Cyclicity in the Dirichlet Spaces and Extremal Polynomials

Catherine Bénéteau

University of South Florida

Montréal, August 28, 2013

This talk is based on a joint paper of the same title with Alberto Condori, Conni Liaw, Daniel Seco, and Alan Sola, Journal D'Analyse (to appear), and on some preliminary joint work with these same authors.

Outline

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Definition

For $-\infty < \alpha < \infty$, the space D_α consists of all analytic functions $f : \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \to \mathbb{C}$ whose Taylor coefficients in the expansion

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad z \in \mathbb{D},$$

satisfy

$$\|f\|_\alpha^2 = \sum_{k=0}^{\infty} (k+1)^\alpha |a_k|^2 < \infty.$$

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The spaces D_α become smaller as α increases, and $f \in D_\alpha$ if and only if the derivative $f' \in D_{\alpha-2}$.

Examples

- $\alpha = -1$ corresponds to *the Bergman space* A^2, consisting of functions with

$$\int_{\mathbb{D}} |f(z)|^2 dA(z) < \infty, \quad dA(z) = \frac{dxdy}{\pi},$$

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$$\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty,$$
- $\alpha = 1$: *the (classical) Dirichlet space* D of functions whose derivatives have finite area integral:
$$\int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty.$$

An equivalent norm

A description like that of the Dirichlet space, in terms of an integral is possible for the D_α spaces for $\alpha < 2$. Indeed, $f \in D_\alpha$ if and only if

$$|f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^{1-\alpha} dA(z) < \infty.$$

This last expression defines the square of an equivalent norm for $f \in D_\alpha$, which we will use when convenient.

Definition of Cyclicity

A function $f \in D_\alpha$ is said to be *cyclic* in D_α if the subspace generated by polynomials multiples of f,

$$[f] = \overline{\mathrm{span}\{z^k f : k = 0, 1, 2, \ldots\}}$$

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So, f is cyclic means that the z-invariant subspace generated by f is the whole space. One of the motivations for studying cyclic functions is to understand the structure of the space in question.

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- $f(z) = 1 - z$ is cyclic in the classical ($\alpha = 1$) Dirichlet space.
- For the classical Dirichlet space, not having "too many zeros" on the circle seems to be the key.

Logarithmic Capacity

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Theorem (Brown and Shields, 1984)

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Conjecture (Brown and Shields)

If f is outer and $cap(\mathcal{Z}(f^*)) = 0$, then f is cyclic.

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- In 2006, El-Fallah, Kellay, and Ransford developed this idea further and gave examples of uncountable Bergman-Smirnov exceptional sets. In a series of papers (2006, 2009, 2010), these authors narrowed the gap in the Brown and Shields conjecture.
- In an earlier paper (1992), Richter and Sundberg discussed multipliers and invariant subspaces, and this leads, for instance, to a proof that non-vanishing univalent functions in the Dirichlet space are cyclic.

Main Theme of this talk

Suppose $f \in D_\alpha$ is cyclic. The main theme of this talk will be to obtain information, when possible, about the polynomials p_n such that $\|p_n f - 1\|_\alpha \to 0$.

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- Can we obtain an explicit sequence of polynomials $\{p_n\}$ such that
$$\|p_n f - 1\|_\alpha \overset{n\to\infty}{\longrightarrow} 0?$$
- Can we give an estimate on the rate of decay of these norms as $n \to \infty$?

Definition

Let $f \in D_\alpha$. We say that a polynomial p_n of degree at most n is an *optimal approximant* of order n to $1/f$ if p_n minimizes $\|pf - 1\|_\alpha$ among all polynomials p of degree at most n.

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In other words, p_n is an optimal polynomial of order n to $1/f$ if

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$$V_n = \{pf : p \text{ is a polynomial}, \ \deg p \leq n\}.$$

Therefore, optimal approximants p_n always exist and are unique for any nonzero function f, and any degree $n \geq 1$.

A model example and a first try.

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Therefore the rate of decay of $\mathrm{dist}_{D_\alpha}(1, f \cdot \mathrm{Pol}_n)$ is exponentially fast.

For the next theorem...

We will use the integral norm representation, namely,

$$\|f\|_\alpha^2 = |f(0)|^2 + D_\alpha(f),$$

where

$$D_\alpha(f) = \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^{1-\alpha}\, dA(z)$$

For simplicity of notation, we let $d\mu_\alpha(z) = (1-|z|^2)^{1-\alpha}\, dA(z)$.

Theorem

Let $n \in \mathbb{N}$, $0 \leq \alpha \leq 1$, $f \in D_\alpha \setminus \{0\}$. Suppose $1 \notin f \cdot \mathrm{Pol}_n$ and let M denote the $n \times n$ matrix with entries $\langle (z^k f)', (z^j f)' \rangle_{L^2(\mu_\alpha)}$. Then the unique $p_n \in \mathrm{Pol}_n$ satisfying

$$\|p_n f - 1\|_\alpha = \mathrm{dist}_{D_\alpha}(1, f \cdot \mathrm{Pol}_n)$$

is given by

$$p_n(z) = p_n(0) + \sum_{k=1}^{n} p_n(0) \frac{\det M_k}{\det M} z^k, \tag{1}$$

where M_k denotes the $n \times n$ matrix obtained from M by replacing the kth column of M with the column with entries $-\langle f', (z^j f)' \rangle_{L^2(\mu_\alpha)}$, $1 \leq j \leq n$.

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Notation for the next few slides:

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \approx \log n$$

The optimal approximants are modified Cesàro mean/Riesz mean polynomials

$$p_n(z) = p_n(0)\left(\sum_{k=0}^{n}\left(1 - \frac{k + H_k}{n + 1 + H_{n+1}}\right) z^k\right)$$

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$$p_n(z) = p_n(0)\left(1+\sum_{k=1}^{n}\left(1-\frac{k(k+7)+4H_k}{(n+1)(n+8)+4H_{n+1}}\right)z^k\right)$$

(for the Bergman Space)

When do Cesàro means work?

In general, it would be very interesting to know what conditions on $1/f$ are associated with the optimal approximants being some kind of Cesàro mean.

Rate of convergence of optimal approximants

Definition

Let $f \in D_\alpha$. The *optimal norm* of degree n associated with f is

$$N_n(f) = \|p_n f - 1\|_\alpha^2,$$

where p_n is the optimal approximant of $1/f$ of degree n.

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Notation for the rate function for the next theorem:

Definition

For $\alpha < 1$, we set $\phi_\alpha(t) = t^{\alpha-1}$, $t \in \mathbb{N}$. In the case $\alpha = 1$, we take $\phi(t) = \phi_1(t) = 1/H_t$, $t \in \mathbb{N}$.

Lemma

Let $\alpha \leq 1$. If $f(z) = \zeta - z$, for $\zeta \in \mathbb{T}$, then $dist^2_{D_\alpha}(1, f \cdot \mathrm{Pol}_n)$ is comparable to $\varphi_\alpha^{-1}(n+1)$ for all sufficiently large n.

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Theorem

Let $\alpha \leq 1$. If f is a polynomial whose zeros lie in $\mathbb{C} \setminus \mathbb{D}$, then there exists a constant $C = C(\alpha, f)$ such that

$$dist^2_{D_\alpha}(1, f \cdot \mathrm{Pol}_m) \leq C\varphi_\alpha^{-1}(m+1)$$

holds for all sufficiently large m. Moreover, this estimate is sharp in the sense that if such a polynomial f has at least one zero on $\mathbb{T}$, then there exists a constant $\tilde{C} = \tilde{C}(\alpha, f)$ such that

$$\tilde{C}\varphi_\alpha^{-1}(m+1) \leq dist^2_{D_\alpha}(1, f \cdot \mathrm{Pol}_m).$$

Theorem

Let $\alpha \leq 1$. If f is a function admitting an analytic continuation to the closed unit disk and whose zeros lie in $\mathbb{C} \setminus \mathbb{D}$, then there exists a constant $C = C(\alpha, f)$ such that

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What about other kinds of boundary zeros?

It would be interesting to get optimal approximants and rates of decay for functions like

$$f(z) = (1 - z)^\beta.$$

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Question: How far can we push the invertibility? Suppose f is bounded and in D_α, and suppose $\log f \in D_\alpha$. Can we say f is cyclic?

Remark: The statement is true (and easy) for $\alpha > 1$ or $\alpha = 0$.

Lemma

Suppose $f \in D_\alpha$ and $\log f \in D_\alpha$. Then, for any $\tau \in (0,1]$, we have

$$D_\alpha(f^\tau) \leq \tau^2 \left(D_\alpha(f) + D_\alpha(\log f)\right),$$

and consequently, $f^\tau \in D_\alpha$.

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Theorem (Richter and Sundberg, 1992)

If $f \in D$ is an outer function, and if $\tau > 0$ is such that $f^\tau \in D$, then $[f] = [f^\tau]$.

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Theorem (Richter and Sundberg, 1992)

If $f \in D$ is an outer function, and if $\tau > 0$ is such that $f^\tau \in D$, then $[f] = [f^\tau]$.

Richter and Sundberg applied this theorem by showing that if f is univalent and non-vanishing, then $f^\tau \in D$, and hence is cyclic.

Theorem

Suppose $f \in D$ and $\log f \in D$. Then f is cyclic in the Dirichlet space.

Proof.

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Next, by the lemma, $f^\tau \in D$ for all $\tau > 0$, and so by Richter and Sundberg, $[f] = [f^\tau]$ for each τ.
Since the lemma also implies $f^\tau \to 1$ in D as $\tau \to 0$, we have $[f] = [1]$, and the assertion follows.

$\square$

For other values of α,

Theorem (Vague)

Let $f \in H^\infty$ and $q = \log f \in D_\alpha$. Suppose a certain technical condition on the growth of approximating polynomials q_n holds. Then f is cyclic in D_α.

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Brown and Cohn proved (1985) that for any closed set of logarithmic capacity zero $E \subset \partial\mathbb{D}$, there exists a cyclic function f in D such that $Z(f) = E$.

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Let $f \in H^\infty$ and $q = \log f \in D_\alpha$. Suppose a certain technical condition on the growth of approximating polynomials q_n holds. Then f is cyclic in D_α.

These technical conditions might be interesting because ...
Brown and Cohn proved (1985) that for any closed set of logarithmic capacity zero $E \subset \partial\mathbb{D}$, there exists a cyclic function f in D such that $Z(f) = E$.
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We consider a scale of Hilbert spaces of holomorphic functions on the bidisk

$$\mathbb{D}^2 = \{(z_1, z_2) \in \mathbb{C}^2 \colon |z_1| < 1, |z_2| < 1\}$$

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indexed by a parameter $\alpha \in (-\infty, \infty)$. We say that a holomorphic function $f \colon \mathbb{D}^2 \to \mathbb{C}$ belongs to the *Dirichlet-type space* $\mathfrak{D}_\alpha$ if its power series expansion

$$f(z_1, z_2) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{k,l} z_1^k z_2^l$$

satisfies

$$\|f\|_\alpha^2 = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (k+1)^\alpha (l+1)^\alpha |a_{k,l}|^2 < \infty.$$

A Pair of Shift Operators

A natural pair of bounded linear operators acting on the spaces $\mathfrak{D}_\alpha$ are the *shift operators* S_1 and S_2 are defined by, for $f \in \mathfrak{D}_\alpha$

$$S_1 f(z_1, z_2) = z_1 f(z_1, z_2) \quad \text{and} \quad S_2 f(z_1, z_2) = z_2 f(z_1, z_2).$$

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The invariant subspaces of interest are the closed subspaces $\mathcal{M} \subset \mathfrak{D}_\alpha$ such that

$$S_1 \mathcal{M} \subset \mathcal{M} \quad \text{and} \quad S_2 \mathcal{M} \subset \mathcal{M}.$$

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- There are analogous definitions for inner and outer functions
- There is no "Beurling" theorem for these invariant subspaces
- These subspaces may contain no bounded elements, and may fail to be finitely generated

Cyclic Functions

Definition

We say f is cyclic if

$$[f] = \overline{\mathrm{span}\{z_1^k z_2^l f : k = 0, 1, \ldots; l = 0, 1, \ldots\}} = \mathfrak{D}_\alpha.$$

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It is easy to see that there exists at least one cyclic function in each $\mathfrak{D}_\alpha$, namely the function $f(z_1, z_2) = 1$. This follows from the fact that polynomials in two variables are dense in $\mathfrak{D}_\alpha$.

For "separable" functions, our one variable results extend

Proposition

Let $\alpha \in \mathbb{R}$, and let $f(z_1, z_2) = g(z_1)h(z_2)$, where $g, h, \in D_\alpha$. Then f is cyclic in $\mathfrak{D}_\alpha$ if and only if g and h are cyclic in D_α.

Theorem

Let $\alpha \leq 1$. Suppose g and h admit analytic continuations to $\overline{\mathbb{D}}$ and have no zeros in $\mathbb{D}$. Define $f(z_1, z_2) = g(z_1)h(z_2)$. Then there exits a constant $C = C(g, h, \alpha)$ such that

$$dist^2_{\mathfrak{D}_\alpha}(1, f \cdot \mathcal{P}_n) \leq C\varphi_\alpha^{-1}(n+1),$$

for all sufficiently large n. Moreover, this estimate is sharp in the sense that if h has at least one zero on $\mathbb{T}$ and g has no zeros in the closed disk $\mathbb{D}$, (or vice versa), then there exists a constant $\tilde{C} = \tilde{C}(g, h, \alpha)$ such that

$$\tilde{C}\varphi_\alpha^{-1}(n+1) \leq dist^2_{\mathfrak{D}_\alpha}(1, f \cdot \mathcal{P}_n).$$

Examples

The last theorem shows that $f(z_1, z_2) = (1 - z_1)(1 - z_2)$ is cyclic in $\mathfrak{D}_\alpha$ for all $\alpha \leq 1$, and in particular in $\mathfrak{D}$, the analogue of the (classical) Dirichlet space.

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- Some polynomials with no zeros in the bidisk are not be cyclic for the Dirichlet space of the bidisk
- The "size" of the zero set doesn't seem to play the same role in cyclicity as it does in one variable.
- Understanding even which polynomials are cyclic in this context seems like an interesting question.

Thank you for your attention!