Cyclicity in the Dirichlet Spaces and Extremal Polynomials

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This talk is based on a joint paper of the same title with Alberto Condori, Conni Liaw, Daniel Seco, and Alan Sola, Journal D’Analyse (to appear), and on some preliminary joint work with these same authors.
Outline

The $D_\alpha$ Spaces
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Logarithmic Capacity and the Brown and Shields Conjecture

Optimal Approximants

Growth Estimates

An Alternate Question?

Functions of several variables

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Definition

For \(-\infty < \alpha < \infty\), the space \(D_\alpha\) consists of all analytic functions \(f : \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \to \mathbb{C}\) whose Taylor coefficients in the expansion

\[
f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad z \in \mathbb{D},
\]

satisfy

\[
\|f\|^2_\alpha = \sum_{k=0}^{\infty} (k + 1)^\alpha |a_k|^2 < \infty.
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$$\|f\|_\alpha^2 = \sum_{k=0}^{\infty} (k + 1)^\alpha |a_k|^2 < \infty.$$

The spaces $D_\alpha$ become smaller as $\alpha$ increases, and $f \in D_\alpha$ if and only if the derivative $f' \in D_{\alpha-2}$.
Examples

- \( \alpha = -1 \) corresponds to the **Bergman space** \( A^2 \), consisting of functions with

\[
\int_{\mathbb{D}} |f(z)|^2 dA(z) < \infty, \quad dA(z) = \frac{dxdy}{\pi},
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$$\int_{\mathbb{D}} |f(z)|^2 dA(z) < \infty, \quad dA(z) = \frac{dxdy}{\pi},$$

- $\alpha = 0$: the **Hardy space** $H^2$, consisting of functions with

$$\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty,$$
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  \]

- $\alpha = 1$: the (classical) Dirichlet space $D$ of functions whose derivatives have finite area integral:
  \[
  \int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty.
  \]
An equivalent norm

A description like that of the Dirichlet space, in terms of an integral is possible for the $D_\alpha$ spaces for $\alpha < 2$. Indeed, $f \in D_\alpha$ if and only if

$$|f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^{1-\alpha} dA(z) < \infty.$$ 

This last expression defines the square of an equivalent norm for $f \in D_\alpha$, which we will use when convenient.
Definition of Cyclicity

A function $f \in D_\alpha$ is said to be cyclic in $D_\alpha$ if the subspace generated by polynomials multiples of $f$,

$$[f] = \text{span}\{z^k f : k = 0, 1, 2, \ldots\}$$

coincides with $D_\alpha$. 
**Definition of Cyclicity**

A function \( f \in D_\alpha \) is said to be *cyclic* in \( D_\alpha \) if the subspace generated by polynomials multiples of \( f \),

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Equivalently: there exists a sequence of polynomials \( \{p_n\}_{n=1}^\infty \) such that

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\|p_n f - 1\|_\alpha \to 0, \quad \text{as } n \to \infty.
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$$\|p_n f - 1\|_\alpha \to 0, \quad \text{as} \quad n \to \infty.$$

So, $f$ is cyclic means that the $z$-invariant subspace generated by $f$ is the whole space. One of the motivations for studying cyclic functions is to understand the structure of the space in question.
Miscellaneous notes about cyclic functions

- Cyclic functions in any $D_\alpha$ space cannot have zeros inside the disk!
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- $f(z) = 1 - z$ is cyclic in the classical ($\alpha = 1$) Dirichlet space.
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- $f(z) = 1 - z$ is cyclic in the classical ($\alpha = 1$) Dirichlet space.
- For the classical Dirichlet space, not having “too many zeros” on the circle seems to be the key.
Logarithmic Capacity

For any \( f \in D \), the non-tangential limit \( f^*(\zeta) = \lim_{z \to \zeta} f(z) \) exists \textit{quasi-everywhere}, that is, outside a set of logarithmic capacity zero (Beurling).
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Theorem (Brown and Shields, 1984)

If the (boundary) zeros of $f$,

$$\mathcal{Z}(f^*) = \{ \zeta \in \mathbb{T}: f^*(\zeta) = 0 \},$$

form a set of positive logarithmic capacity, then $f$ cannot be cyclic.
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**Theorem (Brown and Shields, 1984)**

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form a set of positive logarithmic capacity, \textit{then} \( f \) \textit{cannot be cyclic}.

**Conjecture (Brown and Shields)**

\textit{If} \( f \) \textit{is outer and} \( \text{cap}(\mathcal{Z}(f^*)) = 0 \), \textit{then} \( f \) \textit{is cyclic}. 
A lot of work and partial results on this conjecture.

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- In 2006, El-Fallah, Kellay, and Ransford developed this idea further and gave examples of uncountable Bergman-Smirnov exceptional sets. In a series of papers (2006, 2009, 2010), these authors narrowed the gap in the Brown and Shields conjecture.
- In an earlier paper (1992), Richter and Sundberg discussed multipliers and invariant subspaces, and this leads, for instance, to a proof that non-vanishing univalent functions in the Dirichlet space are cyclic.
Main Theme of this talk

Suppose $f \in D_\alpha$ is cyclic. The main theme of this talk will be to obtain information, when possible, about the polynomials $p_n$ such that $\|p_n f - 1\|_\alpha \to 0$. 
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Main Theme of this talk

Suppose $f \in D_\alpha$ is cyclic. The main theme of this talk will be to obtain information, when possible, about the polynomials $p_n$ such that $\|p_n f - 1\|_\alpha \to 0$.

- Can we obtain an explicit sequence of polynomials $\{p_n\}$ such that
  \[ \|p_n f - 1\|_\alpha \xrightarrow{n \to \infty} 0? \]

- Can we give an estimate on the rate of decay of these norms as $n \to \infty$?
Definition
Let \( f \in D_\alpha \). We say that a polynomial \( p_n \) of degree at most \( n \) is an optimal approximant of order \( n \) to \( 1/f \) if \( p_n \) minimizes \( \|pf - 1\|_\alpha \) among all polynomials \( p \) of degree at most \( n \).
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In other words, $p_n$ is an optimal polynomial of order $n$ to $1/f$ if

$$\|p_nf - 1\|_\alpha = \text{dist}_{D_\alpha}(1, f \cdot \text{Pol}_n),$$

where $\text{Pol}_n$ denotes the space of polynomials of degree at most $n$. 
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Note: \( p_n f \) is the orthogonal projection of \( 1 \) onto the subspace

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V_n = \{pf : p \text{ is a polynomial, } \deg p \leq n\}.
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Note: $p_n f$ is the orthogonal projection of $1$ onto the subspace

$$V_n = \{pf : p \text{ is a polynomial, } \deg p \leq n\}.$$

Therefore, optimal approximants $p_n$ always exist and are unique for any nonzero function $f$, and any degree $n \geq 1$. 
A model example and a first try.

Let \( f(z) = 1 - z \).
A model example and a first try.

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It’s easy to see that 

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so $\|T_n(z)f(z) - 1\|_{H^2} = 1$ (no good); $\|T_n(z)f(z) - 1\|^2_D = n + 2$ (even worse!); $\|T_n(z)f(z) - 1\|^2_{A^2} = 1/(n + 2) \to 0$. 

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Cyclicity in the Dirichlet Spaces and Extremal Polynomials
What about the other extreme?

Suppose $f$ and $1/f$ are analytic in the closed disk.

\[ ||T_n f - 1||_\alpha \leq ||f(T_n - 1/f)||_\alpha \leq ||f||_{M(D_\alpha)} ||T_n - 1/f||_\alpha \]

(f and all its derivatives are bounded, so $f$ is a multiplier for $D_\alpha$) and it is not hard to see that $||T_n - 1/f||_\alpha$ decays exponentially.

Therefore the rate of decay of $\text{dist}(D_\alpha, f \cdot \text{Pol}_n)$ is exponentially fast.

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($f$ and all its derivatives are bounded, so $f$ is a multiplier for $D_\alpha$) and it is not hard to see that $\| T_n - 1/f \|_\alpha$ decays exponentially.

Therefore the rate of decay of $\text{dist}_{D_\alpha}(1, f \cdot \text{Pol}_n)$ is exponentially fast.
For the next theorem...

We will use the integral norm representation, namely,

\[ \|f\|_\alpha^2 = |f(0)|^2 + D_\alpha(f), \]

where

\[ D_\alpha(f) = \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^{1-\alpha} \, dA(z) \]

For simplicity of notation, we let \( d\mu_\alpha(z) = (1 - |z|^2)^{1-\alpha} \, dA(z) \).
Theorem

Let $n \in \mathbb{N}$, $0 \leq \alpha \leq 1$, $f \in D_\alpha \setminus \{0\}$. Suppose $1 \notin f \cdot \text{Pol}_n$ and let $M$ denote the $n \times n$ matrix with entries $\langle (z^k f)', (z^j f)' \rangle_{L^2(\mu_\alpha)}$. Then the unique $p_n \in \text{Pol}_n$ satisfying

$$\|p_n f - 1\|_\alpha = \text{dist}_{D_\alpha}(1, f \cdot \text{Pol}_n)$$

is given by

$$p_n(z) = p_n(0) + \sum_{k=1}^{n} p_n(0) \frac{\det M_k}{\det M} z^k,$$

where $M_k$ denotes the $n \times n$ matrix obtained from $M$ by replacing the $k$th column of $M$ with the column with entries $-\langle f', (z^j f)' \rangle_{L^2(\mu_\alpha)}$, $1 \leq j \leq n$. 
What is the theorem good for?

- If $f(z)$ is a polynomial of degree $t$, then $M$ is “$2t + 1$” diagonal.
What is the theorem good for?

- If \( f(z) \) is a polynomial of degree \( t \), then \( M \) is “\( 2t + 1 \)” diagonal.
- Allows for explicit construction of optimal approximants for simple examples like \( f(z) = 1 - z \) and numerical constructions for functions like \( f_\beta(z) = (1 - z)^\beta \) (\( \beta > 0 \)).
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- Allows for explicit construction of optimal approximants for simple examples like \( f(z) = 1 - z \) and numerical constructions for functions like \( f_\beta(z) = (1 - z)^\beta \) (\( \beta > 0 \)).

Notation for the next few slides:

\[
H_n = \sum_{k=1}^{n} \frac{1}{k} \approx \log n
\]
The optimal approximants are modified Cesàro mean/Riesz mean polynomials

\[ p_n(z) = p_n(0) \left( \sum_{k=0}^{n} \left( 1 - \frac{k + H_k}{n + 1 + H_{n+1}} \right) z^k \right) \]

(for the Hardy space)
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(for the Hardy space)

\[ p_n(z) = p_n(0) \left( \sum_{k=0}^{n} \left( 1 - \frac{H_k}{H_{n+1}} \right) z^k \right) \]

(for the Dirichlet space)

\[ p_n(z) = p_n(0) \left( 1 + \sum_{k=1}^{n} \left( 1 - \frac{k(k + 7) + 4H_k}{(n + 1)(n + 8) + 4H_{n+1}} \right) z^k \right) \]

(for the Bergman Space)
When do Cesàro means work?

In general, it would be very interesting to know what conditions on $1/f$ are associated with the optimal approximants being some kind of Cesàro mean.
Rate of convergence of optimal approximants

Definition
Let $f \in D_\alpha$. The *optimal norm* of degree $n$ associated with $f$ is

$$N_n(f) = \|p_nf - 1\|_\alpha^2,$$

where $p_n$ is the optimal approximant of $1/f$ of degree $n$. 
Rate of convergence of optimal approximants

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N_n(f) = \| p_n f - 1 \|_\alpha^2,
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where \( p_n \) is the optimal approximant of \( 1/f \) of degree \( n \).

Notation for the rate function for the next theorem:

Definition
For \( \alpha < 1 \), we set \( \phi_\alpha(t) = t^{\alpha-1} \), \( t \in \mathbb{N} \). In the case \( \alpha = 1 \), we take \( \phi(t) = \phi_1(t) = 1/H_t \), \( t \in \mathbb{N} \).
Lemma

Let $\alpha \leq 1$. If $f(z) = \zeta - z$, for $\zeta \in \mathbb{T}$, then $\text{dist}_{D_\alpha}^2(1, f \cdot \text{Pol}_n)$ is comparable to $\varphi^{-1}_\alpha(n + 1)$ for all sufficiently large $n$. 
Lemma

Let $\alpha \leq 1$. If $f(z) = \zeta - z$, for $\zeta \in \mathbb{T}$, then $\text{dist}^2_{D\alpha}(1, f \cdot \text{Pol}_n)$ is comparable to $\varphi^{-1}_\alpha(n + 1)$ for all sufficiently large $n$.

Theorem

Let $\alpha \leq 1$. If $f$ is a polynomial whose zeros lie in $\mathbb{C} \setminus \mathbb{D}$, then there exists a constant $C = C(\alpha, f)$ such that

$$\text{dist}^2_{D\alpha}(1, f \cdot \text{Pol}_m) \leq C\varphi^{-1}_\alpha(m + 1)$$

holds for all sufficiently large $m$. Moreover, this estimate is sharp in the sense that if such a polynomial $f$ has at least one zero on $\mathbb{T}$, then there exists a constant $\tilde{C} = \tilde{C}(\alpha, f)$ such that

$$\tilde{C}\varphi^{-1}_\alpha(m + 1) \leq \text{dist}^2_{D\alpha}(1, f \cdot \text{Pol}_m).$$
Theorem

Let $\alpha \leq 1$. If $f$ is a function admitting an analytic continuation to the closed unit disk and whose zeros lie in $\mathbb{C} \setminus \mathbb{D}$, then there exists a constant $C = C(\alpha, f)$ such that

$$\text{dist}_{D_\alpha}^2(1, f \cdot \text{Pol}_m) \leq C \varphi_\alpha^{-1}(m + 1)$$

holds for all sufficiently large $m$. Moreover, this estimate is sharp in the sense that if such a function $f$ has at least one zero on $\mathbb{T}$, then there exists a constant $\tilde{C} = \tilde{C}(\alpha, f)$ such that

$$\tilde{C} \varphi_\alpha^{-1}(m + 1) \leq \text{dist}_{D_\alpha}^2(1, f \cdot \text{Pol}_m).$$
What about other kinds of boundary zeros?

It would be interesting to get optimal approximants and rates of decay for functions like

$$f(z) = (1 - z)^\beta.$$
What kind of invertibility?

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**Question:** How far can we push the invertibility?
What kind of invertibility?

Cyclicity is a kind of “weak” invertibility. Brown showed (1990) that if $f \in D$ and $1/f \in D$, then $f$ is cyclic.

**Question:** How far can we push the invertibility? Suppose $f$ is bounded and in $D_\alpha$, and suppose $\log f \in D_\alpha$. Can we say $f$ is cyclic?
What kind of invertibility?

Cyclicity is a kind of “weak” invertibility. Brown showed (1990) that if \( f \in D \) and \( 1/f \in D \), then \( f \) is cyclic.

**Question:** How far can we push the invertibility? Suppose \( f \) is bounded and in \( D_\alpha \), and suppose \( \log f \in D_\alpha \). Can we say \( f \) is cyclic?

**Remark:** The statement is true (and easy) for \( \alpha > 1 \) or \( \alpha = 0 \).
Lemma

Suppose $f \in D_\alpha$ and $\log f \in D_\alpha$. Then, for any $\tau \in (0, 1]$, we have

$$D_\alpha(f^\tau) \leq \tau^2 (D_\alpha(f) + D_\alpha(\log f)),$$

and consequently, $f^\tau \in D_\alpha$. 
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Theorem (Richter and Sundberg, 1992)

If \( f \in D \) is an outer function, and if \( \tau > 0 \) is such that \( f^\tau \in D \), then \([f] = [f^\tau] \).
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Richter and Sundberg applied this theorem by showing that if \( f \) is univalent and non-vanishing, then \( f^\tau \in D \), and hence is cyclic.
Theorem

Suppose \( f \in D \) and \( \log f \in D \). Then \( f \) is cyclic in the Dirichlet space.

Proof.

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Since the lemma also implies \( f^\tau \to 1 \) in \( D \) as \( \tau \to 0 \), we have \([f] = [1]\), and the assertion follows.
For other values of $\alpha$,

**Theorem (Vague)**

*Let $f \in H^\infty$ and $q = \log f \in D_\alpha$. Suppose a certain technical condition on the growth of approximating polynomials $q_n$ holds. Then $f$ is cyclic in $D_\alpha$.**
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We consider a scale of Hilbert spaces of holomorphic functions on the bidisk

\[ D^2 = \left\{ (z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1 \right\} \]

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indexed by a parameter \( \alpha \in (-\infty, \infty) \). We say that a holomorphic function \( f : \mathbb{D}^2 \to \mathbb{C} \) belongs to the \textit{Dirichlet-type space} \( \mathcal{D}_\alpha \) if its power series expansion

\[ f(z_1, z_2) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{k,l} z_1^k z_2^l \]

satisfies

\[ \|f\|_{\alpha}^2 = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (k + 1)^\alpha (l + 1)^\alpha |a_{k,l}|^2 < \infty. \]
A Pair of Shift Operators

A natural pair of bounded linear operators acting on the spaces $D_\alpha$ are the shift operators $S_1$ and $S_2$ are defined by, for $f \in D_\alpha$

$$S_1 f(z_1, z_2) = z_1 f(z_1, z_2) \quad \text{and} \quad S_2 f(z_1, z_2) = z_2 f(z_1, z_2).$$
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The invariant subspaces of interest are the closed subspaces $\mathcal{M} \subset \mathcal{D}_\alpha$ such that

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S_1 \mathcal{M} \subset \mathcal{M} \quad \text{and} \quad S_2 \mathcal{M} \subset \mathcal{M}.
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- There are analogous definitions for inner and outer functions
- There is no “Beurling” theorem for these invariant subspaces
- These subspaces may contain no bounded elements, and may fail to be finitely generated
Cyclic Functions

Definition
We say $f$ is cyclic if

$$[f] = \text{span}\{z_1^k z_2^l f : k = 0, 1, \ldots; l = 0, 1, \ldots\} = \mathcal{D}_\alpha.$$
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It is easy to see that there exists at least one cyclic function in each $\mathcal{D}_\alpha$, namely the function $f(z_1, z_2) = 1$. This follows from the fact that polynomials in two variables are dense in $\mathcal{D}_\alpha$. 
For “separable” functions, our one variable results extend

**Proposition**

Let $\alpha \in \mathbb{R}$, and let $f(z_1, z_2) = g(z_1)h(z_2)$, where $g, h, \in D_{\alpha}$. Then $f$ is cyclic in $D_{\alpha}$ if and only if $g$ and $h$ are cyclic in $D_{\alpha}$. 
Theorem
Let $\alpha \leq 1$. Suppose $g$ and $h$ admit analytic continuations to $\bar{D}$ and have no zeros in $D$. Define $f(z_1, z_2) = g(z_1)h(z_2)$. Then there exists a constant $C = C(g, h, \alpha)$ such that

$$\text{dist}_{D^\alpha}^2(1, f \cdot P_n) \leq C \varphi_\alpha^{-1}(n + 1),$$

for all sufficiently large $n$. Moreover, this estimate is sharp in the sense that if $h$ has at least one zero on $T$ and $g$ has no zeros in the closed disk $D$, (or vice versa), then there exists a constant $\tilde{C} = \tilde{C}(g, h, \alpha)$ such that

$$\tilde{C} \varphi_\alpha^{-1}(n + 1) \leq \text{dist}_{D^\alpha}^2(1, f \cdot P_n).$$
Examples

The last theorem shows that $f(z_1, z_2) = (1 - z_1)(1 - z_2)$ is cyclic in $D_\alpha$ for all $\alpha \leq 1$, and in particular in $D$, the analogue of the (classical) Dirichlet space.
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However, we have also shown that \( f(z_1, z_2) = 1 - z_1 z_2 \) is only cyclic in \( D_\alpha \) for \( \alpha \leq 1/2 \), and so is NOT cyclic in \( D \).
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- Some polynomials with no zeros in the bidisk are not be cyclic for the Dirichlet space of the bidisk.
- The “size” of the zero set doesn’t seem to play the same role in cyclicity as it does in one variable.
- Understanding even which polynomials are cyclic in this context seems like an interesting question.
Thank you for your attention!