z), Y(b, w)] = 0$ for $N \gg 0$

(conformal) $Y(a, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$, where coeffs satisfy Virasoro relations

II. Calculus of formal distributions

Notation U vect space / \mathbb{C}

$U[z]$ polynomials w/ coeffs in U

$U[z, z^{-1}]$ Laurent polyn. " " " "

$U[[z]]$ formal series " " " " (no negative powers)

$U((z))$ Laurent " " " " (only fin. many neg. powers)

$U[[z, z^{-1}]]$ formal distributions w/ coeffs in U (no restrictions on powers)

} We accept multiplication
of z occurring } only for products
involving only finite
sums in coeffs.

And in several variables,

$U[z, w]$

$U\{[z, z^{-1}, w, w^{-1}, \dots]\}$, etc.

For $\varphi(z) = \sum_{n \in \mathbb{Z}} u_n z^n \in U\{[z, z^{-1}]\}$, let $\text{Res}_z \varphi(z) = u_{-1}$.

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Rem. (1) Formal distributions are distributions for the space $\mathbb{C}[[z, z^{-1}]]$ of "test functions":

$$\begin{array}{ccc} \mathcal{U}[[z, z^{-1}]] & \times & \mathbb{C}[[z, z^{-1}]] \\ (f, \varphi) & \longmapsto & \langle f, \varphi \rangle := \text{Res}_z f(z) \varphi(z). \end{array}$$

(2) Integration by parts (Exercise 1):

Let $\partial_z : \mathcal{U}[[z, z^{-1}]] \longrightarrow \mathcal{U}[[z, z^{-1}]]$

$$\sum u_n z^n \longmapsto \sum n u_n z^{n-1}$$

If $a, b \in \mathcal{U}[[z, z^{-1}]]$ s.t. a, b is well-defined,
then $\text{Res}_z (a(\partial_z b)) = -\text{Res}_z (b(\partial_z a))$.

Example δ -function:

$$\delta(z) := \sum_{n \in \mathbb{Z}} z^n \in \mathbb{C}[[z, z^{-1}]]$$

Then for $\varphi(z) = \sum_m u_m z^m \in \mathcal{U}[[z, z^{-1}]]$, we have

$$\begin{aligned} \langle \delta, \varphi \rangle &= \text{Res}_z (\delta(z) \varphi(z)) \\ &= \text{Res}_z \left(\sum_{n \in \mathbb{Z}} z^n \sum_m u_m z^m \right) \\ &= \sum_m u_m = \varphi(1). \end{aligned}$$

Rem. (1) There is a non-uniqu. additive notation (e.g. in Kac's book):

$$\delta(z-w) = z^{-1} \sum_{n \in \mathbb{Z}} \left(\frac{w}{z}\right)^n \Rightarrow \text{Res}_z \delta(z-w) \varphi(z) = \varphi(w).$$

In our notation, $\delta(z-w)$ is not only a function of w , it is not defined when $w=0$, for instance)

$$\text{Res}_z (z^{-1} \delta(z) \varphi(z)) = \varphi(\infty).$$

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(2) $\delta\left(\frac{\omega}{z}\right)$ can be multiplied by any elt of $U[[z^{\pm 1}], \omega^{\pm 1}]]$ or $U[[\omega^{\pm 1}]]$.
 (but usually not both)

Some sample calculations:

$$\delta\left(\frac{\omega}{z}\right) = \sum_{n \in \mathbb{Z}} \omega^n z^{-n} = \sum_{n \in \mathbb{Z}} \omega^{-n} z^n = \delta\left(\frac{z}{\omega}\right),$$

$$\begin{aligned} z^a \omega^b \delta\left(\frac{\omega}{z}\right) &= \sum_n \omega^{n+b} z^{-n+a} \\ &= \sum_n z^{a+b} \omega^{n+b} z^{-n-b} \\ &= z^{a+b} \delta\left(\frac{\omega}{z}\right). \end{aligned}$$

In particular, $(z - \omega) z^{-1} \delta\left(\frac{\omega}{z}\right) = 0$.

Prop. Let $f(z, \omega) \in U[[z^{\pm 1}, \omega^{\pm 1}]]$. Then

$(z - \omega)^N f(z, \omega) = 0$ for some $N > 0 \iff f(z, \omega)$ can be written in the form

$$\sum_{j=0}^{N-1} c^j(\omega) z^{-1} \partial_\omega^{(j)} \delta\left(\frac{\omega}{z}\right), \quad \text{for some}$$

$$c^j(\omega) \in U[[\omega^{\pm 1}]], \text{ where } \partial_\omega^{(j)} = \frac{1}{j!} \frac{d^j}{d\omega^j}.$$

Moreover, the choice of $c^j(\omega)$ is unique. Explicitly, $c^j(\omega) = \operatorname{Res}_z f(z, \omega) (z - \omega)^j$.

Pf. (\Leftarrow) By Exercise 3, $(z - \omega) \partial_\omega^{(j)} \delta\left(\frac{\omega}{z}\right) = \partial_\omega^{(j-1)} \delta\left(\frac{\omega}{z}\right)$, and since

$(z - \omega)^N (z - \omega)^{-1} \delta\left(\frac{\omega}{z}\right) = 0$, we have

$$(z - \omega)^N \sum_{j=0}^{N-1} c^j(\omega) z^{-1} \partial_\omega^{(j)} \delta\left(\frac{\omega}{z}\right) = 0.$$

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Pf (\Rightarrow). Suppose $(z-w)^N f(z, w) = 0$,

$$\text{Write } f(z, w) = \sum_{n,m} f_{n,m} z^n w^m.$$

$$\text{Then } (z-w) f(z, w) = \sum_{n,m} f_{n,m} (z^{n+1} w^m - z^n w^{m+1})$$

$$= \sum_{n,m} (f_{n-1,m} - f_{n,m-1}) z^n w^m.$$

$$\text{Thus } (z-w) f(z, w) = 0 \Leftrightarrow f_{n-1,m} = f_{n,m-1} \quad \forall n, m$$

$$\Leftrightarrow f_{n,m} = f_{n-1,m+1}$$

$$\Leftrightarrow f_{n,m} = f_{0,m+n} \quad \forall n, m.$$

$$\Leftrightarrow f(z, w) = \sum_n f_{0,n} \underbrace{w^n \delta\left(\frac{w}{z}\right)}_{\text{all the monomials with total degree } n} = \sum_n f_{0,n} w^{n+1} z^{-1} \delta\left(\frac{w}{z}\right) \quad (*)$$

Now induct on N . Done for $N=1$. In general, we have

$$0 = (z-w)^N f(z, w) = (z-w)^{N-1} (z-w) f(z, w), \quad \text{so by induction hyp.,}$$

$$(z-w) f(z, w) = \sum_{j=0}^{N-2} b^j(w) z^{-1} \partial_w^{(j)} \delta\left(\frac{w}{z}\right) \quad \text{for some } b^j(w)$$

$$= (z-w) \sum_{j=0}^{N-2} b^j(w) z^{-1} \partial_w^{(j+1)} \delta\left(\frac{w}{z}\right),$$

Exercise 2

$$\text{so } (z-w) \left[f(z, w) - \sum_{j=0}^{N-2} b^j(w) z^{-1} \partial_w^{(j+1)} \delta\left(\frac{w}{z}\right) \right] = 0, \quad \text{so by } (*),$$

$$f(z, w) - \sum_{j=0}^{N-2} b^j(w) z^{-1} \partial_w^{(j+1)} \delta\left(\frac{w}{z}\right) = \cancel{C(w) z^{-1} \delta\left(\frac{w}{z}\right)}$$

$$\text{so } f(z, w) = C(w) z^{-1} \delta\left(\frac{w}{z}\right) + \sum_{j=0}^{N-2} b^j(w) z^{-1} \partial_w^{(j+1)} \delta\left(\frac{w}{z}\right)$$

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Finally, if $f(z, w) = \sum_{j=0}^{N-1} c^j(w) z^{-j} \partial_w^{(j)} \delta\left(\frac{w}{z}\right)$, then

$$\begin{aligned} \operatorname{Res}_z f(z, w) (z-w)^l &= \operatorname{Res}_z (z-w)^l \sum_{j=0}^{N-1} c^j(w) z^{-j} \partial_w^{(j)} \delta\left(\frac{w}{z}\right) \\ &= \sum_{j=l}^{N-1} c^j(w) z^{-j} \underbrace{\partial_w^{(j-l)} \delta\left(\frac{w}{z}\right)}_{\partial_w^{(j-l)} = 0 \text{ if } j-l < 0} \end{aligned}$$

Exercise 3

But $\underbrace{\partial_w^{(j-l)} \sum w^n z^{-n}}_{\text{coeff of } z^0 \text{ is }} \stackrel{\star}{=} \partial_w^{(j-l)} w^0 = \begin{cases} 0 & \text{if } j \neq l \\ 1 & \text{if } j = l \end{cases}$

Thus $\operatorname{Res}_z f(z, w) (z-w)^l = c^l(w)$. \square

Recall locality axiom:

$$(z-w)^N [Y(a, z), Y(b, w)] = 0 \quad \text{for } N \gg 0.$$

This means

$$[Y(a, z), Y(b, w)] = \sum_{j=0}^{N-1} c^j(w) z^{-j} \partial_w^{(j)} \delta\left(\frac{w}{z}\right),$$

where $c^j(w) = \operatorname{Res}_z [Y(a, z), Y(b, w)] (z-w)^j$

$$= \operatorname{Res}_z \left[\sum_n a_{n,j} z^{-n-1}, \sum_m b_{m,j} w^{-m-1} \right] \sum_{\ell=0}^j \binom{j}{\ell} z^\ell (-w)^{j-\ell}$$

~~$\sum_{n,m} a_{n,j} b_{m,j} \int dz z^{-n-1} (-z)^{\ell} z^{-m-1} (-z)^{j-\ell}$~~

from which one can calculate ~~explictly~~ $c^j(w)$

The $c^j(w)$ are called operator product expansion (OPE) coefficients. explicitly.
We'll come back to them later.

25 fév. (XII)

III. Defn of VOA

Defns still vary, ~~but~~, but are becoming more standard.

Virasoro Lie algebra: Univ. central ext ~~of~~ $V_{\text{irr}} = \text{Span} \{ L_n : n \in \mathbb{Z} \} \oplus \mathbb{C} \epsilon$
of Witt alg $\{ \text{polyn. vect. fields on } \mathbb{C}^x \}$, where

$$L_n = -t^{n+1} \frac{d}{dt} : \mathbb{C}[t, t^{-1}] \rightarrow \mathbb{C}[t, t^{-1}].$$

$$\text{Explicitly, } [L_n, L_m] = (n-m)L_{n+m} + \frac{n^3 - n}{12} \delta_{n+m, 0} \epsilon \\ [\epsilon, V_i] = 0,$$

A vertex algebra consists of the following data:

(space of states) $V = \bigoplus_{n \in \mathbb{Z}} V_n$ \mathbb{Z} -graded v.s./ \mathbb{C} with $\dim V < \infty$ V_n

(vacuum vector) $|1\rangle \in V_0$

(translation operator) $T: V \rightarrow V$ linear op.

(vertex operator) $Y: V \rightarrow \text{End } V[[z^\pm]]$ l.n. op.

$$a \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1},$$

where $a_{(n)}: V_k \rightarrow V_{k+m-n-1}$, $a \in V_m$ $\left(\begin{array}{l} \text{if } Y(a, z) \\ \text{is of conformal} \\ \text{dimension } m \end{array} \right)$

and ~~such that we have~~ $a_{(n)} b = 0$ for $n \gg 0$
($Y(a, z)$ is a field)

25 fév. (XII)

Satisfying 3 axioms:

$$(\text{vacuum}) \quad Y(1, z) = (\text{id})_z.$$

$$Y(a, z) \mathbb{1} \in (\text{End } V)[[z]] \quad \text{and} \quad Y(a, z) \mathbb{1} \Big|_{z=0} = a \quad \forall a \in V$$

$$(\text{translation}) \quad T(1) = 0$$

$$[T, Y(a, z)] = \partial_z Y(a, z) \quad \forall a \in V$$

$$(\text{locality}) \quad \forall a, b \in V \quad \exists N > 0 \text{ s.t.}$$

$$(z - w)^N [Y(a, z), Y(b, w)] = 0 \quad (\text{i.e. } Y(a, z) \text{ & } Y(b, z) \text{ are mutually local})$$

If a vertex algebra V contains an element $w \in V_z$ (called a conformal vector)

s.t.

$$(1) \quad V_{(n)} \longrightarrow \text{End } V \quad \text{is a rep of } V_{(n)}$$

$$L_n \longmapsto w_{(n+1)}$$

$$(2) \quad w_{(0)} \# = T$$

$$(3) \quad w_{(1)} v = nv \quad \forall v \in V_n$$

and $V_n = 0$ for $n < 0$, then V is a vertex operator algebra (VOA).

Rem. Locality is equivalent to the Jacobi identity:

$$z_0^{-1} \delta\left(\frac{z_1 - z_2}{z_0}\right) Y(a, z_1) Y(b, z_2) - z_0^{-1} \delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(b, z_2) Y(a, z_1)$$

$$= z_2^{-1} \delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(a, z_0)b, z_2)$$

from which useful identities can be obtained by specialization.

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Associativity : $Y(a, z) Y(b, \omega) = Y(Y(a, z - \omega)b, \omega) \in (End(V)(\omega))((z - \omega))$

$$= \sum_{n \in \mathbb{Z}} Y(a_{(n)} b, \omega) (z - \omega)^{-n-1}$$

Skew-symmetry : $Y(a, z)b = e^{zT} Y(b, -z)a$

Borchardt:

$$\begin{aligned} & \sum_{i=0}^{\infty} (-1)^i \binom{i}{i} \left[a_{(\ell+m-i)}^{\circ} b_{(n+i)} - (-1)^{\ell} b_{(\ell+n-i)}^{\circ} a_{(m+i)} \right] \\ &= \sum_{i=0}^{\infty} \binom{m}{i} (a_{(\ell+i)} b)_{(m+n-i)} \end{aligned}$$

25 Feb (XIV)

Example 1: Holomorphic vertex algebras.

A vertex algebra V is holomorphic if $\sum_{n \in \mathbb{Z}} a_n z^{-n-1} \in \text{End } V[[z]]$,
 i.e. $a_{n_0} = 0 \quad \forall n \geq 0$.

Then the OPE coefficients from $[Y(a, z), Y(b, w)]$ are

$$c^{\psi}(w) = \underbrace{\text{Res}_z [Y(a, z), Y(b, w)]}_{\text{no neg. power of } z} (z-w)^j = 0,$$

so $Y(a, z) Y(b, w) = Y(b, w) Y(a, z)$ and V is commutative!

In fact, V is a commutative and associative algebra with unit $\mathbf{1}$ derivation.

$$\begin{array}{ccc} V \times V & \longrightarrow & V \\ (a, b) & \longmapsto & ab = a_{(-1)} b \in V \end{array}$$

Commutativity: $a_{(-1)} b_{(-1)} = b_{(-1)} a_{(-1)}$, so

$$ab = a_{(-1)} b = a_{(-1)}(b_{(-1)} \mathbf{1}) = b_{(-1)}(a_{(-1)} \mathbf{1}) = b_{(-1)} a = ba$$

Associativity: $a(bc) = a_{(-1)}(b_{(-1)} c) = b_{(-1)}(a_{(-1)} c)$,

$$\text{so } a(bc) = a(cb) = a(ab)c = (ab)c$$

Unit: $\mathbf{1} a = a \mathbf{1} = a \quad \forall a$

Derivation: By Exercise 3, $Y(a, z)\mathbf{1} = e^{zT}(a)$, so

$$\begin{aligned} Y(a, z) Y(b, w)\mathbf{1} &= Y(a, z) e^{wT}(b) \\ &= Y(b, w) e^{zT}(a) \end{aligned}$$

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Taking $w=0$, we get

$$Y(a, z)b = b_{(-1)} e^{zT(a)},$$

Then $[T, Y(a, z)] = \partial_z Y(a, z)$ means that

$$\begin{aligned} T(e^{zT(a)}_{(-1)} b) - e^{zT(a)}_{(-1)} T(b) &= \partial_z (e^{zT(a)}_{(-1)} b) \\ &= T(e^{zT(a)}_{(-1)} b). \end{aligned}$$

Let $z=0$:

$$T(a_{(-1)} b) - a_{(-1)} T(b) = T(a)_{(-1)} b,$$

$$\text{so } T(ab) = (Ta)b + a(Tb).$$

Conversely, any \mathbb{Z} -graded commutative, associative unital algebra with dimensions of fin dim graded components is a holomorphic alg. (Exercise 5).