Mini-cours
« Sous-espaces invariants pour l'opérateur shift »
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Mini-course
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Dirichlet space

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The name Dirichlet space derives from its definition in terms of the so-called Dirichlet integral, arising in Dirichlet's method for solving Laplace's equation. The first appearance of the Dirichlet space under that name dates back to the late 1950s, but in fact the notion existed and had been studied at least since Beurling's thesis, which was published in 1933 and written even a little earlier. In the years that followed, Beurling and Carleson laid the foundations of the theory, and, after their pioneering work, many other distinguished mathematicians made important contributions. Here are a few reasons for studying the Dirichlet space.

(1) The Hardy space corresponds to ℓ^2, the Hilbert space of square-summable sequences. One of the main advantages of thinking of it as a function space is that the shift operator on ℓ^2 becomes simply multiplication by z. If one is interested in weighted shifts on ℓ^2, then one should consider multiplication by z on a weighted function space. The two most basic non-constant weights lead one immediately to the Dirichlet and Bergman spaces.

(2) The Dirichlet integral of a holomorphic function f has a natural geometric interpretation as the area of the image of f, counted according to multiplicity. Seen this way, it is obviously invariant under precomposition with every automorphism of the unit disk. It is a remarkable fact that this invariance property characterizes the Dirichlet space among all Hilbert function spaces on the disk.

(3) The Dirichlet space is closely related to logarithmic potential theory. In particular, the notions of energy and logarithmic capacity play a prominent role in the theory. This reflects Beurling's vision of the subject, and yields interesting interactions with physics.

(4) From many points of view, the Dirichlet space is a borderline case. For example, it is nearly an algebra, but not quite. This borderline nature makes it an interesting and challenging example of a function space. Several basic questions remain open, and the Dirichlet space is still an active area of research. To get a quick idea of the topics to be

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covered in this mini-course, imagine being presented with a function space on the unit disk. Several standard questions naturally arise. For example:

- What can be said about the boundary behaviour of functions in the space?
- Are there simple characterizations of zero sets and uniqueness sets?
- What can we say about interpolation?
- Is the space an algebra? If not, then what are the multipliers?
- How rich is the operator theory on this function space? For example, can we classify the shift-invariant subspaces? Which functions are cyclic?

In the case of the Hardy space, the answers to all these questions are well known and important. By contrast, in the Dirichlet space, some have been only partially answered, and even where the complete answers are known, they are more subtle. This is the subject of these lectures.

Survey articles

[1] William T. Ross, *The classical Dirichlet space*, Recent Advances in Operator-Related Function Theory, Contemp. Math., vol. 393, Amer. Math. Soc., Providence, RI, 2006, pp. 171–197.

[2] Nicola Arcozzi, Richard Rochberg, Eric T. Sawyer, and Brett D. Wick, *The Dirichlet space: a survey*, New York J. Math. **17A** (2011), 45–86.

Forthcoming book

[3] O. El-Fallah, K. Kellay, J. Mashreghi, and T. Ransford, *A primer on the Dirichlet space*, Cambridge Univ. Press, to appear.