THE CAT(0) PROPERTY FOR THE MANIFOLD OF RIEMANNIAN METRICS

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Abstract. I will discuss the metric geometry of the manifold of all Riemannian metrics over a closed manifold, in particular sketching a proof that the L^2 metric on this space has nonpositive curvature as a metric space, at least after passing to the completion. I will also discuss some general consequences of nonpositive curvature and potential applications in the context of spaces of metrics.

1. Background

For the entire talk: M a smooth, closed, oriented manifold, $\dim M = n$. Fix a smooth reference Riemannian metric $\bar{g}$.

A tale of two spaces:

$\mathrm{Met}_x(M)$	$\mathrm{Met}(M)$
$\mathrm{Met}_x(M) := S^2_+ T^*_x M = \{$positive definite scalar products on $T_x M\}$	$\mathrm{Met}(M) := \Gamma(S^2_+ T^* M) = \{$Riemannian metrics on $M\}$
finite-dimensional	infinite-dimensional
$T_a \mathrm{Met}_x(M) \cong \mathcal{S}_x = S^2_+ T^* M$	$\mathcal{T}_g \mathrm{Met}_x(M) = \mathcal{S} = \Gamma(S^2 T^* M)$
$\langle b, c \rangle_a = \mathrm{tr}_a(bc) \sqrt{\bar{g}(x)^{-1} a}$, where $\mathrm{tr}_a(bc) = \mathrm{tr}(a^{-1} b a^{-1} c) = a^{ij} b_{jl} a^{lm} c_{mi}$ (for $b, c \in T_a \mathrm{Met}_x(M)$	$(h, k)_g = \int_M \mathrm{tr}_g(hk)\, d\mu_g$ (L^2 or *Ebin metric*)
$d_x :=$ Riemannian distance function of $\langle \cdot, \cdot \rangle$	$d :=$ Riemannian distance function of $(\cdot, \cdot)$

Freed–Groisser ('89), Gil-Medrano–Michor ('91): The (C^∞)-local geometry of $\mathrm{Met}(M)$ can be understood through the local geometry of $\mathrm{Met}_x(M)$.

Given a vector field h defined along a curve, we have

$$\frac{\nabla^{\mathrm{Met}(M)}}{dt} h \bigg|_x = \frac{\nabla^{\mathrm{Met}_x(M)}}{dt} h(x).$$

This implies that curvature and geodesics are given "fiberwise": e.g., a curve $g(t)$ in $\mathrm{Met}(M)$ is a geodesic if and only if $g(t)(x)$ is a geodesic in $\mathrm{Met}_x(M)$ for each x (this automatically glues together smoothly in x).

Using this one can:

- Compute the curvature of $\mathrm{Met}(M)$—it is nonpositive, since the curvature of $\mathrm{Met}_x(M)$ is nonpositive;
- Solve the geodesic equation of $\mathrm{Met}(M)$ explicitly in terms of elementary functions.

On the other hand, $(\cdot,\cdot)$ is a *weak Riemannian metric* $\Longrightarrow$ knowledge of geodesics gives no *a priori* information about the distance function, even locally (i.e., in a small d-ball).

In the case of $(\mathrm{Met}(M), (\cdot,\cdot))$, this is indeed a very large problem:

$$\mathrm{im}(\exp_{g_0}) = \left\{ g = g_0 \exp H \in \mathrm{Met}(M) \;\middle|\; H \in \Gamma(\mathrm{End}(M)),\ \mathrm{tr}(H_T^2) < \frac{16\pi^2}{n} \right\}$$

This is a pointwise condition, while d induces an integral-type distance. A nearly direct consequence is that the image of the exponential mapping contains no open d-ball.

This breaks the most important meta-tool in Riemannian geometry: the local$\longrightarrow$infinitesimal reduction allowed by the differential nature of the metric.

An additional complication: $\mathrm{Met}(M)$ is incomplete, with essentially two "flavors" of incompleteness arising:

- There is a geodesic boundary, i.e., some geodesics exit the space in finite time (by losing positive definiteness at some points): Geodesic with initial data $g(0) = g_0$, $\dot{g}(0) = \rho g_0$ for some $\rho \in C^\infty(M)$ is
$$g(t) = \left(1 + \frac{n}{4}\rho t\right)^{4/n} g_0.$$
- Metrics in the completion may lose smoothness in x: such metrics are not reachable by geodesics.

2. Results

The incompleteness complication was dealt with in my thesis:

Theorem 1 (C. '09)**.** *Let $\mathrm{Met}_f(M)$ denote the space of all measurable, symmetric, finite-volume $(0,2)$-tensor fields on M that induce a positive semidefinite scalar product on each tangent space of M. For $g_0, g_1 \in \mathrm{Met}_f(M)$ and $x \in M$, we say $g_0 \sim g_1$ if and only if the following statement holds almost surely:*

If at least one of $g_0(x)$ or $g_1(x)$ is nondegenerate, then $g_0(x) = g_1(x)$.

Then there is a natural identification $\overline{\mathrm{Met}(M)} \cong \mathrm{Met}_f(M)/\sim$.

(Note the parallel with the result of Shnirelman ('85): for $n \geq 3$, the completion of the space of diffeomorphisms of the n-cube w.r.t. its L^2 metric is given by the set of all measure-preserving endomorphisms of the cube.)

The lack of local$\longrightarrow$infinitesimal reduction is overcome by an infinite-dimensional$\longrightarrow$finite-dimensional reduction:

Theorem 2 (C. '12)**.** *Let*

$$\Omega_2(g_0, g_1) := \left(\int_M d_x(g_0(x), g_1(x))^2 \, d\,\mathrm{vol}_{\bar{g}} \right)^{1/2}.$$

Then $d = \Omega_2$.

This allows for a very complete understanding of the Ebin distance d. In particular:

Theorem 3. $\overline{(\mathrm{Met}(M), d)}$ *is a CAT(0) space.*

The CAT(0) property is a nonpositive curvature property for *path metric spaces.*

The CAT(0) property has some important implications:

- *Existence of harmonic maps (Korevaar–Schoen ('93), Jost ('94...)):* Let X be a closed manifold with $\Gamma \subseteq \mathrm{Isom}(X)$ a subgroup, and let Y be a CAT(0) space. Suppose $\rho : \Gamma \to \mathrm{Isom}(Y)$ is a *reductive*[1] homomorphism. Then there exists a ρ-equivariant harmonic map in every homotopy class of maps $X \to Y$.
  - Used, e.g., by Korevaar–Schoen (unpublished) with $\mathrm{Met}_\mu(M)$ to study existence of invariant Riemannian metrics for actions of groups with Property (T) on closed manifolds.
- *Existence of barycenters (centers of mass):* Given a measure μ with finite second moment on a CAT(0) space, there exists a unique barycenter q with:

  $$\int d^2(q, x) \, d\mu(x) = \inf_p \int d^2(p, x) \, d\mu(x).$$

  This is an important technical tool, and can be used to prove...
- *Ergodic theorems:* e.g., Austin ('11) uses the barycenter mapping of a CAT(0) space to show existence of functions $f : \Omega \to X$ (Ω: probability space; X: CAT(0) space) invariant under a measure-preserving action on Ω. (See also Lior Silberman's talk for another striking application of CAT(0) geometry on $\mathrm{Met}_\mu(M)$.)

3. Proof sketches

Sketch as much of the proofs of Theorems 2 and 3 as possible.

Important technical tool for these proofs:

[1] A subgroup $G \subseteq \mathrm{Isom}(Y)$ is reductive if whenever (p_n) is an unbounded sequence with $d(p_n, \gamma p_n) \leq const.$ for all $\gamma \in \Gamma$ (with a constant depending on γ), then G stabilizes a totally geodesic flat subspace.

Proposition 4 (C. '11). *Let $g_0, g_1 \in \operatorname{Met}_f(M)$ and $A := \operatorname{carr}(g_1 - g_0)$. Then*

$$d(g_0, g_1) \leq C(n) \left(\sqrt{\operatorname{Vol}(A, g_0)} + \sqrt{\operatorname{Vol}(A, g_1)} \right),$$

where $C(n)$ is a constant depending only on $n = \dim M$.

Theorem 2: Sketch of proof. To prove this theorem, we need to find a path of smooth (in x) metrics in $\operatorname{Met}(M)$ from g_0 to g_1 that has near-minimal length *pointwise*.

First reduction: by smoothing, we can just look for a continuous path.

A naive approach: construct a near-minimal path from $g_0(x)$ to $g_1(x)$ at each x and glue these together. Problem: no clear way to insure that such paths will glue together continuously, particularly when trying to "close up" the manifold.

Solution:

- At a point $x_0 \in M$, choose a near-minimal path from $g_0(x_0)$ to $g_1(x_0)$.
- In a ball with radius smaller than $\operatorname{inj}(M)$, can extend this path by parallel transport along radial geodesics (draw picture).
- In a small enough ball, can do this so that the path is still almost minimal at each point of the ball
- Such a "small enough" ball can be chosen in a uniform way and M can be covered up to a small subset by such balls.
- Interpolation gives a path from g_0 to some metric g_1^ϵ that agrees with g_1 over most of the manifold; Proposition 4 finishes the job.

□

Theorem 2 implies, in particular, that geodesics (in the metric space sense) in $\overline{\operatorname{Met}(M)}$ are exactly those that are geodesic in $\overline{\operatorname{Met}_x(M)}$ at each point.

Theorem 3: Sketch of proof. Essence of proof:

(1) Showing that Riemannian geodesics in $\operatorname{Met}_x(M)$ are minimizing in $\overline{\operatorname{Met}_x(M)}$;
(2) Establishing geodesics to points not connected by a Riemannian geodesic (concatenations of straight segments through $[0]$);
(3) Using Freed–Groisser/Gil-Medrano–Michor to establish CAT(0) inequality locally for $\operatorname{Met}_x(M)$;
(4) Analyzing triangles involving geodesics that pass through $[0]$.

Once the CAT(0) inequality is established for $\overline{\operatorname{Met}_x(M)}$, it can be integrated to Ω_2 (and hence d).

Points 1) and 2) above involve investigating the explicit expressions for geodesics in $\operatorname{Met}_x(M)$ and establishing that the closest point to a_0 on the boundary of $\operatorname{im}(\exp_{a_0})$ is $[0]$, and the shortest path to [0] is the straight segment. Uses a low-regularity version of a foundational result in Riemannian geometry (minimality of radial geodesics among paths lying in $\mathrm{im}(\exp_{g_0})$). $\square$