Pairwise Comparison Dynamics for Games with Continuous Strategy Space

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Evolutionary dynamics with finite strategy space

Incentives and aggregate behavior

- Nash stationarity (NS): rest points of the dynamic coincide with the set of Nash equilibria
- positive correlation (PC): whenever the dynamic is not at rest, the covariance between strategies' payoffs and growth rates is positive under the uniform probability distribution on strategies

Convergence results for pairwise comparison dynamics (PCD)

- potential games: all ω-limit points of the dynamic are NE
- contractive games: set of NE is globally asymptotically stable

Usually, the underlying games studied are normal form games with finite number of strategies.

However, the restriction on the dimension of strategy space limits the use of population games in applications whose strategy spaces are naturally modeled as continuous, e.g., auctions, bargaining games, games of timing, oligopoly games.

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New technical issues:

- infinite dimensional state space
- choice of topology matters
    - existence/uniqueness of solutions
    - definition/analysis of stability

Literature

- Bomze (1990, 1991)
  - defines replicator dynamic in Banach space of finite signed measures with variational norm
  - shows replicator dynamic is well-defined if certain Lipschitz continuity conditions are satisfied for mean payoff function
- Oechssler and Riedel (2001, 2002)
  - show Bomze's conditions are always satisfied in pairwise encounters if underlying pairwise payoff function is bounded
  - introduce evolutionary robustness
- Hofbauer, Oechssler and Riedel (2009)
  - show Nash stationarity (NS) is satisfied for BNN dynamic
  - provide stability results for doubly symmetric games ($\subseteq$ potential games) and contractive games

This paper

- studies PCD for population games with cts strategy space
- shows PCD is well-defined under certain mild Lipschitz continuity conditions
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- studies PCD for population games with cts strategy space
- shows PCD is well-defined under certain mild Lipschitz continuity conditions
- establishes Nash stationarity (NS) and positive correlation (PC)
- defines potential games with cts strategy space, and provides a global convergence result for general deterministic evolutionary dynamics in potential games
- defines contractive games with cts strategy space, and provides a global asymptotic stability result for PCD in contractive games

Settings

The set of strategies is $\mathcal{S}$, which is compact and convex, with metric d.

There is a unit mass of agents. Each chooses a pure strategy from $\mathcal{S}$.

Thus, a population state (distribution over strategies) is described by a probability measure $\mu \in \mathcal{M}_1^+(\mathcal{S})$

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Payoffs

A population game is a map $F : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{C}_b(\mathcal{S})$ that is continuous with respect to the weak topology. (Thus, $F(\mu) \in \mathcal{C}_b(\mathcal{S})$.)

$F_x(\mu)$ is the payoff of pure strategy $x \in \mathcal{S}$ at state μ.

Example: Matching to play a 2-player symm game

- a unit mass of agents are matched to play a 2-player symm game
- $\mathcal{S}$: continuous strategy set, convex and compact
- each pair of agents is matched exactly once
- $h(x, y)$: single match payoff of an agent playing $x \in \mathcal{S}$ against an opponent playing $y \in \mathcal{S}$, assume h is bounded and continuous
- $F_x(\mu) = \int_{\mathcal{S}} h(x, y)\, \mu(dy)$, average payoff of $x \in \mathcal{S}$ at state μ

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- $F_x(\mu) = \int_{\mathcal{S}} h(x, y)\, \mu(dy)$, average payoff of $x \in \mathcal{S}$ at state μ
- $\mu \mapsto F_x(\mu)$ is continuous wrt weak topology
- $x \mapsto F_x(\mu)$ is bounded continuous
- thus, $F : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{C}_b(\mathcal{S})$ defines a population game

Weak topology vs strong topology

- Strong topology $\Rightarrow$ existence/uniqueness of solutions
    - $\|\varphi\| := \sup_g |\int g\, d\varphi|$, the variational norm on $\mathcal{M}(\mathcal{S})$, where the sup is taken over all mble $g : \mathcal{S} \to \mathbb{R}$ with $|g| \leq 1$
    - $(\mathcal{M}(\mathcal{S}), \|\cdot\|)$ is a Banach space $\Rightarrow$ allows us to show existence/uniqueness of solutions

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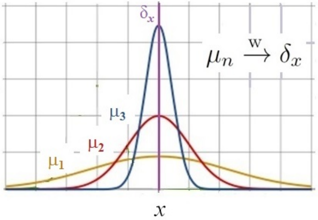
  - $(\mathcal{M}(\mathcal{S}), \| \cdot \|)$ is a Banach space $\Rightarrow$ allows us to show existence/uniqueness of solutions
- Weak topology gives a more natural notion of "closeness" for definitions of local stability
  - strong topology $\Rightarrow$ only consider perturbations to states by (possibly large) change of strategic play from small fraction of players (e.g., change from δ_x to $(1-\varepsilon)\delta_x + \varepsilon\delta_u$)
  - in applications, also want to consider perturbations to states by small change of strategic play from large fraction of players (e.g., change from δ_x to δ_u with $d(x, u) < \varepsilon$)

Weak topology vs strong topology (con't)

- Weak topology provides the more natural notion of convergence for sequences of population states (ex.: $\mu_n \sim \mathcal{N}(x, \sigma_n^2)$ with $\sigma_n \to 0$), and is needed to ensure existence of ω-limit points



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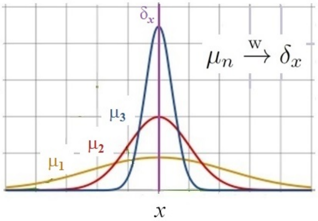

- $\mu_n \to \mu$ in strong topology $\Rightarrow$ $\mu_n \to \mu$ in weak topology
- $F : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{C}_b(\mathcal{S})$
  continuous in weak topology $\Rightarrow$ continuous in strong topology

Pairwise comparison dynamics (PCD)

Let λ be a fixed probability measure that has full support.

Special case: Smith dynamic (SD) (1984)

$$\begin{aligned}\dot{\mu}(A) &= \int_{z\in\mathcal{S}}\int_{y\in A}[F_y(\mu)-F_z(\mu)]_+\,\lambda(dy)\,\mu(dz) \\ &\quad - \int_{z\in\mathcal{S}}\int_{y\in A}[F_z(\mu)-F_y(\mu)]_+\,\mu(dy)\,\lambda(dz), \qquad \text{(SD)}\end{aligned}$$

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Remark: All results apply to the more general pairwise comparison dynamics (PCD).

Existence and uniqueness of solutions

Theorem: Under mild Lipschitz continuity conditions, solutions to SD exist and are unique, and are continuous in initial conditions.

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Corollary: For the ex. of matching to play a 2-player symm game, solutions to SD exist and are unique, and are continuous in initial conditions.

Nash stationarity (NS) and positive correlation (PC)

- relate dynamics to underlying game
- provide tools for analysis of convergence
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Proposition: Every pairwise comparison dynamic satisfies Nash stationarity (NS) and positive correlation (PC).

Potential games (Monderer and Shapley (1996), Sandholm (2001))

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Definition: Population game $F : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{C}_b(\mathcal{S})$ is a potential game if there exists a continuous (wrt weak topology) function $f : \mathcal{M}(\mathcal{S}) \to \mathbb{R}$ s.t. f is Fréchet-differentiable with gradient ∇f satisfying

$$\nabla f(\mu) = F(\mu), \quad \forall \mu \in \mathcal{M}_1^+(\mathcal{S}). \tag{PG}$$

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Denote by Ω the set of all ω-limit points of all solution trajectories.

Theorem: Let F be a potential game with potential function f, and $\dot{\mu} = V^F(\mu)$ be SD for F. Then $\Omega = RP(V^F) = NE(F)$.

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Remark: For general deterministic dynamics, if only (PC) is satisfied but not (NS), then we still have $\Omega = RP(V^F)$.

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$$\langle F(\mu)-F(\psi), \mu-\psi\rangle \equiv \int_{\mathcal{S}} (F(\mu)-F(\psi))\, d(\mu-\psi) \leq 0, \ \ \forall \mu, \psi \in \mathcal{M}_1^+(\mathcal{S}). \quad \text{(CG)}$$

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- null contractive game: ineq always binds

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Theorem: Let $F : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{C}_b(\mathcal{S})$ be C^1 contractive game, $\dot{\mu} = V^F(\mu)$ be SD, and $\eta_y(d) := \int_0^d [r]_+ \, dr$. Define $H : \mathcal{M}_1^+(\mathcal{S}) \to \mathbb{R}_+$ by

$$H(\mu) := \int_{y \in \mathcal{S}} \int_{z \in \mathcal{S}} \eta_y(F_y(\mu) - F_z(\mu))\, \mu(dz)\, \lambda(dy).$$

Then $H^{-1}(0) = NE(F)$. Furthermore, H acts as a decreasing strict Lyapunov function, and so $NE(F)$ is globally asymptotically stable.

Summary

- studied PCD for population games with cts strategy space
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Fréchet-differentiability and gradient

Suppose $f : \mathcal{M}(\mathcal{S}) \to \mathbb{R}$ is Fréchet-differentiable when $\mathcal{M}(\mathcal{S})$ is endowed with strong topology.[1] The Fréchet-derivative of f at $\mu \in \mathcal{M}(\mathcal{S})$ is a cts linear map

$$Df(\mu) : \mathcal{M}(\mathcal{S}) \to \mathbb{R}$$

that maps tangent vectors $\zeta \in \mathcal{M}(\mathcal{S})$ to rates of change in the value of f when one moves from μ in the direction ζ. Since $Df(\mu)$ is a linear map from $\mathcal{M}(\mathcal{S})$ to $\mathbb{R}$, by Riesz representation theorem, there is an element $\nabla f(\mu)$ of $\mathcal{C}_b(\mathcal{S})$ (the dual space of $\mathcal{M}(\mathcal{S})$) that represents $Df(\mu)$ in the sense that

$$Df(\mu)\zeta = \int \nabla f(\mu) d\zeta \equiv \langle \nabla f(\mu), \zeta \rangle.$$

This $\nabla f(\mu)$ is called the gradient of f at μ.

[1] If X and Y are Banach spaces, we say $g : X \to Y$ is Fréchet-differentiable at x if $\exists$ a continuous linear map $T : X \to Y$ such that $g(x+\vartheta) = g(x) + T\vartheta + o(\|\vartheta\|)$ for all ϑ in some nbd of zero in X. If it exists, this T is called the Fréchet-derivative of g at x, and is written as $Dg(x)$.

Weak topology vs strong topology (picture)

close in both weak and strong topologies

close in weak topology but not in strong topology