On the sustainability of cooperation in games with heterogeneous agents: two economic applications

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Preliminary comments

In this talk:

- Models are in continuous time.
- Toy models are studied.

Summary of contents

Motivation

Let us consider the following basic problem of exploitation of a nonrenewable natural resource:

- $x(t)$: stock of natural resource at time t,
- $c_i(t)$: extraction rate at time t of player i, for $i = 1, \ldots, N$,
- The state dynamics is described by the equation

$$\dot{x}(s) = -\sum_{i=1}^{2} c_i(s) \ , \quad x(t) = x_t \ , \quad \text{and}$$

- The intertemporal utility function for player i is given by

$$J_i = \int_t^{\infty} e^{-\rho_i (s-t)} \ln c_i(s) \, ds \ .$$

Motivation

Let $V_i^{nc}(x)$, for $x \leq x_0$, the payment for Player i in the Markovian Nash Equilibrium.

Let $V_i^c(x)$, for $x \leq x_0$, the payment for Player i in the equilibrium of the problem with partial (intragenerational, not intergenerational) cooperation.

Two questions:

1. Is it guaranteed that $V_1^c(x) + V_2^c(x) > V_1^{nc}(x) + V_2^{nc}(x)$?
2. If at initial time $V_1^c(x_0) + V_2^c(x_0) > V_1^{nc}(x_0) + V_2^{nc}(x_0)$, is it guaranteed the sustainability of cooperation? That is, $V_i^c(x) \geq V_i^{nc}(x)$, for every $x \leq x_0$?.

For the Nash bargaining solution in a general model, see Haurie, JOTA 1976.

Motivation

For this model, the noncooperative payoffs are given by

$$V_i^{nc}(x) = \alpha_i \ln x + \beta_i^{nc} \ ,$$

and the payoffs with partial cooperation are of the form

$$V_i^{c}(x) = \alpha_i \ln x + \beta_i^{c} \ .$$

Answers to the previous questions:

1. $\beta_1^c + \beta_2^c \geq \beta_1^{nc} + \beta_2^{nc}$, hence the joint payments are higher in the case of partial cooperation. And for more general models?
2. In the case of symmetric players, $\beta^c > \beta^{nc}$ for both players. However, if $\rho_1 > 1.409507769\, \rho_2$, then $\beta_1^c < \beta_1^{nc}$ and it is not profitable for Player 1 to cooperate at every instant of time with Player 2 (Breton and Keoula, 2010).

Possible ways to maintain cooperation: to transfer utilities (if possible) or to bargain at every instant of time the weight of each player into the whole coalition (nonconstant weights?).

The general problem with 1-decision maker: A review

The Discounted Utility (DU) Model

Idea: along any specified time interval, individuals search for maximizing the sum of all future utilities (Samuelson (1937)).
At $\tau = 0$, the decision-maker looks for maximizing

$$U_0 = \int_0^{\infty} L(x(t), c(t), t)\, dt\ .$$

In general, at time τ, the agent maximizes

$$U_\tau = \int_\tau^{\infty} L(x(t), c(t), t, x_\tau, \tau)\, dt\ .$$

State equation: $\dot{x}(t) = f(x(t), c(t), t)$, $t \in [0, T]$, with $x(0) = x_0$.

In addition, $L(x(t), c(t), t, x_\tau, \tau) = D(t-\tau)u(x(t), c(t))$, with $D(s) = \delta^s$ (discrete time setting) or $D(s) = e^{-\rho s}$ (continuous time setting). Hence, At $\tau = 0$, the decision-maker looks for maximizing

$$U_0 = \int_0^\infty e^{-\rho t} u(x(t), c(t))\, dt \ .$$

In general, at time τ, the agent maximizes

$$U_\tau = \int_\tau^\infty e^{-\rho(t-\tau)} u(x(t), c(t))\, dt = e^{\rho\tau} \int_\tau^T e^{-\rho t} u(x(t), c(t))\, dt \ .$$

Advantages and drawbacks of the standard (DU) model

1. Advantages:
   - Time consistency.
   - Time preferences remain unchanged along time.
   - Relatively easy mathematical treatment.
2. Drawbacks: non realistic. In real life, time preferences change. Empirical evidence.

General discount functions

An agent taking decisions at time t (the t-agent) aims to maximize

$$J(x,u,t)=\int_t^{\infty} d(s,t)L(x(s),u(s),s)\,ds \ ,$$

with

$$\dot{x}^i(s)=g^i(x(s),u(s),s) \ , \ \ x^i(t)=x_t^i \ , \ \text{for } i=1,\ldots,n \ .$$

If $u^*(s)=\phi(s,x(s))$ is the equilibrium rule, then the value function is given by

$$V(x,t)=\int_t^{\infty} d(s,t)L(x(s),\phi(x(s),s),s)\,ds \ .$$

The general problem with 1-decision maker: A review

Next, for $\epsilon > 0$, let us consider the variations

$$u_\epsilon(s) = \begin{cases} v(s) & \text{if} \quad s \in [t, t+\epsilon] \,, \\ \phi(x,s) & \text{if} \quad s > t+\epsilon \,. \end{cases}$$

If the t-agent has the ability to precommit her behavior during the period $[t, t+\epsilon]$, the value function for the perturbed control path u_ϵ is given by

$$V_\epsilon(x,t) = \max_{\{v(s),\, s\in[t,t+\epsilon]\}} \left\{ \int_t^{t+\epsilon} d(s,t) L(x(s), v(s), s)\, ds + \right.$$

$$\left. + \int_{t+\epsilon}^{\infty} d(s,t) L(x(s), \phi(x(s),s), s)\, ds + d(T,t) S(x(T), T) \right\} .$$

Definition

A decision rule $u^*(s) = \phi(s, x(s))$ is called an equilibrium rule if

$$\lim_{\epsilon \to 0^+} \frac{V(x,t) - V_\epsilon(x,t)}{\epsilon} \geq 0 \,.$$

The general problem with 1-decision maker: A review

If there exists a bounded value function of class C^1 solving the integral equation

$$V(x,t) = \int_t^\infty d(s,t) L(x(s), \phi(x(s),s), s)\, ds$$

where

$$u^* = \phi(x,t) = \text{argmax}_u \left[L(x,u,t) + \nabla_x V(x,t) \cdot g(x,u,t) \right] ,$$

then $u^* = \phi(x,t)$ is an equilibrium rule.

Remark: Note that, if we differentiate the value function $V(x,t)$, we do not obtain a partial differential equation.

N-players: Noncooperative case

The state of the game at time t is described by a vector $x \in X \subseteq \mathbf{R}^n$. The initial state is fixed, $x(0) = x_0$. There are N players. Let $u_i(t) \in U_i \subseteq \mathbf{R}^{m_i}$ be the control variables of player i. Each agent i at time t seeks to maximize in u_i her objective functional

$$J_i(x, t, u_1(s), \ldots, u_N(s)) = \int_t^{\infty} d_i(s,t) L_i(x(s), u_1(s), \ldots, u_N(s), s) ds$$

subject to

$$\dot{x}(s) = g(x(s), u_1(s), \ldots, u_N(s), s), \qquad x(t) = x_t \ ,$$

Agents are heterogeneous in the sense that they have different preferences represented by different utility functions and different discount functions.

N-players: Noncooperative case

The General Problem. Markovian Nash Equilibria

Let $(\phi_1^{nc}, \ldots, \phi_N^{nc})$ be a N-tuple of functions $\phi_i^{nc} \colon X \times [0, \infty) \to \mathbf{R}^{m_i}$ such that the following assumptions are satisfied:

- There exists a unique absolutely continuous curve $x \colon [0, \infty) \to X$ solution to

$$\dot{x}(t) = g(x(t), \phi_1^{nc}(x(t), t), \ldots, \phi_N^{nc}(x(t), t)) \ , \quad x(0) = x_0 \ ,$$

- For all $i = 1, 2, \ldots, N$, there exists a bounded continuously differentiable function $V_i \colon X \times [0, \infty) \to \mathbf{R}$ verifying

$$V_i^{nc}(x, t) = \int_t^{\infty} d_i(s, t) L_i(x(s), \phi_1^{nc}(x(s), s), \ldots, \phi_N^{nc}(x(s), s), s)\, ds \ ,$$

N-players: Noncooperative case

where $u_i^{nc} = \phi_i^{nc}(x,t)$ solves

$$u_i^{nc} = \phi_i^{nc}(x,t) =$$

$$\text{argmax}_{\{u_i\}} \left\{ L_i(x, \phi_1^{nc}(x,t), \ldots, \phi_{i-1}^{nc}(x,t), u_i, \phi_{i+1}^{nc}(x,t), \ldots, \phi_N^{nc}(x,t), t) + \right.$$

$$\left. \nabla_x V_i^{nc}(x,t) \cdot g(x, \phi_1^{nc}(x,t), \ldots, \phi_{i-1}^{nc}(x,t), u_i, \phi_{i+1}^{nc}(x,t), \ldots, \phi_N^{nc}(x,t), t) \right\} .$$

Then the strategy $(\phi_1^{nc}(x,t), \ldots, \phi_N^{nc}(x,t))$ is a time-consistent Markov Nash equilibrium, and $V_i^{nc}(x,t)$, $i = 1, \ldots, N$, are the corresponding value functions.

N-players: Partial (not intergenerational) cooperation

We tackle the problem of maximizing

$$J^c = \sum_{i=1}^{N} \lambda_i(x_t, t) \int_t^{\infty} d_i(s,t) L_i\left(x(s), u_1(s), \ldots, u_N(s), s\right) \, ds \; ,$$

where $\lambda_i(x_t, t) \geq 0$, for every $i = 1, \ldots, N$. Coefficients $\lambda_i(x_t, t)$ represent the bargaining power of agent i at time t.
If $u_i^c(s) = \phi_i^c(s, x(s))$, $i = 1, \ldots, N$, is the equilibrium rule, then the joint value function is

$$V^c(x,t) = \sum_{i=1}^{N} \lambda_i(x,t) \int_t^{\infty} d_i(s,t) L_i(x(s), \phi_1^c(x(s), s), \ldots, \phi_N^c(x,s), s) \, ds \; .$$

We assume that the value function is finite (i.e. the integral converges).

N-players: Partial (not intergenerational) cooperation

If there exists a value function of class C^1 solving

$$V_i^c(x,t) = \int_t^\infty d_i(s,t) L_i(x(s), \phi_1^c(x(s),s), \ldots, \phi_N^c(x(s),s)\, ds \ ,$$

where

$$(u_1^c, \ldots, u_N^c) = (\phi_1^c(x,t), \ldots, \phi_N^c(x,t)) =$$

$$\mathrm{argmax}_{\{u_1,\ldots,u_N\}} \left\{ \sum_{i=1}^N \lambda_i(x,t) \left(L_i(x, u_1, \ldots, u_N, t) + \right.\right.$$

$$\left.\left. \nabla_x V_i^c(x,t) \cdot g(x, u_1, \ldots, u_N, t) \right) \right\} \ ,$$

and there exists a unique absolutely continuous curve $x \colon [0,\tau] \to X$ solution to $\dot{x}(t) = g(x(t), \phi_1^c(x(t),t), \ldots, \phi_N^c(x(t),t))$, $x(0) = x_0$, then $(u_1^c, \ldots, u_N^c) = (\phi_1^c(x,t), \ldots, \phi_N^c(x,t))$ is an equilibrium rule.

A Linear State Differential Game of Pollution Control

There are N countries coordinating their pollution strategies to optimize their joint payoff. Let us denote by $E_i(t)$, for $i = 1, \ldots, N$, the emissions of country i at time t. The evolution of the stock of pollution $S(t)$ is described by the differential equation

$$\dot{S}(\tau) = \sum_{i=1}^{N} E_i(\tau) - \delta S(\tau) \, , \quad S(0) = S_0 \, ,$$

where $\delta > 0$ represents the natural absorption rate of pollution. The emissions are assumed to be proportional to the production and hence the revenue from production can be expressed as a function of the emissions. In particular, the revenue function of country i is assumed to be logarithmic. The damage cost is a linear function on the stock of pollution. The intertemporal utility function for player i is given by

$$J_i = \int_t^{\infty} \theta_i(\tau - t)\big(\ln E_i(\tau) - \varphi_i S(\tau)\big) \, d\tau \, .$$

Noncooperative Markovian Nash equilibrium

In this case, player i aims to maximize

$$\max_{\{E_i\}} \left\{ \ln E_i - \varphi_i S + (V_i^{nc}(S))'(E_i + \sum_{j \neq i} \phi_j^{nc}(S) - \delta S) \right\}$$

where $E_j^{nc} = \phi_j^{nc}(S)$ is the equilibrium rule. Then $E_i^{nc} = (-(V_i^{nc})'(S))^{-1}$. We look for a value function of the form $V_i(S) = \alpha_i^{nc} S + \beta_i^{nc}$. Then $E_i^{nc} = \phi_i^{nc} = (-\alpha_i^{nc})^{-1}$. Therefore,

$$S(\tau) = e^{-\delta(\tau - t)} S_t - \sum_{j=1}^{N} \frac{1}{\delta \alpha_j^{nc}} \left(1 - e^{-\delta(\tau - t)}\right) .$$

A Linear State Differential Game of Pollution Control

By identifying the value functions we obtain

$$\alpha_i^{nc} S + \beta_i^{nc} = \int_t^{\infty} \theta_i(\tau - t) \left[\ln \phi_i^{nc}(S(\tau)) - \varphi_i S \right] d\tau =$$

$$\int_t^{\infty} \theta_i(\tau - t) \left[-\ln(-\alpha_i^{nc}) - \varphi_i \left(e^{-\delta(\tau - t)} S - \sum_{j=1}^{N} \frac{1}{\delta \alpha_j^{nc}} (1 - e^{-\delta(\tau - t)}) \right) \right] d\tau$$

By simplifying we obtain

$$\alpha_i^{nc} = -\varphi_i \int_0^{\infty} \theta_i(\tau) e^{-\delta\tau} \, d\tau \,,$$

$$\beta_i^{nc} = -\ln \left(\varphi_i \int_0^{\infty} \theta_i(\tau) e^{-\delta\tau} \, d\tau \right) \int_0^{\infty} \theta_i(\tau) \, d\tau$$

$$-\frac{\varphi_i}{\delta} \sum_{j=1}^{N} \frac{1}{\varphi_j \int_0^{\infty} \theta(\tau) e^{-\delta\tau}} \left(\int_0^{\infty} \theta_i(\tau)(1 - e^{-\delta\tau}) \, d\tau \right) .$$

Partial cooperation

- We consider the case of nonconstant weights for this problem.
- Since the decision rule for linear state games is typically independent on the state variable (the pollution stock), it seems natural to restrict our attention to weights $\lambda_i(t)$, i.e. independent on the state variable. This simplification allows to solve the model.
- The payoff for the grand coalition is given by

$$J^c = \sum_{j=1}^{N} \lambda_j^c(t) \int_t^{\infty} \theta_j(\tau - t)(\ln E_j(\tau) - \varphi_j S(\tau))\, d\tau \,.$$

- We must solve

$$\max_{\{E_1,\ldots,E_N\}} \left\{ \sum_{j=1}^{N} \lambda_j^c(t) \left[\ln E_j - \varphi_j S + (V_j^c(S))' \left(\sum_{i=1}^{N} E_i - \delta S \right) \right] \right\} .$$

A Linear State Differential Game of Pollution Control

We look for a family of value functions of the form $V_j(S) = \alpha_j^c(t)S + \beta_j^c(t)$, for $j = 1, \ldots, N$. Then the emission rules become

$$E_i = -\frac{\lambda_i(t)}{\sum_{j=1}^N \lambda_j(t)\alpha_j^c(t)} \, .$$

The solution to

$$\dot{S}(\tau) = -\frac{\sum_{j=1}^N \lambda_j(t)}{\sum_{i=1}^N \lambda_i(t)\alpha_i^c(t)} - \delta S(\tau) = -\frac{N}{\sum_{i=1}^N \lambda_i(t)\alpha_i(t)} - \delta S(\tau) \, ,$$

with the initial condition $S(t) = S_t$ is given by

$$S(\tau) = e^{-\delta(\tau - t)} S_t - \int_t^\tau \frac{e^{-\delta(\tau - z)}}{\sum_{i=1}^N \lambda_i(z)\alpha_i^c(z)} \, dz \, .$$

A Linear State Differential Game of Pollution Control

$$\alpha_i(t)S + \beta_i(t) = \int_t^{\infty} \theta_i(\tau - t) \left[\ln \left(-\frac{\lambda_i(\tau)}{\sum_{j=1}^{N} \lambda_j(\tau)\alpha_j^c(\tau)} \right) - \right.$$

$$\left. \varphi_i \left(e^{-\delta(\tau - t)} S_t - \int_t^{\tau} \frac{e^{-\delta(\tau - z)}}{\sum_{j=1}^{N} \lambda_j(z)\alpha_j^c(z)} \, dz \right) \right] d\tau \ .$$

$$\alpha_i^c(t) = -\varphi_i \int_0^{\infty} \theta_i(\tau) e^{-\delta\tau} \, d\tau \ ,$$

$$\beta_i^c(t) = \int_t^{\infty} \theta_i(\tau - t) \left[\ln \frac{\lambda_i(\tau)}{\sum_{j=1}^{N} \varphi_j \int_0^{\infty} \theta_j(z) e^{-\delta z} \, dz} \right.$$

$$\left. -\varphi_i \int_t^{\tau} \frac{e^{-\delta(\tau - z)}}{\sum_{j=1}^{N} \varphi_j \int_0^{\infty} \theta_j(s) e^{-\delta s} \, ds} \, dz \right] d\tau \ .$$

We have:

1. $\alpha_i^c = \alpha_i^{nc}$. Hence, in order to check the time consistency (and agreeability) of the equilibrium, as in the case of constant and equal discount rates (Jorgensen et al, JOTA 2003), it suffices to check if $\beta_i^c \geq \beta_i^{nc}$.
2. The emission rules of country i in the noncooperative and partial cooperative case are

$$E_i^{nc} = \frac{1}{\varphi_i \int_0^\infty \theta_i(\tau) e^{-\delta\tau}\, d\tau} \ ,$$

$$E_i^c(t) = \frac{\lambda_i(t)}{\sum_{j=1}^N \lambda_j(t) \varphi_j \int_0^\infty \theta_j(\tau) e^{-\delta\tau}\, d\tau} \ .$$

Clearly, $E_i^{nc} > E_i^c$.

A Common Property Resource Game.

Let us consider the problem of exploitation of a renewable natural resource in which, if $x(t)$ represents the stock of natural resource at time t, and $h_i(t)$ is the harvest rate at time t of player i, for $i = 1, \ldots, N$, the state dynamics is described by the equation

$$\dot{x}(s) = x(s)(a - b \ln x(s)) - \sum_{i=1}^{N} h_i(s) \; , \quad x(t) = x_t \; .$$

Players have logarithmic instantaneous utility functions depending just on their harvest rates, and they discount the future according to different distance-based nonconstant discount rates of time preference. Hence, the intertemporal utility function for player i is given by

$$J_i = \int_t^{\infty} \theta_i(s - t) \ln h_i(s) \, ds \; .$$

Noncooperative Markovian Nash equilibrium

If players do not cooperate, let us look for stationary strategies. Player i aims to look for the solution to

$$\max_{\{h_i\}} \left\{ \ln h_i + (V_i^{nc}(x))'[x(a - b\ln x) - h_i - \sum_{j\neq i} \phi_j^{nc}(x)] \right\},$$

where $\phi_j^{nc}(x)$, $j = 1, \ldots, N$, denotes the equilibrium strategy of player j in feedback form. We look for a value function of the form $V_i^{nc}(x) = \alpha_i^{nc}\ln x + \beta_i^{nc}$, for $i = 1, \ldots, N$. Then $h_i^{nc} = \phi_i^{nc}(x) = (\alpha_i^{nc})^{-1}x$ and

$$x(s) = \exp\left[\left(\ln x_t + \frac{\sum_{j=1}^N \frac{1}{\alpha_j^{nc}} - a}{b}\right)e^{-b(s-t)} + \frac{a - \sum_{j=1}^N \frac{1}{\alpha_j^{nc}}}{b}\right].$$

By solving we obtain

$$\alpha_i^{nc} = \int_0^\infty \theta_i(s) e^{-bs}\, ds \ ,$$

and

$$\beta_i^{nc} = \frac{1}{b}\left(a - \sum_{j=1}^{N} \frac{1}{\int_0^\infty \theta_j(s) e^{-bs}\, ds} \right) \int_0^\infty \theta_i(s)[1 - e^{-bs}]\, ds$$

$$- \ln \left(\int_0^\infty \theta_i(s) e^{-bs}\, ds \right) \int_0^\infty \theta_i(s)\, ds \ ,$$

for $i = 1, \ldots, N$.

Partial cooperation

We restrict our attention to stationary strategies and constant weights.

$$\max_{\{h_1,\ldots,h_N\}} \left\{ \sum_{j=1}^{N} \lambda_j \ln h_j + \left(\sum_{i=1}^{N} \lambda_i (V_i^c(x))' \right) [x(a - b\ln x) - h_i - \sum_{j\neq i} \phi_j^c(x)] \right\}$$

We look for a set of value functions of the form $V_i^c(x) = \alpha_i^c \ln x + \beta_i^c$.

$$h_j^c = \phi^c(x) = \frac{\lambda_j x}{\sum_{i=1}^N \lambda_i \alpha_i^c} \quad \text{and}$$

$$x(s) = \exp\left[\left(\ln x_t + \frac{\sum_{j=1}^N \lambda_j - a\sum_{j=1}^N \lambda_j \alpha_j^c}{b\sum_{j=1}^N \lambda_j \alpha_j^c} \right) e^{-b(s-t)} - \frac{\sum_{j=1}^N \lambda_j - a\sum_{j=1}^N \lambda_j \alpha_j^c}{b\sum_{j=1}^N \lambda_j \alpha_j^c} \right].$$

Proceeding as in the noncooperative case, we easily obtain

$$\alpha_i^c = \int_0^\infty \theta_i(s) e^{-bs}\, ds ,$$

and

$$\beta_i^c = \frac{a \sum_{j=1}^N \lambda_j \int_0^\infty \theta_j(s)\, ds - N}{b \sum_{j=1}^N \lambda_j \int_0^\infty \theta_j(s)\, ds} \int_0^\infty \theta_i(s) [1 - e^{-bs}]\, ds$$

$$- \ln \left(\frac{\sum_{j=1}^N \lambda_j \int_0^\infty \theta_j(s) e^{-bs}\, ds}{\lambda_i} \right) \int_0^\infty \theta_i(s)\, ds ,$$

for $i = 1, \ldots, N$.

A Common Property Resource Game

We have:

1. $\alpha_i^c = \alpha_i^{nc}$. Hence, in order to check the time consistency (and agreeability) of the equilibrium, it suffices to check if $\beta_i^c \geq \beta_i^{nc}$.
2. The harvest rules of player i in the noncooperative and partial cooperative case are

$$h_i^{nc}(x) = \frac{x}{\int_0^\infty \theta_i(s) e^{-bs}\, ds},$$

$$h_i^c(x) = \frac{\lambda_i x}{\sum_{j=1}^N \lambda_j \int_0^\infty \theta_j(s) e^{-bs}\, ds}.$$

   Clearly, $h_i^{nc} > h_i^c$.
3. There exists constant weights guaranteeing the sustainability of cooperation. For example, in the extrem case with $\rho_1 = 0.9$, $\rho_2 = 0.1$, take $\lambda_1 \in (0.9466565788, 0.9836663427)$.

A Common Property Resource Game

More general utility functions, $b = 0$

Let us consider the following problem::

- The intertemporal utility function for player i is given by

$$J_i = \int_t^\infty \theta_i(s-t) \frac{c_i^{1-\sigma_i}(s) - 1}{1-\sigma_i} \, ds \, .$$

- The state dynamics is described by the equation

$$\dot{x}(s) = ax - \sum_{i=1}^{2} c_i(s) \, , \quad x(t) = x_t \, .$$

Marginal elasticities and discount functions are different.

Markovian Nash equilibrium

1. Linear equilibrium rules of the form $c_i^{nc}(x) = A_i x$, with A_i constant, exist.
2. The corresponding value functions are of the form $V_i^{nc} = \alpha_i^{nc} x + \beta_i^{nc}$, with $\alpha_i^{nc}, \beta_i^{nc}$ constant numbers.

Partial cooperation

1. If weights are constant, no linear equilibrium rules exist.
2. In the case of general nonconstant weights, stationary linear equilibrium rules exist if, and only if, weights are such that
$$\frac{\lambda_i(x)}{\lambda_j(x)} = \frac{A_i}{A_j} x^{\sigma_i - \sigma_j} \ .$$
3. In Sorger, JEDC 2006, recursive Nash bargaining solution was introduced for a dynamic game with two asymmetric players in a discrete time setting and partial cooperation. According to this procedure, $\lambda(x) = \dfrac{\lambda_2(x)}{\lambda_1(x)} = Ax^{\sigma_2 - \sigma_1}$.
4. Although both procedures are essentially different, maybe the set of recursive Nash bargaining solutions for 2-player games are particular realizations of partial cooperative equilibria with nonconstant weights. This would provide a support to the *a priori* arbitrary choice of this particular class of nonconstant weights.